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Kyoto University
A Pincer Randomization Method for Valuing American Options

北海道大学・経済学研究科 木村 俊一 (Toshikazu Kimura)
北海道大学・経済学研究科 鈴木 健雄 (Takeyoshi Suzuki)
Graduate School of Economics and Business Administration
Hokkaido University

1 Introduction

Variety has come to the options market nowadays since Black & Scholes (1973) and Merton (1973) published the seminal paper. In particular, the valuation of American options (i.e., options which can be exercised before the pre-specified date) written on dividend-paying assets is an important issue in the market due to that they have a much broader range of applications. Many academics and practitioners have attempted to resolve the value of American option analytically since McKean (1965) and Merton (1973) formulated the option value as a free boundary problem. However, there have been no closed-form formula and analytical solutions. The difficulties in such pricing options originate from the possibility of early exercise and the early exercise boundary not known prior must be determined as a part of the solution. Researchers have also made further efforts toward developments of numerical approximation methods for pricing American options.

A brand-new approximation is the randomization method proposed by Carr (1998), which is based on an American option with a random maturity. The random maturity follows the \(n\)-stage Erlangian distribution with mean equal to the pre-specified maturity. Although the idea is easy to understand, the probability density function (pdf) of Erlangian distribution is not suitable for obtaining a simple formula for the \(n\)-th approximation. Actually, Carr's formula for the \(n\)-th approximation of the American put value is given by a recursion of complex triple sums. To improve this shortcoming, an alternative randomization method has been recently developed by Kimura (2004), which used an order statistic for the random maturity. The order statistic also plays a key role in our new randomization method in this thesis, and hence the details of his method will be specified later. Kimura's approximation not only has a much simpler expression than Carr's one, but also its numerical results have almost the same accuracy as Carr's. However, computational results sometimes behave unstably under a certain condition. Improving this inadequacy is a principal goal of our randomization method, which we call a pincer randomization. The primal focus of this thesis is on the American put option because the call case can be analyzed by put-call symmetry relations.

The rest of this thesis is organized as follows: In Section 2, we provide some preliminaries for the analysis. Section 3 provides an idea of the pincer randomization method. To examine the accuracy of our method, numerical comparisons with other approximations are shown in Section 4. Finally, we give a conclusion and some comments on future research in Section 5.

2 Preliminaries

2.1 Basic Framework

Assume that the stock price is a risk-neutralized process governed by the stochastic differential equation

\[
\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t,
\]

(2.1)

where \(W \equiv \{W_t : t \in [0, T]\}\) is a standard Brownian motion process on a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) where \((\mathcal{F}_t)_{t \geq 0}\) is the natural filtration corresponding to \(W\) and the probability measure \(\mathbb{P}\) is chosen so that the stock has mean rate of return \(r\). Here, \(r\) is the risk-free rate of interest, \(\delta\) is the dividend rate, and \(\sigma\) is the volatility coefficient of the asset price.

We define the value of American put option with maturity date \(T\) and exercise price \(K\), which is expressed as \(C(S_t, t)\) through this article, satisfies the Black and Scholes (1973) partial differential equation (PDE)

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P(S, t)}{\partial S^2} + (r - \delta)S \frac{\partial P(S, t)}{\partial S} + \frac{\partial P(S, t)}{\partial t} - rP(S, t) = 0
\]

(2.2)
subject to the boundary conditions

\[
\begin{align*}
\lim_{S \downarrow 0} P(S, t) &= 0, \\
\lim_{S \uparrow B_t} P(S, t) &= K - B_t, \\
\lim_{S \uparrow B_t} \frac{\partial P(S, t)}{\partial S} &= -1,
\end{align*}
\]

and the terminal condition

\[
P(S_T, T) = (K - S_T)^+.
\]

Equation (2.4) is usually called the "value matching" condition and Equation (2.5) is the "smooth pasting" condition. These conditions guarantee that premature exercise strategy on the early exercise boundary \( B_t \) will be optimal.

### 2.2 Randomization Methods

#### 2.2.1 Carr's randomization

Carr’s randomization method consists of the following three steps:

1. Randomize the maturity \( T \) by an exponentially distributed random variable \( \tilde{T} \) with mean \( \mathbb{E}[\tilde{T}] = \lambda^{-1} = T \) in order to value the so-called Canadian option.

2. Extend the result to the case that \( \tilde{T} \) is distributed as the \( n \)-stage Erlangian distribution with the same mean \( \mathbb{E}[\tilde{T}] = \lambda^{-1} = T \).

3. Take the limit of the randomized option value by letting \( n \to \infty \) to obtain the underlying American option value.

Figure 1 illustrates the \( n \)-stage Erlangian distribution converges to Dirac’s delta function concentrated at the mean \( \lambda^{-1} = T \).

