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Schauder's Fixed Point Theorems in Complete Metric Spaces and Fuzzy Boundary Value Problems on an Infinite Interval

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Abstract
Aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder's fixed point theorem. Finally we discuss a sufficient conditions for the existence of solutions of fuzzy differential equations on an infinite interval with boundary conditions.

1 Complete Metric Space of Fuzzy Numbers

Denote $I = [0,1]$. The following definition means that a fuzzy number can be identified with a membership function.

\textbf{Definition 1} Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by

$$
F_{b}^{st} = \{ \mu : \mathbb{R} \rightarrow I \text{ satisfying (i)-(iv) below} \}.
$$

(i) $\mu$ has a unique number $m \in \mathbb{R}$ such that $\mu(m) = 1$ (normality);
(ii) $\text{supp}(\mu) = cl(\{\xi \in \mathbb{R} : \mu(\xi) > 0\})$ is bounded in $\mathbb{R}$ (bounded support);
(iii) $\mu$ is strictly fuzzy convex on $\text{supp}(\mu)$ as follows:

(a) if $\text{supp}(\mu) \neq \{m\}$, then \(\mu(\lambda \xi_1 + (1 - \lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]\) for $\xi_1, \xi_2 \in \text{supp}(\mu)$ with $\xi_1 \neq \xi_2$ and $0 < \lambda < 1$;
(b) if $\text{supp}(\mu) = \{m\}$, then $\mu(m) = 1$ and $\mu(\xi) = 0$ for $\xi \neq m$;
(iv) $\mu$ is upper semi-continuous on $\mathbb{R}$ (upper semi-continuity).

It follows that $\mathbb{R} \subset F_{b}^{st}$. Because $m$ has a membership function as follows:

$$
\mu(m) = 1; \quad \mu(\xi) = 0 (\xi \neq m)
$$

(1.1)

Then $\mu$ satisfies the above (i)-(iv).
In usual case a fuzzy number $x$ satisfies fuzzy convex on $\mathbb{R}$, i.e.,

$$\mu(\lambda \xi_1 + (1-\lambda) \xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)]$$

(1.2)

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbb{R}$. Denote $\alpha$-cut sets by

$$L_\alpha(\mu) = \{\xi \in \mathbb{R} : \mu(\xi) \geq \alpha\}$$

for $\alpha \in I$. When the membership function is fuzzy convex, then we have the following remarks.

**Remark 1** The following statements (1) - (4) are equivalent each other, provided with (i) of Definition 1.

1. (1.2) holds;
2. $L_\alpha(\mu)$ is convex with respect to $\alpha \in I$;
3. $\mu$ is non-decreasing in $\xi \in (-\infty, m)$, non-increasing in $\xi \in [m, +\infty)$, respectively;
4. $L_\alpha(\mu) \subseteq L_\beta(\mu)$ for $\alpha > \beta$.

**Remark 2** The above condition (iii) is stronger than (1.2). From (iii) it follows that $\mu(\xi)$ is strictly monotonously increasing in $\xi \in [\min \text{supp}(\mu), m]$. Suppose that $\mu(\xi_1) \geq \mu(\xi_2)$ for $\xi_1 < \xi_2 \leq m$. From Remark 1(iii), it follows that $\mu(\xi_1) = \mu(\xi_2)$ for some $\xi_1 < \xi_2$, so we get $\mu(\xi) = \mu(\xi_1) = \mu(\xi_2)$ for $\xi \in [\xi_1, \xi_2]$. This contradicts with Definition 1 (iiia). Thus $\mu$ is strictly monotonously increasing. In the similar way $\mu$ is strictly monotonously decreasing in $\xi \in [m, \max \text{supp}(\mu)]$. This condition plays an important role in Theorem 1.

We introduce the following parametric representation of $\mu \in F_b^a$ as

$$x_1(\alpha) = \min L_\alpha(\mu),$$

$$x_2(\alpha) = \max L_\alpha(\mu)$$

for $0 < \alpha \leq 1$ and

$$x_1(0) = \min \text{supp}(\mu),$$

$$x_2(0) = \max \text{supp}(\mu).$$

In the following example we illustrate typical types of fuzzy numbers.

