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SEVERAL REVERSE INEQUALITIES OF OPERATORS

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ABSTRACT. In this report, we show reverse inequalities to Araki’s inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and L"owner-Heinz inequalities. Among others, we show that if $A$ and $B$ are positive operators on a Hilbert space $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p) \|BAB\|^p \leq \|B^p A^p B^p\|$$
for all $0 < p < 1$,

where $K(m, M, p)$ is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let $A$ and $B$ be positive operators on a Hilbert space $H$. The equivalence among Cordes and L"owner-Heinz inequalities was discussed by many authors. In [8], Furuta showed that the Cordes inequality

$$\|A^p B^p\| \leq \|AB\|^p$$
for $0 < p < 1$
is equivalent to the L"owner-Heinz inequality (cf.[14])

$$A \geq B \geq 0 \implies A^p \geq B^p$$
for $0 < p < 1$

(cf. [5]). In [1], Araki showed a trace inequality which entails the following inequality:

$$\|B^p A^p B^p\| \leq \|BAB\|^p$$
for $0 < p < 1$.

Moreover, it was shown in [6, 2] that the Cordes inequality (1) is equivalent to Araki’s inequality (3).

On the other hand, Furuta [9] showed the following Kantorovich type inequalities: If $A$ and $B$ are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \implies K(m, M, p) A^p \geq B^p$$
for $p > 1$,

where a generalized Kantorovich constant $K(m, M, p)$ [3, 7, 11] is defined as

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left( \frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p$$
for all real numbers $p$.

We here cite Furuta’s textbook [10] as a pertinent reference to Kantorovich inequalities.

Also, Yamazaki [16] showed the following difference type reverse inequalities of the L"owner-Heinz inequality: If $A$ and $B$ are positive operators with $0 < mI \leq B \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \implies C(m, M, p) + A^p \geq B^p$$
for $p > 1$,
where the constant $C(m, M, p)$ [12, 16] is defined as

\[(7)\]
\[C(m, M, p) = (p - 1) \left( \frac{M^p - m^p}{p(M - m)} \right)^{\frac{1}{p-1}} + \frac{Mm^p - mM^p}{M - m} \quad \text{for all real numbers } p.\]

In this report, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If $A$ and $B$ are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then the following inequalities hold

\[(8)\]
\[K(m, M, p) \|BAB\|^p \leq \|B^pA^pB^p\| \quad \text{for } 0 < p < 1,\]

\[(9)\]
\[K(m^2, M^2, p)^{1/2} \|AB\|^p \leq \|A^pB^p\| \quad \text{for } 0 < p < 1,\]

respectively. We moreover show that reverse inequalities (4), (8) and (9) are mutually equivalent.

2. MAIN RESULTS

First of all, we present our main theorem which is a reverse inequality to Araki's inequality (3).

**Theorem 1.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then for each $\alpha > 0$

\[(10)\]
\[\|BAB\|^p \leq \alpha \|B^pA^pB^p\| + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|B\|^{2p} \quad \text{for all } 0 < p < 1,\]

or equivalently

\[(11)\]
\[\|B^pA^pB^p\|^{\frac{1}{p}} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha) \|B\|^2 \quad \text{for all } p > 1,\]

where

\[(12)\]
\[\beta(m, M, p, \alpha) = \begin{cases} \frac{p-1}{p} \left( \frac{M^p - m^p}{\alpha p(M - m)} \right)^{\frac{1}{p-1}} + \frac{p-1}{p} \left( \frac{M^p - m^p}{\alpha p(M - m)} \right)^{\frac{1}{p-1}} & \text{if } \frac{M^p - m^p}{pM^{p-1}(M - m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M - m)}, \\ (1 - \alpha)M & \text{if } 0 < \alpha \leq \frac{M^p - m^p}{pM^{p-1}(M - m)}, \\ (1 - \alpha)m & \text{if } \alpha \geq \frac{M^p - m^p}{pm^{p-1}(M - m)}. \end{cases}\]

If we choose $\alpha$ satisfying $\beta(m, M, p, \alpha) = 0$ in Theorem 1, then we have the following ratio type reverse inequalities.

