Schauder's Fixed Point Theorems in Complete Metric Spaces and Fuzzy Boundary Value Problems on an Infinite Interval

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Abstract
Aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder's fixed point theorem. Finally we discuss a sufficient conditions for the existence of solutions of fuzzy differential equations on an infinite interval with boundary conditions.

1 Complete Metric Space of Fuzzy Numbers

Denote \( I = [0,1] \). The following definition means that a fuzzy number can be identified with a membership function.

**Definition 1** Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by

\[ F_{b}^{st} = \{ \mu : \mathbb{R} \rightarrow I \text{ satisfying (i)-(iv) below} \} \]

(i) \( \mu \) has a unique number \( m \in \mathbb{R} \) such that \( \mu(m) = 1 \) (normality);
(ii) \( \text{supp}(\mu) = \text{cl}(\{ \xi \in \mathbb{R} : \mu(\xi) > 0 \}) \) is bounded in \( \mathbb{R} \) (bounded support);
(iii) \( \mu \) is strictly fuzzy convex on \( \text{supp}(\mu) \) as follows:
   (a) if \( \text{supp}(\mu) \neq \{m\} \), then
   \[ \mu(\lambda \xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)] \]
   for \( \xi_1, \xi_2 \in \text{supp}(\mu) \) with \( \xi_1 \neq \xi_2 \) and \( 0 < \lambda < 1 \);
   (b) if \( \text{supp}(\mu) = \{m\} \), then \( \mu(m) = 1 \) and \( \mu(\xi) = 0 \) for \( \xi \neq m \);
(iv) \( \mu \) is upper semi-continuous on \( \mathbb{R} \) (upper semi-continuity).

It follows that \( \mathbb{R} \subset F_{b}^{st} \). Because \( m \) has a membership function as follows:

\[ \mu(m) = 1; \quad \mu(\xi) = 0 (\xi \neq m) \quad (1.1) \]

Then \( \mu \) satisfies the above (i)-(iv).
In usual case a fuzzy number \( x \) satisfies \textit{fuzzy convex on} \( \mathbb{R} \), i.e.,

\[
\mu(\lambda \xi_1 + (1-\lambda) \xi_2) \geq \min\{\mu(\xi_1), \mu(\xi_2)\}
\]  

(1.2)

for \( 0 \leq \lambda \leq 1 \) and \( \xi_1, \xi_2 \in \mathbb{R} \). Denote \( \alpha \)-cut sets by

\[
L_\alpha(\mu) = \{ \xi \in \mathbb{R} : \mu(\xi) \geq \alpha \}
\]

for \( \alpha \in I \). When the membership function is fuzzy convex, then we have the following remarks.

\textbf{Remark 1} \quad The following statements (1) - (4) are equivalent each other, provided with (i) of Definition 1.

1. (1.2) holds;
2. \( L_\alpha(\mu) \) is convex with respect to \( \alpha \in I \);
3. \( \mu \) is non-decreasing in \( \xi \in (-\infty,m) \), non-increasing in \( \xi \in [m, +\infty) \), respectively;
4. \( L_\alpha(\mu) \subseteq L_\beta(\mu) \) for \( \alpha > \beta \).

\textbf{Remark 2} \quad The above condition (iiiia) is stronger than (1.2). From (iiiia) it follows that \( \mu(\xi) \) is strictly monotonously increasing in \( \xi \in [\min \text{supp}(\mu), m] \). Suppose that \( \mu(\xi_1) \geq \mu(\xi_2) \) for \( \xi_1 < \xi_2 \leq m \). From Remark 1(3), it follows that \( \mu(\xi_1) = \mu_1(\xi_1) \) for some \( \xi_1 < \xi_2 \), so we get \( \mu(\xi) = \mu(\xi_1) = \mu(\xi_2) \) for \( 0 \leq \xi \leq \xi_2 \). This contradicts with Definition 1 (iiiia). Thus \( \mu \) is strictly monotonously increasing. In the similar way \( \mu \) is strictly monotonously decreasing in \( \xi \in [m, \max \text{supp}(\mu)] \). This condition plays an important role in Theorem 1.