**Example 1** Consider the following $L-R$ fuzzy number $x \in F_b^a$ with a membership function as follows:

$$\mu(\xi) = \begin{cases} 
L(\frac{m-\xi}{\ell})_+ & (\xi \leq m) \\
R(\frac{\xi-m}{r})_+ & (\xi > m)
\end{cases}$$

Here it is said that $m \in \mathbb{R}$ is a center and $\ell > 0, r > 0$ are spreads. $L, R$ are $I$-valued functions. Let $L(\xi)_+ = \max(L(\xi), 0)$ etc. We identify $\mu$ with $x = (x_1, x_2)$. As long as there exist $L^{-1}$ and $R^{-1}$, we have $x_1(\alpha) = m - L^{-1}(\alpha)\ell$ and $x_2(\alpha) = m + R^{-1}(\alpha)r$.

Let $L(\xi) = -c_1\xi + 1$, where $c_1 > 0$ and $|x_1 - m| \leq \ell$. We illustrate the following cases (i)-(iv).

(i) Let $R(\xi) = -c_2\xi + 1$, where $c_2 > 0$. Then $c_2\ell(x_2 - m) = c_1r(m - x_1)$.

(ii) Let $R(\xi) = -c_2\sqrt{\xi} + 1$, where $c_2 > 0$. Then $c_2\ell(x_2 - m)^2 = c_1r^2(m - x_1)$.

(iii) Let $R(\xi) = -c_2\xi^2 + 1$, where $c_2 > 0$. Then $c_2^2\ell^2(x_2 - m) = c_1^2r^2(x_1 - m)^2$. 

(iv) Let $c$ be a real number such that $0 < c < 1$. Denote

$$L(\xi) = \begin{cases} 
1 & (\xi = 0) \\
-c\xi + c & (0 < \xi \leq 1)
\end{cases}$$

and let $R(\xi) = L(\xi)$. Then we have $\ell(x_2 - m) = r(m - x_1)$ for $|x_1 - m| \leq \ell$. The representation of $x = (x_1, x_2)$ is as follows:

$$x_1(\alpha) = m - (1 - \frac{\alpha}{c})\ell, \\
x_2(\alpha) = m + (1 - \frac{\alpha}{c})r \quad (0 \leq \alpha < c) \\
x_1(\alpha) = x_2(\alpha) = m \quad (c \leq \alpha \leq 1)$$

The membership function is given by as follows:

$$\mu(\xi) = \begin{cases} 
0 & (\xi < x_1(0), \xi > x_2(0)) \\
x_1^{-1}(\xi) & (x_1(0) \leq \xi < m) \\
1 & (\xi = m) \\
x_2^{-1}(\xi) & (m < \xi \leq x_2(0))
\end{cases}$$

Denote by $C(I)$ the set of all the continuous functions on $I$ to $\mathbb{R}$. The following theorem shows a membership function is characterized by $x_1, x_2$.

**Theorem 1** Denote the left-, right-end points of the $\alpha$--cut set of $\mu \in \mathcal{F}_b^{st}$ by $x_1(\alpha), x_2(\alpha)$, respectively. Here $x_1, x_2 : I \rightarrow \mathbb{R}$. The following properties (i)-(iii) hold.

(i) $x_1, x_2 \in C(I)$;

(ii) $\max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1)$;

(iii) $x_1, x_2$ are non-decreasing, non-increasing on $I$, respectively, as follows:

(a) there exists a positive number $c \leq 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c)$ and that $x_1(\alpha) = m = x_2(\alpha)$ for $\alpha \in [c, 1]$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$;

Conversely, under the above conditions (i) - (iii), if we denote

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\}$$

for $\xi \in \mathbb{R}$, then $\mu \in \mathcal{F}_b^{st}$.

**Remark 3** From the above Condition (i) a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $\mathbb{R}^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$.