**Corollary 2.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\[(13)\]
\[K(m, M, p) \|BAB\|^p \leq \|B^pA^pB^p\| \quad \text{for } 0 < p < 1,\]

or equivalently

\[(14)\]
\[\|BAB\|^p \leq K(m, M, p) \|B^pA^pB^p\| \quad \text{for } p > 1,\]

where $K(m, M, p)$ is defined as (5) in §1.
If we put $\alpha = 1$ in Theorem 1, then we have the following difference type reverse inequalities.

**Corollary 3.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\begin{equation}
\|BAB\|^{p} - \|B^{p}A^{p}B^{p}\| \leq -C(m, M, p)\|B\|^{2p} \quad \text{for } 0 < p < 1,
\end{equation}

or equivalently

\begin{equation}
\|B^{p}A^{p}B^{p}\|^{\frac{1}{p}} - \|BAB\| \leq -C(m^{p}, M^{p}, \frac{1}{p})\|B\|^{2} \quad \text{for } p > 1,
\end{equation}

where $C(m, M, p)$ is defined as (7) in §1.

As special cases of Corollary 2 and Corollary 3, we have the following corollary.

**Corollary 4.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\begin{align*}
(17) & \quad \|B^{2}A^{2}B^{2}\| \leq \frac{(M + m)^{2}}{4Mm}\|BAB\|^{2}. \\
(18) & \quad \|B^{2}A^{2}B^{2}\|^{\frac{1}{2}} - \|BAB\| \leq \frac{(M - m)^{2}}{4(M + m)}\|B\|^{2}. \\
(19) & \quad \frac{2\sqrt{Mm}}{\sqrt{M} + \sqrt{m}}\|BAB\|^{\frac{1}{2}} \leq \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \\
(20) & \quad \|BAB\|^{\frac{1}{2}} - \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{(\sqrt{M} - \sqrt{m})^{2}}{4(\sqrt{M} + \sqrt{m})}\|B\|.
\end{align*}

Since $\|X^{*}X\| = \|X\|^{2}$ for an operator $X$, we obtain the following reverse inequality to the Cordes inequality by Corollary 2.

**Theorem 5.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\begin{equation}
K(m^{2}, M^{2}, p)^{\frac{1}{2}}\|AB\|^{p} \leq \|A^{p}B^{p}\| \quad \text{for all } 0 < p < 1,
\end{equation}

or equivalently

\begin{equation}
\|A^{p}B^{p}\| \leq K(m^{2}, M^{2}, p)^{\frac{1}{2}}\|AB\|^{p} \quad \text{for all } p > 1.
\end{equation}

In particular,

\begin{equation}
\sqrt{\frac{2\sqrt{Mm}}{M + m}}\|AB\|^{\frac{1}{2}} \leq \|A^{\frac{1}{2}}B^{\frac{1}{2}}\|.
\end{equation}

and

\begin{equation}
\|A^{2}B^{2}\| \leq \frac{M^{2} + m^{2}}{2Mm}\|AB\|^{2}
\end{equation}

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.
**Theorem 6.** For a given $p > 1$, the following are mutually equivalent: For all $A, B \geq 0$ and $0 < mI \leq A \leq MI$

(A) \quad A \geq B \geq 0 \implies K(m, M, p)A^p \geq B^p.
(B) \quad \|A^pB^p\| \leq K(m^2, M^2, p)^{1/2}\|AB\|^p.
(C) \quad \|B^pA^pB^p\| \leq K(m, M, 1/p)\|BAB\|^p.

(B') \quad K(m^2, M^2, 1/p)^{1/2}\|AB\|^p \leq \|A^pB^p\|.
(C') \quad K(m, M, 1/p)\|BAB\|^p \leq \|B^pA^pB^p\|.

3. **Lemmas**

We start with the following three lemmas before we give proofs of the results in §2.

