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Functions related to some geometrical properties of Banach Spaces

Abstract. We introduce some functions $\varphi_X(\tau)$ as a generalization (or refinement) of the Schäffer constant $S(X)$ of Banach spaces $X$, and investigate geometrical properties of Banach spaces such as uniform non-squareness and uniform convexity in terms of those functions. The normal structure coefficient $N(X)$ is also estimated by the function $\varphi_X(\tau)$.

1. Definitions

(i) $X$ is called uniformly non-square in the sense of James when there exists $\delta > 0$ such that

$$\min(||x+y||, ||x-y||) \leq 2(1-\delta) \text{ if } ||x|| = ||y|| = 1.$$ 

(ii) The James constant is defined by

$$J(X) := \sup \{ \min(||x+y||, ||x-y||) : ||x|| = ||y|| = 1 \}.$$ 

(iii) $X$ is called uniformly non-square in the sense of Schäffer when there exists $\lambda > 1$ such that

$$\max(||x+y||, ||x-y||) \geq \lambda \text{ if } ||x|| = ||y|| = 1.$$
(iv) The Schäffer constant is defined by
\[
S(X) := \inf \{ \max(\|x + y\|, \|x - y\|) : \|x\| = \|y\| = 1 \}.
\]

(v) The von Neumann-Jordan (NJ-) constant of a Banach space $X$ is the smallest constant $C$ for which
\[
\frac{1}{C} \leq \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C \quad \text{for all } (x, y) \neq (0, 0)
\]
holds; we denote it by $C_{NJ}(X)$.

(vi) The modulus of convexity of $X$ is defined by
\[
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\} \quad (0 \leq \epsilon \leq 2).
\]

$X$ is called uniformly convex if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon < 2$, and $q$-uniformly convex $(2 \leq q < \infty)$ if there is $C > 0$ such that $\delta_X(\epsilon) \geq C\epsilon^q$ for all $0 < \epsilon \leq 2$.

It is obvious that $X$ is uniformly non-square in the sense of James, resp., Schäffer if and only if $J(X) < 2$, resp., $S(X) > 1$. Since $J(X)S(X) = 2$ for any Banach space $X$ (cf.[3,12]), these two notions are equivalent. It is known that $X$ is uniformly non-square if and only if $C_{NJ}(X) < 2$ (cf.[16]). Let us recall that $X$ is super-reflexive if any Banach space finitely representable in $X$ is reflexive. It is well-known that if $X$ is uniformly convex, or more generally, uniformly non-square, then $X$ is reflexive. It is easy to see that if $X$ is uniformly non-square, then any Banach space finitely representable in $X$ is uniformly non-square. Thus, any uniformly non-square Banach space is super-reflexive (cf.[9]). Enflo [5] showed that $X$ is super-reflexive if and only if $X$ admits an equivalent uniformly convex norm. Pisier [13] also showed that if $X$ is super-reflexive, then $X$ admits an equivalent $q$-uniformly convex norm for some $2 \leq q < \infty$.

2. Definitions (i) A Banach space $X$ is said to have normal structure if $r(K) < \text{diam}(K)$ for every non-singleton closed bounded convex subset $K$ of $X$, where $\text{diam}(K) := \sup\{\|x - y\| : x, y \in K\}$ and $r(K) := \inf\{\sup\{\|x - y\| : y \in K\} : x \in K\}$.

(ii) The normal structure coefficient of $X$ (Bynum [2]) is the number:
\[
N(X) = \inf\{\text{diam}(K)/r(K) : K \subset X \text{ bounded and convex, diam}(K) > 0\}.
\]

Obviously, $1 \leq N(X) \leq 2$. The space $X$ is said to have uniform normal structure if $N(X) > 1$. It is well-known that if $X$ has uniform normal structure, then $X$ has fixed point property (cf.[8]). Gao and Lau [7] showed that if $J(X) < 3/2$, then $X$ has uniform normal structure. Prus [14] even estimated the normal structure coefficient $N(X)$ by $J(X)$. Kato, Maligranda and Takahashi [12] also estimated the normal
structure coefficient $N(X)$ by $C_{NJ}(X)$, and showed that if $C_{NJ}(X) < 5/4$, then $X$ as well as its dual $X'$ have the uniform normal structure.