In what follows we denote $\mu = (x_1, x_2)$ for $\mu \in \mathcal{F}_b^{st}$. The parametric representation of $\mu$ is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be an $\mathbb{R}$--valued function. The corresponding binary operation of two fuzzy numbers $x, y \in \mathcal{F}_b^{st}$ to $g(x, y) : \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \rightarrow \mathcal{F}_b^{st}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x, y)}$ of $g$ is as follows:

$$\mu_{g(x, y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))$$
Here $\xi, \xi_1, \xi_2 \in \mathbb{R}$ and $\mu_x, \mu_y$ are membership functions of $x, y$, respectively. From the extension principle, it follows that, in case where $g(x, y) = x + y,$

$$
\begin{align*}
\mu_{x+y}(\xi) &= \max_{\xi = \xi_1 + \xi_2} \min_{i=1,2} (\mu_{i}(\xi_i)) \\
&= \max\{ \alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i), i = 1, 2 \} \\
&= \max\{ \alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)] \}.
\end{align*}
$$

Thus we get $x + y = (x_1 + y_1, x_2 + y_2)$. In the similar way $x - y = (x_1 - y_2, x_2 - y_1)$.

Denote a metric by

$$
 d_{\infty}(x, y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)
$$

for $x = (x_1, x_2), y = (y_1, y_2) \in F_b^{st}$.

**Theorem 2** $F_b^{st}$ is a complete metric space in $C(I)^2$.

### 2 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions $\mu_x, \mu_y$ of $x, y \in F_b^{st}$ and $\lambda \in \mathbb{R}$, the following addition and a scalar product are given as follows:

$$
\begin{align*}
\mu_{x+y}(\xi) &= \sup\{ \alpha \in [0,1] : \xi = \xi_1 + \xi_2, \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y) \}; \\
\mu_{\lambda x}(\xi) &= \left\{ \begin{array}{ll}
\mu_x(\xi/\lambda) & (\lambda \neq 0) \\
0 & (\lambda = 0, \xi \neq 0) \\
\sup_{\eta \in \mathbb{R}} \mu_x(\eta) & (\lambda = 0, \xi = 0)
\end{array} \right.
\end{align*}
$$

In [5] they introduced the following equivalence relation $(x, y) \sim (u, v)$ for $(x, y), (u, v) \in F_b^{st} \times F_b^{st}$, i.e.,

$$(x, y) \sim (u, v) \iff x + v = u + y.$$  \hspace{1cm} (2.4)

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (2.4) means that the following equations hold.

$$
x_i + v_i = u_i + y_i \quad (i = 1, 2)
$$

Denote an equivalence class by $[x, y] = \{(u, v) \in F_b^{st} \times F_b^{st} : (u, v) \sim (x, y) \}$ for $x, y \in F_b^{st}$ and the set of equivalence classes by

$$
F_b^{st}/\sim = \{[x, y] : x, y \in F_b^{st}\}
$$

such that one of the following cases (i) and (ii) hold:

(i) if $(x, y) \sim (u, v)$, then $[x, y] = [u, v]$;

(ii) if $(x, y) \not\sim (u, v)$, then $[x, y] \cap [u, v] = \emptyset$.

Then $F_b^{st}/\sim$ is a linear space with the following addition and scalar product

$$
[x, y] + [u, v] = [x+u, y+v] \hspace{1cm} (2.5)
$$

$$
\lambda[x, y] = \left\{ \begin{array}{ll}
[\lambda x, \lambda y] & (\lambda \geq 0) \\
[(-\lambda)y, (-\lambda)x] & (\lambda < 0)
\end{array} \right.
$$  \hspace{1cm} (2.6)
for $\lambda \in \mathbb{R}$ and $[x, y], [u, v] \in \mathcal{F}_b^{et}/\sim$. They denote a norm in $\mathcal{F}_b^{et}/\sim$ by
\[
\| [x, y] \| = \sup_{\alpha \in I} d_H(L_\alpha(x), L_\alpha(y)).
\]
Here $d_H$ is the Hausdorff metric is as follows:
\[
d_H(L_\alpha(x), L_\alpha(y)) = \max \left( \sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|, \sup_{\eta \in L_\alpha(\mu_y)} \inf_{\xi \in L_\alpha(\mu_x)} |\xi - \eta| \right).
\]
It can be easily seen that $\| [x, y] \| = d_\infty(x, y)$.