![Figure 1: \( n \)-stage Erlangian probability density functions (\( n = 1, 2, 4, 8, 16, 32 \))](image)

Let \( g_n^*(T) = \mathbb{E}[g(\tilde{T})] \) for a continuous function \( g \). Then, we have

\[
g_n^*(T) = \int_0^\infty g(t) \frac{(nt/T)^{n-1}}{(n-1)!} \frac{n}{T} e^{-nt/T} dt,
\]

for which we obtain

\[
\lim_{n \to \infty} g_n^*(T) = g(T)
\]

that is the mathematical essence of Carr’s randomization method.
Kimura’s randomization

Instead of the \( n \)-stage Erlangian distribution, Kimura (2004) used an order statistic for the random maturity. In much the same way as in Carr's randomization, his method consists of the following three steps:

1. Randomize the maturity \( T \) by an exponentially distributed random variable \( \tilde{T} \) with mean \( \mathbb{E}[\tilde{T}] = \lambda^{-1} = T \) in order to value the Canadian option.
2. Extend the result to the case that \( \tilde{T} \) is distributed as an order statistic with the same mean \( \mathbb{E}[\tilde{T}] = \lambda^{-1} = T \).
3. Take the limit of the randomized option value by letting \( n,m \to \infty \) to obtain the underlying American option value.

Let \( X_1, \ldots, X_{n+m} \) be independent and exponentially distributed random variables with parameter \( \alpha \) \((>0)\), and let \( X_{(i)} \) denote the \( i \)-th smallest of these random variables \( (i = 1, \ldots, n+m) \). The pdf of \( X_{(n+1)} \) is given by

\[
f(t) = \frac{(n+m)!}{n!(m-1)!} \left(1 - e^{-\alpha t}\right)^n \alpha e^{-m\alpha t}, \quad t \geq 0.
\] (2.9)

The mean and variance of \( X_{(n+1)} \) are given by

\[
\mathbb{E}[X_{(n+1)}] = \frac{1}{\alpha} \sum_{i=0}^{n} \frac{1}{m+i}, \quad \mathbb{V}[X_{(n+1)}] = \frac{1}{\alpha^2} \sum_{i=0}^{n} \frac{1}{(m+i)^2}.
\] (2.10)

In addition, the modal value of \( X_{(n+1)} \) is given by

\[
\mathbb{M}[X_{(n+1)}] = \text{argmax}_t f(t) = \frac{1}{\alpha} \log \frac{n+m}{m}.
\] (2.11)

Figure 2: Probability density functions of the order statistic \((n = m = 1, 2, 4, 8, 16, 32)\)

Figure 2(a) and 2(b) show the convergence of the pdf as \( n (=m) \to \infty \). There is not all that much difference between these figures and they converge to Dirac's delta function concentrated at the mean \( \mathbb{E}[X_{(n+1)}] = 1 \). By setting either \( \mathbb{E}[X_{(n+1)}] = T \) or \( \mathbb{M}[X_{(n+1)}] = T, X_{(n+1)} \) can be another candidate for the random maturity \( \tilde{T} \), because \( \lim_{n,m \to \infty} \mathbb{V}[X_{(n+1)}] = 0 \).

Kimura (2004) adopted the mode-matching \( \mathbb{M}[X_{(n+1)}] = T \) in his randomization for computational convenience, because there is no significant difference between the two matchings. For the mode-matching, \( \alpha \) can be determined by

\[
\alpha = \frac{1}{T} \log \frac{n+m}{m}.
\] (2.12)

For a continuous function \( g(t) \ (t \geq 0) \), let \( g_{n,m} = g_{n,m}(T) = \mathbb{E}[g(\tilde{T})] \), then

\[
g_{n,m}(T) = \frac{(n+m)!}{n!(m-1)!} \int_{0}^{\infty} g(t)(1 - e^{-\alpha t})^n \alpha e^{-m\alpha t} dt,
\] (2.13)
for which we have
\[
\lim_{n,m \to \infty} g_{n,m}^*(T) = g(T). \tag{2.14}
\]

**Proposition 1 (Kimura)** The sequence \((g_{n,m}^*)_{n,m \geq 1}\) satisfies the recursion
\[
\begin{cases}
  g_{0,m}^* = \int_0^\infty mae^{-mt}g(t)dt \\
  g_{n,m}^* = \frac{n+m}{n}g_{n-1,m}^* - \frac{m}{n}g_{n-1,m+1}^*, \quad n \geq 1.
\end{cases} \tag{2.15}
\]

Let \(L^*(m)\) denote a root of the equation for the early exercise boundary of Canadian options. The \(N\)-th randomized approximation \(g_{N,N}^* \equiv \beta_N \approx B_t (N \geq 1)\) can be obtained by the algorithm named OS-Random.