3. Definitions (Schäffer type constants): We define for $\tau \geq 0$

$$S_{X,p}(\tau) = \begin{cases} \inf \left\{ \left( \frac{\|x + \tau y\|^p + \|x - \tau y\|^p}{2} \right)^{1/p} : \|x\| = \|y\| = 1 \right\} & \text{if } 1 < p < \infty, \\
\inf \left\{ \max(\|x + \tau y\|, \|x - \tau y\|) : \|x\| = \|y\| = 1 \right\} & \text{if } p = \infty. \end{cases}$$

Let $X$ be a Banach space (of dimension at least 2). Let $\varphi$ be a strictly convex and strictly increasing function defined on $[0, \infty)$ with values in $[0, \infty)$ (such a function is continuous on $[0, \infty)$). For simplicity, we assume that $\varphi(0) = 0$, $\varphi(1) = 1$.

4. Definition (Generalized Schäffer type constant): For $\tau \geq 0$ let

$$\varphi_X(\tau) = \inf \left\{ \frac{\varphi(\|x + \tau y\|) + \varphi(\|x - \tau y\|)}{2} : \|x\| = \|y\| = 1 \right\}$$

5. Remark If $\varphi(t) = t^p$, $1 < p < \infty$, then $\varphi^{-1}(\varphi_X(\tau)) = S_{X,p}(\tau)$, where $S_{X,p}(\tau)$ is the Schäffer type constant.

6. Proposition $\varphi_X(\tau)$ is continuous and non-decreasing for $0 \leq \tau < \infty$.

7. Theorem $X$ is uniformly non-square if and only if $\varphi_X(\tau) > 1$ for some $0 < \tau < 1$.

8. Corollary Let $1 < p \leq \infty$. The following are equivalent.

1. $X$ is uniformly non-square.
2. $S_{X,p}(1) > 1$.
3. $S_{X,p}(\tau) > 1$ (0 < $\exists \tau < 1$).
4. $S_{X,p}(\tau) > \tau$ (1 < $\exists \tau < \infty$).

9. Theorem $X$ is uniformly convex if and only if $\varphi_X(\tau) > 1$ for any $0 < \tau < 1$.

10. Corollary Let $1 < p \leq \infty$. The following are equivalent.

1. $X$ is uniformly convex.
2. $S_{X,p}(\tau) > 1$ (0 < $\forall \tau < 1$).
3. $S_{X,p}(\tau) > \tau$ (1 < $\forall \tau < \infty$).
11. **Theorem** \( N(X) \geq \varphi^{-1}(\varphi_X(1/2)) \). In particular, if \( \varphi_X(1/2) > 1 \), then \( X \) has uniform normal structure.

12. **Corollary** Let \( 1 < p \leq \infty \). Then

\[
N(X) \geq S_{X,p}(1/2).
\]

It is easy to see that if \( C_{NJ}(X) < 5/4 \), then \( S_{X,2}(1/2) \geq 1 \). Since \( C_{NJ}(X) = C_{NJ}(X') \), we have

13. **Corollary** If \( C_{NJ}(X) < 5/4 \), then \( X \) as well as \( X' \) have the uniform normal structure.

14. **Theorem** Let \( 1 < p \leq \infty \) and \( 2 \leq q < \infty \). The following are equivalent.
   (1) \( X \) is \( q \)-uniformly convex.
   (2) There is \( C > 0 \) such that

\[
S_{X,p}(\tau) \geq (1+C\tau^q)^{1/q} \quad \text{for all } \tau \geq 0.
\]

15. **Theorem** Let \( 2 \leq p < \infty \). Then the following are equivalent.
   (1) \( X \) is isometric to a Hilbert space.
   (2) \( S_{X,p}(\tau) = (1+\tau^2)^{1/2} \) for all \( \tau \geq 0 \).

16. **Remark** If \( X \) is a Hilbert space, then for all \( \tau \geq 0 \)

\[
S_{X,p}(\tau) = \left( \frac{|1+\tau|^{r} + |1-\tau|^{r}}{2} \right)^{1/r} \quad \text{if } 1 < p < 2.
\]

Hence, the above theorem is false if \( 1 < p < 2 \). Finally we calculate \( S_{X,p}(\tau) \) in \( L_{r} \)-spaces.

17. **Theorem** Let \( X \) be an \( L_{r} \)-space with \( \dim X \geq 2 \).
   (1) Let \( 1 < r \leq 2 \) and \( 1/r + 1/r' = 1 \). Then for all \( \tau \geq 0 \)

\[
S_{X,p}(\tau) = \left( \frac{|1+\tau|^{r} + |1-\tau|^{r}}{2} \right)^{1/r} \quad \text{if } r \leq p \leq \infty.
\]

   (2) Let \( 2 \leq r < \infty \) and \( 1/r + 1/r' = 1 \). Then for all \( \tau \geq 0 \)

\[
S_{X,p}(\tau) = (1+\tau^r)^{1/r} \quad \text{if } r' \leq p \leq \infty.
\]
参考文献