Let $A$ be a positive operator on a Hilbert space $H$ and $x$ a unit vector in $H$. Then it follows from Hölder-McCarthy inequality that

\[ (Ax, x) \leq (A^p x, x)^{\frac{1}{p}} \quad \text{for all } p > 1. \]  

By using the Mond-Pečarić method [12, 13], we have the following reverse inequality of (25) [15, 4]:

**Lemma 7.** If $A$ is a positive operator on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$

\[ (A^p x, x)^{\frac{1}{p}} \leq \alpha A(x, x) + \beta(m, M, p, \alpha) \quad \text{for all } p > 1 \]

holds for every unit vector $x \in H$, where $\beta(m, M, p, \alpha)$ is defined as (12) in Theorem 1.

**Proof.** For the sake of reader’s convenience, we give a proof. Put $\beta = \beta(m, M, p, \alpha)$ and $f(t) = (at + b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p - m^p}{M - m}$ and $b = \frac{Mm^p - m^M}{M - m}$, then we have $f'(t) = \frac{a}{p}(at + b)^{\frac{1}{p} - 1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{1}{a}(\frac{\alpha p}{a})^{\frac{p}{1-p}} - \frac{b}{a}$. If $m \leq t_0 \leq M$, then we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2M^{p-1}}(at + b)^{\frac{1}{p} - 2} < 0$ and the condition $m \leq t_0 \leq M$ is equivalent to the condition

\[ \frac{M^p - m^p}{pM^{p-1}(M - m)} \leq \frac{M^p - m^p}{pm^{p-1}(M - m)}. \]

If $M \leq t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1 - \alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1 - \alpha)m$ for $t_0 = m$ if $t_0 \leq m$. Hence it follows that

\[ (at + b)^{\frac{1}{p}} - \alpha t \leq \beta \quad \text{for all } t \in [m, M]. \]

Since $t^p$ is convex for $p > 1$, it follows that $t^p \leq at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \leq aA + b$ and hence $(A^p x, x) \leq a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

\[ (A^p x, x)^{\frac{1}{p}} - \alpha(Ax, x) \leq (a(Ax, x) + b)^{\frac{1}{p}} - \alpha(Ax, x) \leq \max_{m \leq t \leq M} f(t) = \beta(m, M, p, \alpha). \]

$\square$
By Lemma 7, we have the following estimates of both the difference and the ratio in the inequality (25).

**Lemma 8.** If $A$ is a positive operator on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $p > 1$

\[
(A^p x, x)^{\frac{1}{p}} \leq K(m, M, p)^{\frac{1}{p}} (Ax, x)
\]

and

\[
(A^p x, x)^{\frac{1}{p}} - (Ax, x) \leq -C(m^p, M^p, \frac{1}{p})
\]

hold for every unit vector $x \in H$, where $K(m, M, p)$ is defined as (5) in §1 and $C(m, M, p)$ is defined as (7) in §1.

**Proof.** If we choose $\alpha$ satisfying $\beta(m, M, p, \alpha) = 0$ in Lemma 7, then we have $\alpha = K(m, M, p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 7, then we have $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$.

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

We summarize some important properties of a generalized Kantorovich constant [3, 11].

**Lemma 9.** Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.

(i) $K(m, M, p) = K(M, m, p)$ for all $p \in \mathbb{R}$.

(ii) $K(m, M, p) = K(m, M, 1 - p)$ for all $p \in \mathbb{R}$.

(iii) $K(m, M, 0) = K(m, M, 1) = 1$ for all $p \in \mathbb{R}$.

(iv) $K(m, M, p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$.

(v) $K(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} = K(m^p, M^p, \frac{r}{p})^{-\frac{1}{r}}$ for $pr \neq 0$.

4. PROOF OF RESULTS

Based on Lemmas in the preceding section, we give proofs of the results mentioned in the second section.

