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Kyoto University
Strong Convergence Theorem by the Hybrid and Extragradient Methods for Nonexpansive Nonself-Mappings and Monotone Mappings

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Abstract

In this paper we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive nonself-mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping. The iterative process is based on two well known methods - hybrid and extragradient. We obtain a strong convergence theorem for three sequences generated by this process.

1 Introduction

Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $P_C$ be the metric projection of $H$ onto $C$. A mapping $S$ of $C$ into $H$ is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of $S$. A mapping $A$ of $H$ into itself is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0$$

for all $u, v \in H$. The variational inequality problem is to find some $u \in C$ such that

$$\langle Au - u, v - u \rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. A mapping $A$ of $H$ into itself is called $\alpha$-inverse-strongly-monotone if there exists a positive real number $\alpha$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in H$; see [1], [5]. It is obvious that any $\alpha$-inverse-strongly-monotone mapping $A$ is monotone and Lipschitz-continuous. For finding a common element of $VI(C,A)$ and $F(S)$ under the assumption that the set $C \subset H$ is closed and convex and the mapping $A$ of $H$ into itself is $\alpha$-inverse-strongly-monotone, Iiduka and Takahashi [2] introduced the following iterative scheme by a hybrid method:

$$\begin{align*}
x_0 & \in C \\
y_n & = P_C(Sx_n - \lambda_n ASx_n) \\
C_n & = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\
Q_n & = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\} \\
x_{n+1} & = P_{C_n \cap Q_n}x
\end{align*}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$. They showed that if $F(S) \cap VI(C,A)$ is nonempty, then the sequence $\{x_n\}$ generated by this iterative process, converges strongly to $P_{F(S) \cap VI(C,A)}x$. 
On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that the set $C \subset \mathbb{R}^n$ is closed and convex and the mapping $A$ of $C$ into $\mathbb{R}^n$ is monotone and $k$-Lipschitz-continuous, Korpelevich [4] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \
x_n = P_C(x_n - \lambda Ax_n) \\ x_{n+1} = P_C(x_n - \lambda Ax_n) \end{cases}$$  \hspace{1cm} (1)$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, 1/k)$. He showed that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{x_n\}$, generated by (1), converge to the same point $x \in VI(C, A)$.

In this paper, by an idea of combining hybrid and extragradient methods, we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive nonself-mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. We obtain a strong convergence theorem for three sequences generated by this process.

## 2 Preliminaries

Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$ and let $C$ be a closed convex subset of $H$. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges weakly to $x$ and $x_n \rightharpoonup x$ to indicate that $\{x_n\}$ converges strongly to $x$. For every point $x \in H$ there exists a unique nearest point in $C$, denoted by $P_Cx$, such that $\|x - P_Cx\| \leq \|x - y\|$ for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is a nonexpansive mapping of $H$ onto $C$. It is also known that $P_C$ is characterized by the following properties: $P_Cx \in C$ and

$$\langle x - P_Cx, P_Cx - y \rangle \geq 0;$$

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2$$

for all $x \in H, y \in C$; see [9] for more details. Let $A$ be a monotone mapping of $H$ into $H$. In the context of variational inequality problem this implies

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$  

It is also known that $H$ satisfies Opial's condition [7], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $(x - y, f - g) \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x, f) \in H \times H$, $(x - y, f - g) \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $C$ into $H$ and $N_Cv$ be the normal cone to $C$ at $v \in C$, i.e. $N_Cv = \{w \in H : (v - u, w) \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_Cv, & \text{if } v \in C, \\ \emptyset, & \text{if } v \not\in C. \end{cases}$$

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [8].

## 3 Strong Convergence Theorem

In this section we prove a strong convergence theorem by a combined hybrid-extragradient method for nonexpansive nonself-mappings and monotone, $k$-Lipschitz-continuous mappings.
Theorem 3.1. Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $H$ into itself and $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by

$$
\begin{align*}
x_0 &= x \\
y_n &= P_C(Sx_n - \lambda_n ASx_n) \\
z_n &= P_C(Sx_n - \lambda_n Ay_n) \\
C_n &= \{z \in C : \|x_n - z\| \leq \|x_n - z\|\} \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}
$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S) \cap VI(C, A)} x$.