We introduce the following parametric representation of \( \mu \in \mathcal{F}_b^a \) as

\[
\begin{align*}
x_1(\alpha) &= \min L_\alpha(\mu), \\
x_2(\alpha) &= \max L_\alpha(\mu)
\end{align*}
\]

for \( 0 < \alpha \leq 1 \) and

\[
\begin{align*}
x_1(0) &= \min \text{supp}(\mu), \\
x_2(0) &= \max \text{supp}(\mu).
\end{align*}
\]

In the following example we illustrate typical types of fuzzy numbers.

\textbf{Example 1} \quad Consider the following \( L - R \) fuzzy number \( x \in \mathcal{F}_b^a \) with a membership function as follows:

\[
\mu(\xi) = \begin{cases}
L(\frac{m-\xi}{\ell})_+ & (\xi \leq m) \\
R(\frac{x-m}{r})_+ & (\xi > m)
\end{cases}
\]

Here it is said that \( m \in \mathbb{R} \) is a center and \( \ell > 0, r > 0 \) are spreads. \( L, R \) are \( I \)-valued functions. Let \( L(\xi)_+ = \max(L(\xi), 0) \) etc. We identify \( \mu \) with \( x = (x_1, x_2) \). As long as there exist \( L^{-1} \) and \( R^{-1} \), we have

\[
x_1(\alpha) = m - L^{-1}(\alpha)\ell \quad \text{and} \quad x_2(\alpha) = m + R^{-1}(\alpha)r.
\]

Let \( L(\xi) = -c_1 \xi + 1 \), where \( c_1 > 0 \) and \( |x_1 - m| \leq \ell \). We illustrate the following cases (i)-(iv).

(i) \quad Let \( R(\xi) = -c_2 \xi + 1 \), where \( c_2 > 0 \). Then \( c_2 \ell(x_2 - m) = c_1 r(m - x_1) \).

(ii) \quad Let \( R(\xi) = -c_2 \sqrt{\xi} + 1 \), where \( c_2 > 0 \). Then \( c_2 \ell(x_2 - m)^2 = c_1 r^2(m - x_1) \).

(iii) \quad Let \( R(\xi) = -c_2 \xi^2 + 1 \), where \( c_2 > 0 \). Then \( c_2^2 \ell(x_2 - m)^2 = c_1^2 r^2(x_1 - m)^2 \).
Let \( c \) be a real number such that \( 0 < c < 1 \). Denote
\[
L(\xi) = \begin{cases} 
1 & (\xi = 0) \\
-c\xi + c & (0 < \xi \leq 1)
\end{cases}
\]
and let \( R(\xi) = L(\xi) \). Then we have \( \ell(x_2 - m) = r(m - x_1) \) for \( |x_1 - m| \leq \ell \). The representation of \( x = (x_1, x_2) \) is as follows:
\[
x_1(\alpha) = m - (1 - \frac{\alpha}{c})\ell, \\
x_2(\alpha) = m + (1 - \frac{\alpha}{c})r \\
x_1(\alpha) = x_2(\alpha) = m \quad (c \leq \alpha \leq 1)
\]
The membership function is given by as follows:
\[
\mu(\xi) = \begin{cases} 
0 & (\xi < x_1(0), \xi > x_2(0)) \\
x_1^{-1}(\xi) & (x_1(0) \leq \xi < m) \\
1 & (\xi = m) \\
x_2^{-1}(\xi) & (m < \xi \leq x_2(0))
\end{cases}
\]
Denote by \( C(I) \) the set of all the continuous functions on \( I \) to \( \mathbb{R} \). The following theorem shows a membership function is characterized by \( x_1, x_2 \).

**Theorem 1** Denote the left-, right-end points of the \( \alpha \)--cut set of \( \mu \in F_b^{st} \) by \( x_1(\alpha), x_2(\alpha) \), respectively. Here \( x_1, x_2 : I \rightarrow \mathbb{R} \). The following properties (i)-(iii) hold.