3 Schauder’s Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $\mathcal{F}_b^{et}$ has an induced Banach space.

**Theorem 3** Let $S$ be a bounded closed subset in $\mathcal{F}_b^{et}$. Assume that $S$ contains any segments of $x, y \in S$, i.e., $\lambda x + (1 - \lambda)y \in S$ for $\lambda \in I$. Let $V$ be an into continuous mapping on $S$. Assume that the closure $cl(V(S))$ is compact in $\mathcal{F}_b^{et}$. Then $V$ has at least one fixed point $x$ in $S$, i.e., $V(x) = x$.

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

**Theorem 4** Let $\mathcal{F}$ be a complete metric space with a metric $d$. Assume that $\mathcal{F}$ is closed under addition and scalar product, and that $d(\lambda x, 0) = |\lambda|d(x, 0)$ for the scalar product $\lambda x$ and $\lambda \in \mathbb{R}, x \in \mathcal{F}$. Denote $X = \{[x, 0] : x, 0 \in \mathcal{F}\}$. Here $[x, y]$ for $x, y \in \mathcal{F}$ are equivalence classes of (2.4) and 0 is the origin. Then $X$ is a Banach space concerning addition (2.5), scalar product (2.6) and norm $\| [x, 0] \| = d(x, 0)$ for $[x, 0] \in X$.

Moreover let $S$ be a bounded closed subset in $\mathcal{F}$. Assume that $S$ contains any segments of $x, y \in S$ in the same meaning of Theorem 3. Let $V$ be an into continuous mapping on $S$. Assume that the closure $cl(V(S))$ is compact in $\mathcal{F}$. Then $V$ has at least one fixed point in $S$.

4 FBVP on Infinite Intervals

In this section we deal with the following FBVP on an infinite interval:
\[
\frac{dx}{dt} = p(t)x + f(t, x), \quad x(\infty) = c \tag{4.7}
\]
Here $p : \mathbb{R}_+ \rightarrow \mathcal{F}_b^{et}, \ f : \mathbb{R}_+ \times \mathcal{F}_b^{et} \rightarrow \mathcal{F}_b^{et}$ are continuous functions. Let denote $\mathbb{R}_+ = [0, \infty)$ and $c \in \mathcal{F}_b^{et}$. The following assumptions play important roles in considering the existence of solutions of (4.7).

**Assumption.**

(A1) Assume that
\[
\int_0^\infty d(p(s), 0)ds = K < \infty.
\]

(A2) There exist positive real numbers $a, r, R$ and integrable function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that
\[
d(f(t, x), 0) \leq m(t) \text{ for } (t, x) \in \mathbb{R}_+ \times S_1;
\]
\[
\int_0^\infty m(s)ds \leq rR;
\]
\[
[R + N_p(a + \| L \| R)]K < 1.
\]
Here
\[ S_1 = \{ x \in F_b^\epsilon t : d(x,0) \leq \min(\alpha r, r) \} \]
and \( N_p \) is independent on the function \( p \).

\( L : C^\lim r \to F_b^\epsilon t \) is a linear operator as \( L(x) = x(\infty) \) and
\[ C^\lim r = \{ x \in C(\mathbb{R}_+ : F_b^\epsilon t) : \exists x(\infty), d(x,0) \leq r \}. \]

**A3** There exists no solution of
\[ \frac{dx}{dt} = p(t)x, L(x) = 0 \]
extcept for the zero solution.

We expect the following existence theorem for solutions of FBVP on the infinite interval.
Under assumptions (A1) - (A3) we expect that there exists at least one solution of (4.7) in \( C^\lim r \) for any \( c \in S_1 \) by applying the Schauder's fixed point theorem in \( C^\lim r \).

**References**