Proof. It is obvious that $C_n$ is closed and $Q_n$ is closed and convex for every $n = 0, 1, 2, \ldots$. As $C_n = \{z \in C : \|x_n - x_n\| + 2\|x_n - x, x_n - z\| \leq 0\}$, we also have $C_n$ is convex for every $n = 0, 1, 2, \ldots$. Let $u \in F(S) \cap VI(C, A)$. From (3), monotonicity of $A$ and $u \in VI(C, A)$, we have

$$
\|x_n - u\|^2 \leq \|Sx_n - \lambda_n Ay_n - u\|^2 - \|Sx_n - \lambda_n Ay_n - z_n\|^2
$$

$$
= \|Sx_n - u\|^2 - \|Sx_n - z_n\|^2 + 2\lambda_n \langle Ay_n, u - z_n\rangle
$$

$$
\leq \|Sx_n - u\|^2 - \|Sx_n - z_n\|^2 + 2\lambda_n \|Ay_n, u - y_n\| + \|Ay_n, y_n - x_n\|
$$

$$
\leq \|Sx_n - u\|^2 - \|Sx_n - z_n\|^2 + 2\lambda_n \|Ay_n, y_n - z_n\|
$$

$$
= ||x_n - u||^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle Sx_n - y_n, y_n - x_n\rangle - \|y_n - z_n\|^2
$$

$$
\leq \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle Sx_n - y_n, y_n - x_n\rangle - \|y_n - z_n\|^2
$$

$$
\leq \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\lambda_n \langle Ay_n, y_n - z_n\rangle
$$

$$
= \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\lambda_n \langle Ay_n, y_n - z_n\rangle
$$

Further, since $y_n = P_C(Sx_n - \lambda_n ASx_n)$ and $A$ is $k$-Lipschitz-continuous, we have

$$
\langle Sx_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle
$$

$$
= \langle Sx_n - \lambda_n ASx_n - y_n, z_n - y_n\rangle + \langle \lambda_n ASx_n - \lambda_n Ay_n, z_n - y_n\rangle
$$

$$
\leq \langle \lambda_n ASx_n - \lambda_n Ay_n, z_n - y_n\rangle
$$

$$
\leq \lambda_n k \|Sx_n - y_n\| \|z_n - y_n\|.
$$

So, we have

$$
\|z_n - u\|^2 \leq \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\lambda_n k \|Sx_n - y_n\| \|z_n - y_n\|
$$

$$
\leq \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\lambda_n k \|Sx_n - y_n\| \|z_n - y_n\|
$$

$$
+ \lambda_n^2 k^2 \|Sx_n - y_n\|^2 + \|y_n - z_n\|^2
$$

$$
\leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|Sx_n - y_n\|^2
$$

$$
\leq \|x_n - u\|^2
$$

So, we have

$$
\|z_n - u\| \leq \|x_n - u\|
$$

for every $n = 0, 1, 2, \ldots$ and hence $u \in C_n$. So, $F(S) \cap VI(C, A) \subset C_n$ for every $n = 0, 1, 2, \ldots$. Next, let us show by mathematical induction that $\{z_n\}$ is well-defined and $F(S) \cap VI(C, A) \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$. For $n = 0$ we have $Q_0 = C$. Hence we obtain $F(S) \cap VI(C, A) \subset C_n \cap Q_n$ for some $k \in N$. Since $F(S) \cap VI(C, A)$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of $C$. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{i+1} - z_i, x - x_{i+1}\rangle \geq 0$ for every $z \in C_k \cap Q_k$. 

Since \( F(S) \cap VI(C, A) \subset C_k \cap Q_k \), we have \( (x_{k+1} - x, x - x_{k+1}) \geq 0 \) for \( x \in F(S) \cap VI(C, A) \) and hence \( F(S) \cap VI(C, A) \subset C_{k+1} \cap Q_{k+1} \). Therefore, we obtain \( F(S) \cap VI(C, A) \subset C_n \cap Q_n \), we have

\[
\|x_{n+1} - x\| \leq \|t_0 - x\|
\]

(5)

for every \( n = 0, 1, 2, \ldots \). Therefore, \( \{x_n\} \) is bounded. We also have

\[
\|x_n - u\| \leq \|z_n - u\|
\]

for some \( u \in F(S) \cap VI(C, A) \). So, \( \{z_n\} \) is also bounded. Since \( x_{n+1} \in C_n \cap Q_n \subset Q_n \) and \( x_n = P_{Q_n} x \), we have

\[
\|x_n - x\| \leq \|x_{n+1} - x\|
\]

for every \( n = 0, 1, 2, \ldots \). Therefore, there exists \( c = \lim_{n \to \infty} \|x_n - x\| \). Since \( x_n = P_{Q_n} x \) and \( x_{n+1} \in Q_n \), we have

\[
\|x_{n+1} - x_n\|^2 = \|x_{n+1} - x\|^2 + \|x_n - x\|^2 + 2 (x_{n+1} - x, x - x_n)
\]

\[
= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 (x_n - x_{n+1}, x - x_n)
\]

\[
\leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2
\]

for every \( n = 0, 1, 2, \ldots \). This implies that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

Since \( x_{n+1} \in C_n \), we have \( \|x_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \) and hence

\[
\|x_n - x\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2 \|x_{n+1} - x\|
\]

for every \( n = 0, 1, 2, \ldots \). From \( \|x_{n+1} - x\| \to 0 \), we have \( \|x_n - z_n\| \to 0 \).