(i) \( x_1, x_2 \in C(I) \);

(ii) \( \max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1) \);

(iii) \( x_1, x_2 \) are non-decreasing, non-increasing on \( I \), respectively, as follows:

(a) there exists a positive number \( c \leq 1 \) such that \( x_1(\alpha) < x_2(\alpha) \) for \( \alpha \in [0, c) \) and that \( x_1(\alpha) = m = x_2(\alpha) \) for \( \alpha \in [c, 1] \);

(b) \( x_1(\alpha) = x_2(\alpha) = m \) for \( \alpha \in I \);

Conversely, under the above conditions (i) -(iii), if we denote
\[
\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\}
\]
for \( \xi \in \mathbb{R} \), then \( \mu \in F_b^{st} \).

**Remark 3** From the above Condition (i) a fuzzy number \( x = (x_1, x_2) \) means a bounded continuous curve over \( \mathbb{R}^2 \) and \( x_1(\alpha) \leq x_2(\alpha) \) for \( \alpha \in I \).

In what follows we denote \( \mu = (x_1, x_2) \) for \( \mu \in F_b^{st} \). The parametric representation of \( \mu \) is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let \( g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be an \( \mathbb{R} \)--valued function. The corresponding binary operation of two fuzzy numbers \( x, y \in F_b^{st} \) to \( g(x, y) : F_b^{st} \times F_b^{st} \rightarrow F_b^{st} \) is calculated by the extension principle of Zadeh. The membership function \( \mu_{g(x,y)} \) of \( g \) is as follows:
\[
\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))
\]
Here $\xi, \xi_1, \xi_2 \in \mathbb{R}$ and $\mu_x, \mu_y$ are membership functions of $x, y$, respectively. From the extension principle, it follows that, in case where $g(x, y) = x + y$,

\[
\begin{align*}
\mu_{x+y}(\xi) &= \max_{\xi = \xi_1 + \xi_2} \min_{i=1,2} (\mu_i(\xi_i)) \\
&= \max\{\alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i), i = 1, 2\} \\
&= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}.
\end{align*}
\]

Thus we get $x + y = (x_1 + y_1, x_2 + y_2)$.

In the similar way $x - y = (x_1 - y_2, x_2 - y_1)$.

Denote a metric by

\[
d_\infty(x, y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)
\]

for $x = (x_1, x_2), y = (y_1, y_2) \in F_b^t$.

**Theorem 2** $F_b^t$ is a complete metric space in $C(I)^2$.

## 2 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions $\mu_x, \mu_y$ of $x, y \in F_b^t$ and $\lambda \in \mathbb{R}$, the following addition and a scalar product are given as follows:

\[
\begin{align*}
\mu_{x+y}(\xi) &= \sup\{\alpha \in [0, 1] : \\
&\xi = \xi_1 + \xi_2, \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\}; \\
\mu_{\lambda x}(\xi) &= \begin{cases} \\
\mu_x(\xi/\lambda) & (\lambda \neq 0) \\
0 & (\lambda = 0, \xi \neq 0) \\
\sup_{\eta \in \mathbb{R}} \mu_x(\eta) & (\lambda = 0, \xi = 0)
\end{cases}
\end{align*}
\]

In [5] they introduced the following equivalence relation $(x, y) \sim (u, v)$ for $(x, y), (u, v) \in F_b^t \times F_b^t$, i.e.,

\[
(x, y) \sim (u, v) \iff x + v = u + y.
\]

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (2.4) means that the following equations hold.

\[
x_i + v_i = u_i + y_i \quad (i = 1, 2)
\]

Denote an equivalence class by $[x, y] = \{(u, v) \in F_b^t \times F_b^t : (u, v) \sim (x, y)\}$ for $x, y \in F_b^t$ and the set of equivalence classes by

\[
F_b^t/\sim = \{[x, y] : x, y \in F_b^t\}
\]

such that one of the following cases (i) and (ii) hold:

(i) if $(x, y) \sim (u, v)$, then $[x, y] = [u, v]$;

(ii) if $(x, y) \not\sim (u, v)$, then $[x, y] \cap [u, v] = \emptyset$.