**Proof of Theorem 1.**

For every unit vector $x \in H$, it follows that

\[
((BAB)^p x, x) \leq (BABx, x)^p
\]

by Hölder-McCarthy inequality and $0 < p < 1$

\[
= \left( (A^p)^{\frac{1}{p}} \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|} \right)^p \|Bx\|^{2p}
\]

\[
\leq \left( \alpha(A^p \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}) + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p} \right) \text{ by Lemma 7}
\]

\[
= \alpha \left(B^p A^p Bx, Bx\right) \|Bx\|^{2p-2} + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p}
\]

\[
= \alpha \left(B^p A^p Bx, \frac{B^1x}{\|B^1x\|}, \frac{B^1x}{\|B^1x\|}\right) \|Bx\|^{2p-2} \|B^1x\|^2 + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p}
\]
and

\[ \| Bx \|^{2p-2} \| B^{1-p}x \|^{2} = (B^{2}x, x)^{p-1}(B^{2-2p}x, x) \leq (B^{2}x, x)^{p-1}(B^{2}x, x)^{1-p} = 1 \text{ by } 0 < 1 - p < 1. \]

By combining two inequalities above, we have

\[ \| BAB \|^{p} = \| (BAB)^{p} \| \leq \alpha \| B^{p}A^{p}B^{p} \| + \beta(m^{p}, M^{p}, \frac{1}{p}, \alpha) \| B \|^{2p} \]

and hence we have the desired inequality (10).

Next, we show (10) \( \Rightarrow \) (11). For \( p > 1 \), since \( 0 < \frac{1}{p} < 1 \), it follows from (10) that

\[ \| BAB \|^{\frac{1}{p}} \leq \alpha \| B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}} \| + \beta(m^{\frac{1}{p}}, M^{\frac{1}{p}}, p, \alpha) \| B \|^{\frac{2}{p}}. \]

By replacing \( A \) by \( A^{p} \) and \( B \) by \( B^{p} \) in the above inequality respectively, we have

\[ \| B^{p}A^{p}B^{p} \|^{\frac{1}{p}} \leq \alpha \| B^{p}A^{p}B^{p} \| + \beta(m, M, p, \alpha) \| B^{p} \|^{\frac{2}{p}}. \]

and so we have the desired inequality (11). Similarly we can show (11) \( \Rightarrow \) (10). Therefore (10) is equivalent to (11).

\( \square \)

**Proof of Corollary 2.**

For \( p > 1 \), if we put \( \beta(m, M, p, \alpha) = 0 \) in Theoren 1, then it follows that

\[ \frac{p-1}{p} \left( \frac{M^{p}-m^{p}}{p(M-m)} \right)^{\frac{1}{p-1}} + \alpha^{\frac{1}{p-1}} \frac{(Mm^{p}-mM^{p})}{M^{p}-m^{p}} = 0 \]

and hence

\[ \alpha^{\frac{1}{p-1}} = \frac{p-1}{p} \left( \frac{M^{p}-m^{p}}{p(M-m)} \right)^{\frac{1}{p-1}} \frac{M^{p}-m^{p}}{Mm^{p}-mM^{p}}. \]

Therefore, we have

\[ \alpha^{p} = \frac{M^{p}-m^{p}}{p(M-m)} \left( \frac{p-1}{p} \frac{M^{p}-m^{p}}{mM^{p}-Mm^{p}} \right)^{p-1} = K(m, M, p) \]

and we obtain the desired inequality (14).

For \( 0 < p < 1 \), since \( 1/p > 1 \), it follows from (14) that

\[ \| BAB \|^{\frac{1}{p}} \leq K(m, M, \frac{1}{p}) \| B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}} \|. \]

By replacing \( A \) and \( B \) by \( A^{p} \) and \( B^{p} \) respectively, then we have

\[ \| B^{p}A^{p}B^{p} \|^{\frac{1}{p}} \leq K(m^{p}, M^{p}, \frac{1}{p}) \| BAB \|. \]

Hence it follows from Lemma 9 that

\[ \| B^{p}A^{p}B^{p} \| \leq K(m^{p}, M^{p}, \frac{1}{p}) \| BAB \|^{p} \leq K(m, M, p)^{-1} \| BAB \|^{p}, \]

and we have the desired inequality (13). Similarly we have the implication (13) \( \Rightarrow \) (14).