For \( u \in F(S) \cap VI(C, A) \), from (4) we obtain

\[
\|z_n - u\|^2 \leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|Sx_n - y_n\|^2.
\]

Therefore, we have

\[
\|Sx_n - y_n\|^2 \leq \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\|^2 - \|z_n - u\|^2)
\]

\[
= \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\|^2 - \|z_n - u\|^2) (\|x_n - u\| + \|z_n - u\|)
\]

\[
\leq \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\| + \|z_n - u\|) \|x_n - z_n\|.
\]

Since \( \|x_n - z_n\| \to 0 \), we obtain \( Sx_n - y_n \to 0 \). From (4) we also have

\[
\|z_n - u\|^2 \leq \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2 + 2 \lambda_n k \|Sx_n - y_n\| \|x_n - y_n\|
\]

\[
\leq \|x_n - u\|^2 - \|Sx_n - y_n\|^2 - \|y_n - z_n\|^2
\]

\[
+ \|Sx_n - y_n\|^2 + \lambda_n^2 k^2 \|y_n - z_n\|^2
\]

\[
\leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|y_n - z_n\|^2.
\]

Therefore we have

\[
\|y_n - z_n\|^2 \leq \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\|^2 - \|z_n - u\|^2)
\]

\[
= \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\| - \|x_n - u\|) (\|x_n - u\| + \|z_n - u\|)
\]

\[
\leq \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\| + \|z_n - u\|) \|x_n - z_n\|.
\]
Since $\|x_n - x_n\| \to 0$, we obtain $y_n - z_n \to 0$. From $\|x_n - y_n\| \leq \|x_n - z_n\| + \|z_n - y_n\|$ we also have $x_n - y_n \to 0$. Since A is $k$-Lipschitz-continuous, we have $Ay_n - Ax_n \to 0$. From $\|x_n - Sx_n\| \leq \|x_n - y_n\| + \|y_n - Sx_n\|$ we have $x_n - t_n \to 0$. Since $\|z_n - Sx_n\| = \|z_n - Sx_n\| + \|Sx_n - Sx_n\| \leq \|z_n - Sx_n\| + \|z_n - x_n\|$,
we have $\|z_n - Sx_n\| \to 0$.

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $u$. We can obtain that $u \in F(S) \cap VI(C, A)$. First, we show $u \in VI(C, A)$. Since $x_n - x_n \to 0$ and $x_n - y_n \to 0$, we have $\{z_{n_k}\} \to u$ and $\{y_{n_k}\} \to u$. Let

\[ T_u = \begin{cases} &Av + NCu, \quad \text{if } v \in C, \\ &0, \quad \text{if } v \notin C. \end{cases} \]

Then $T$ is maximal monotone and $0 \in T_u$ if and only if $v \in VI(C, A)$; see [8]. Let $(v, w) \in G(T)$. Then, we have $w \in Tu = Av + NCu$ and hence $w - Au \in NCu$. So, we have $(v - w, w - Au) \geq 0$ for all $v \in C$.

On the other hand, from $x_n = P_C(Sx_n - \lambda_n Ay_n)$ and $v \in C$ we have

\[ \langle Sx_n - \lambda_n Ay_n - z_n, u - v \rangle \geq 0 \]

and hence

\[ \langle v - z_n, \frac{z_n - Sx_n}{\lambda_n} + Ay_n \rangle \geq 0. \]

Therefore from $w - Au \in NCu$ and $z_{n_k} \in C$, we have

\[
(v - z_{n_k}, w) \geq (v - z_{n_k}, Au) \\
\geq (v - z_{n_k}, Au) - \langle v - z_{n_k}, \frac{z_{n_k} - Sx_{n_k}}{\lambda_{n_k}} + Ay_{n_k} \rangle \\
= (v - z_{n_k}, Au - Ax_{n_k}) + \langle v - z_{n_k}, Az_{n_k} - Ay_{n_k} \rangle - \langle v - z_{n_k}, \frac{z_{n_k} - Sx_{n_k}}{\lambda_{n_k}} \rangle \\
\geq \langle v - z_{n_k}, Az_{n_k} - Ay_{n_k} \rangle - \langle v - z_{n_k}, \frac{z_{n_k} - Sx_{n_k}}{\lambda_{n_k}} \rangle.
\]

Hence, we obtain $(v - u, u) \geq 0$ as $i \to \infty$. Since $T$ is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in VI(C, A)$.