Then $F_b^t/\sim$ is a linear space with the following addition and scalar product

\[
[x, y] + [u, v] = [x + u, y + v] \quad (2.5)
\]

\[
\lambda[x, y] = \begin{cases} \\
[(\lambda x, \lambda y)] & (\lambda \geq 0) \\
[(-\lambda)y, (-\lambda)x] & (\lambda < 0)
\end{cases}
\]

(2.6)
for $\lambda \in R$ and $[x, y], [u, v] \in F_b^{st}/\sim$. They denote a norm in $F_b^{st}/\sim$ by

$$\| [x, y] \| = \sup_{\alpha \in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here $d_H$ is the Hausdorff metric is as follows:

$$d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)) = \max \left\{ \sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|, \sup_{\eta \in L_\alpha(\mu_y)} \inf_{\xi \in L_\alpha(\mu_x)} |\xi - \eta| \right\}.$$

It can be easily seen that $\| [x, y] \| = d_\infty(x, y)$. Note that $\| [x, y] \| = 0$ in $F_b^{st}/\sim$ if and only if $x = y$ in $F_b^{st}$.

3 Schauder's Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $F_b^{st}$ has an induced Banach space.

**Theorem 3** Let $S$ be a bounded closed subset in $F_b^{st}$. Assume that $S$ contains any segments of $x, y \in S$, i.e., $\lambda x + (1 - \lambda)y \in S$ for $\lambda \in I$. Let $V$ be an into continuous mapping on $S$. Assume that the closure $\text{cl}(V(S))$ is compact in $F_b^{st}$. Then $V$ has at least one fixed point $x$ in $S$, i.e., $V(x) = x$.

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

**Theorem 4** Let $F$ be a complete metric space with a metric $d$. Assume that $F$ is closed under addition and scalar product, and that $d(\lambda x, 0) = |\lambda|d(x, 0)$ for the scalar product $\lambda x$ and $\lambda \in R, x \in F$. Denote $X = \{[x, 0] : x, 0 \in F\}$. Here $[x, y]$ for $x, y \in F$ are equivalence classes of (2.4) and 0 is the origin. Then $X$ is a Banach space concerning addition (2.5), scalar product (2.6) and norm $\| [x, y] \| = d(x, 0)$ for $[x, 0] \in X$.

Moreover let $S$ be a bounded closed subset in $F$. Assume that $S$ contains any segments of $x, y \in S$ in the same meaning of Theorem 3. Let $V$ be an into continuous mapping on $S$. Assume that the closure $\text{cl}(V(S))$ is compact in $F$. Then $V$ has at least one fixed point in $S$.

4 FBVP on Infinite Intervals

In this section we deal with the following FBVP on an infinite interval:

$$\frac{dx}{dt} = p(t)x + f(t, x), \quad x(\infty) = c \quad (4.7)$$

Here $p : R_+ \to F_b^{st}, f : R_+ \times F_b^{st} \to F_b^{st}$ are continuous functions. Let denote $R_+ = [0, \infty)$ and $c \in F_b^{st}$. The following assumptions play important roles in considering the existence of solutions of (4.7).

**Assumption.**

(A1) Assume that

$$\int_0^\infty d(p(s), 0)ds = K < \infty.$$

(A2) There exist positive real numbers $r, R$ and integrable function $m : R_+ \to R_+$ such that

$$d(f(t, x), 0) \leq m(t) \text{ for } (t, x) \in R_+ \times S_1;$$

$$\int_0^\infty m(s)ds \leq rR;$$

$$[R + N_p(a + \| L \| R)]K < 1.$$
Here

\[ S_1 = \{ x \in \mathcal{F}^{st}_b : d(x, 0) \leq \min(a_0, r) \} \]

and \( N_p \) is independent on the function \( p \).

\( L : C^\text{lim}_r \to \mathcal{F}^{st}_b \) is a linear operator as \( L(x) = x(\infty) \) and

\[ C^\text{lim}_r = \{ x \in C(\mathbb{R}_+ : \mathcal{F}^{st}_b) : \exists x(\infty), d(x, 0) \leq r \} \].

(A3) There exists no solution of

\[ \frac{dx}{dt} = p(t)x, L(x) = 0 \]

except for the zero solution.

We expect the following existence theorem for solutions of FBVP on the infinite interval.

Under assumptions (A1) - (A3) we expect that there exists at least one solution of (4.7) in \( C^\text{lim}_r \) for any \( c \in S_1 \) by applying the Schauder's fixed point theorem in \( C^\text{lim}_r \).

References