\( \square \)
Proof of Corollary 3.
If we put $\alpha = 1$ in Theorem 1, then it follows that
\[
\beta(m^p, M^p, \frac{1}{p}, 1) = \frac{1}{p} - 1 \left( \frac{M - m}{\frac{1}{p} (M^p - m^p)} \right)^{\frac{1}{p} - 1} + \frac{M^p m - m^p M}{M - m}
\]
\[
= (1 - p) \left( \frac{p(M - m)}{M^p - m^p} \right)^{\frac{1}{p}} + \frac{M^p m - m^p M}{M - m}
\]
\[
= -C(m, M, p).
\]
Similarly it follows that $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. Hence we have the equivalence (15) $\iff$ (16). \hfill $\square$

Proof of Corollary 4.
In Corollary 2 and 3, we have only to put $p = 2$ and $p = 1/2$. \hfill $\square$

Proof of Theorem 5
By Corollary 2, it follows that
\[
K(m, M, p) \|A^{\frac{1}{2}}B\|^{2p} \leq \|A^{\frac{p}{2}}B^{p}\|^2.
\]
By replacing $A$ by $A^2$, we have
\[
K(m^2, M^2, p) \|AB\|^{2p} \leq \|A^p B^p\|^2.
\]
Therefore we have (21). Similarly, we have the equivalence (21) $\iff$ (22). \hfill $\square$

Proof of Theorem 6
The proof is divided into three parts, namely the equivalence $(A) \implies (B) \implies (C) \implies (A)$, $(B) \iff (B')$ and $(C) \iff (C')$.

$(A) \implies (B)$. It follows that
\[
(A) \iff \|A^{-\frac{1}{2}}B^{\frac{1}{2}}\| \leq 1 \implies \|A^{-\frac{p}{2}}B^{\frac{p}{2}}\|^2 \leq K(m, M, p)
\]
\[
\iff \|A^{\frac{p}{2}}B^{\frac{p}{2}}\| \leq 1 \implies \|A^{\frac{p}{2}}B^{\frac{p}{2}}\|^2 \leq K(M^{-1}, m^{-1}, p) = K(m, M, p)
\]
\[
\iff \|AB\| \leq 1 \implies \|A^p B^p\| \leq K(m^2, M^2, p).
\]
If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that
\[
\|A^p B_1^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \iff \|A^p B^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^p.
\]

$(B) \implies (C)$. If we replace $A$ by $A^{\frac{1}{2}}$ in $(A)$, then it follows that
\[
\|A^{\frac{p}{2}}B^p\| \leq K(m, M, p)^{\frac{1}{2}} \|A^{\frac{1}{2}}B\|^p.
\]
Square both sides, we have
\[
\|B^p A^p B^p\| \leq K(m, M, p) \|BAB\|^p.
\]

$(C) \implies (A)$. If we replace $B$ by $B^{\frac{1}{2}}$ and $A$ by $A^{-1}$ in $(C)$, then it follows that
\[
\|B^{\frac{p}{2}} A^{-p} B^{\frac{p}{2}}\| \leq K(M^{-1}, m^{-1}, p) \|B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\|^p.
\]
By rearranging it, we have
\[
\|A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}}\| \leq K(m, M, p) \|A^{-\frac{1}{2}} BA^{-\frac{1}{2}}\|^p.
\]
Since $A \geq B \geq 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq 1$ that
\[ \|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \leq K(m, M, p) \]
and hence
\[ B^p \leq K(m, M, p)A^p. \]

$(B) \iff (B')$: If we replace $A$ and $B$ by $A^\frac{1}{p}$ and $B^\frac{1}{p}$ in $(B)$ respectively, then it follows that
\[ (B) \iff \|AB\| \leq K(m^\frac{2}{p}, M^\frac{2}{p}, p)^\frac{1}{2}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\|^p \]
\[ \iff \|AB\|^\frac{1}{p} \leq K(m^\frac{2}{p}, M^\frac{2}{p}, p)^\frac{1}{2}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \]
\[ \iff (B') \]
Similarly we have $(C) \iff (C')$ and so the proof is complete. \(\square\)

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