Let us show $u \in F(S)$. Assume $u \notin F(S)$. From Opial’s condition, we have

\[
\liminf_{i \to \infty} \|z_{n_i} - w\| < \liminf_{i \to \infty} \|z_{n_i} - Su\| \\
= \liminf_{i \to \infty} \|z_{n_i} - Sx_{n_i} + Sx_{n_i} - Su\| \\
\leq \liminf_{i \to \infty} \|Sx_{n_i} - Su\| \\
\leq \liminf_{i \to \infty} \|z_{n_i} - w\|.
\]

This is a contradiction. So, we obtain $u \in F(S)$. This implies $u \in F(S) \cap VI(C, A)$.

From $t_0 = P_{F(S) \cap VI(C, A)}x$, $u \in F(S) \cap VI(C, A)$ and (5), we have

\[
\|t_0 - x\| \leq \|u - x\| \leq \liminf_{i \to \infty} \|z_{n_i} - x\| \leq \limsup_{i \to \infty} \|z_{n_i} - x\| \leq \|t_0 - x\|.
\]

So, we obtain

\[
\lim_{i \to \infty} \|z_{n_i} - x\| = \|u - x\|.
\]

From $x_n - x \to u - x$ we have $x_{n_i} - x \to u - x$ and hence $x_{n_i} \to u$. Since $x_n \in P_{Q_n}x$ and $t_0 \in F(S) \cap VI(C, A) \subset C \cap Q_n \subset Q_n$, we have

\[
\|t_0 - x\| = \langle t_0 - x_n, x - x_n \rangle + \langle t_0 - x_n, x - t_0 \rangle \geq \langle t_0 - x_n, x - t_0 \rangle.
\]

As $i \to \infty$, we obtain $\|t_0 - u\| \geq \langle t_0 - u, x - t_0 \rangle \geq 0$ by $t_0 = P_{F(S) \cap VI(C, A)}x$ and $u \in F(S) \cap VI(C, A)$. Hence we have $u = t_0$. This implies that $z_n \to t_0$. It is easy to see $y_n \to t_0, z_n \to t_0$. □
4 Applications.

Using Theorem 3.1, we prove some theorems in a real Hilbert space.

**Theorem 4.1.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ such that $VI(C,A)$ is nonempty. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by

\[
\begin{align*}
x_0 &= x \in C, \\
y_n &= P_C (x_n - \lambda_n Ax_n) \\
z_n &= P_C (x_n - \lambda_n Ay_n) \\
C_n &= \{z \in C : \|z - z_n\| \leq \|x_n - z\|\} \\
Q_n &= \{z \in C : (x_n - z, x - x_n) \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}
\]

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to $P_{VI(C,A)x}$.

**Proof.** Putting $S = I$, by Theorem 3.1, we obtain the desired result. \qed

**Remark 4.1.** See Iiduka, Takahashi and Toyoda [3] for the case when $A$ is a-inverse-strongly-monotone.

**Theorem 4.2.** Let $C$ be a closed convex subset of a real Hilbert space $H$ and $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

\[
\begin{align*}
x_0 &= x \in C, \\
y_n &= P_C S x_n \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\
Q_n &= \{z \in C : (x_n - z, x - x_n) \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}
\]

for every $n = 0, 1, 2, \ldots$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(S)x}$.

**Proof.** Putting $A = 0$, by Theorem 3.1, we obtain the desired result. \qed

**Theorem 4.3.** Let $H$ be a real Hilbert space. Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $H$ into itself and $S$ be a nonexpansive mapping of $H$ into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

\[
\begin{align*}
x_0 &= x \in C, \\
y_n &= S x_n - \lambda_n A (S x_n - \lambda_n A S x_n) \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\
Q_n &= \{z \in C : (x_n - z, x - x_n) \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}
\]

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(S)\cap A^{-1}0} x$.

**Proof.** We have $A^{-1}0 = VI(H,A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result. \qed

**Remark 4.2.** Notice that $F(S) \cap A^{-1}0 \subset VI(F(S), A)$. See also Yamada [10] for the case when $A$ is a strongly monotone and Lipschitz-continuous mapping of a real Hilbert space $H$ into itself and $S$ is a nonexpansive mapping of $H$ into itself.

**References**