The algorithm also can be applied to computing the option value \(P(t, S)\) by assuming that we have a functional program for computing \(P^*(m, S)\) for a set of the parameters \(\{t, S, K, T, r, \delta, \sigma\}\).

### 2.3 Canadian Options

The randomization methods are based on the value of Canadian option whose maturity is exponentially distributed to introduce not only Carr’s randomization but also the alternative one proposed by Kimura (2004).

**Proposition 2 (Kimura)** The value of the European-style Canadian put option is given by
\[
p^*(\lambda, S) = \begin{cases}
  \xi(S) + \frac{\lambda}{\lambda + \delta}K - \frac{\lambda}{\lambda + \delta}S, & S < K \\
  \eta(S), & S \geq K
\end{cases}, \tag{2.16}
\]
where
\[
\xi(S) = \frac{1}{\theta_+ - \theta_-} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r}\right) K \left(\frac{S}{K}\right)^{\theta_+}, \quad S < K
\]
\[
\eta(S) = \frac{1}{\theta_+ - \theta_-} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r}\right) K \left(\frac{S}{K}\right)^{\theta_-}, \quad S \geq K,
\]
and the parameters \(\theta \pm\) are two roots of the following quadratic equation
\[
\frac{1}{2}\sigma^2 \theta^2 + (r - \delta - \frac{1}{2}\sigma^2) \theta - (\lambda + r) = 0, \tag{2.18}
\]
i.e.,
\[
\theta_\pm = \frac{1}{\sigma^2} \left\{- (r - \delta - \frac{1}{2}\sigma^2) \pm \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)}\right\}, \tag{2.19}
\]

**Proposition 3 (Kimura)** For \(L^* \leq K\), the value of the American-style Canadian put option is given by
\[
P^*(\lambda, S) = \begin{cases}
  K - S, & S \leq L^* \\
  p^*(\lambda, S) + e^*(\lambda, S), & S > L^*.
\end{cases} \tag{2.20}
\]
where
\[
e^*(\lambda, S) = -\frac{1}{\theta_-} \left\{\theta_+ \xi(L^*) + \frac{\delta}{\lambda + \delta} \left(\frac{S}{L^*}\right)^{\theta_-}\right\}, \quad S > L^*. \tag{2.21}
\]
3 A Pincer Randomization Method

Kimura's randomization method is not only much simpler than Carr's one, but also as accurate as Carr's one; however, the method shows unstable behaviors near the expiry under certain conditions. The reasons for the unstability are considered as

(i) the algorithm is sensitive to the precision of the root $L^*$.

(ii) the $(n, m)$-th approximation $g^*_{n,m}$ cannot appropriately satisfies the value matching condition in the recursive procedure.

In this section, we propose a new randomization scheme named a pincer randomization (PR) method to overcome those difficulties. The PR method is based on a pair of lower and upper bounds for a true value (say TRUE), and then TRUE is sandwiched in between the bounds. This methods reflect some fundamental properties of the option Greek Theta and the order statistic. It is generally known that Theta indicates the ratio of the change in an American put option price to the decrease in time to expiration, so that the shorter the remaining time to expiration, the option value is cheaper.

Remark 1 Note that the relation of a pair of lower and upper bounds inverts if and only if the Theta is negative under deep-in-the-money.

3.1 Lower and upper bounds for the option value

Assume that the maturity $T$ is a random variable $\hat{T}$ distributed as the order statistic $X_{(n+1)}$ with mean $E[\hat{T}] = T$, as in the OS-Random algorithm. From Figure 2(a) and the Theta property of American put options that the mean-matching approximation for the option value always underestimates the true value when $n, m$ is not large enough, giving a lower bound. Note that mean-matching approximation for the early exercise boundary provides the upper bound. Figure 3(a) shows that the lower bound is a tight one over the true value derived by the CRR binomial method with $n = 1000$.

In the same manner as the mean-matching case, Figure 2(b) and the Theta property shows that the mode-matching approximation always overestimates the true value when $n, m$ is small, i.e., it is an upper bound. For the early exercise boundary, the mode-matching approximation provides the lower bound. Figure 3(b) shows that the upper bound is less tight than the lower bound, where TRUE values are also computed by the CRR binomial method with $n = 1000$.

Figure 3: Lower and upper bounds ($T = 1.0, S = 100, K = 100, r = 0.05, \sigma = 0.3, \delta = 0$)

3.2 Interpolating lower and upper bounds

Figure 4 illustrates a relationship between the lower and upper bounds for the early exercise boundary. This figure shows that the the TRUE value is appropriately sandwiched in between the bounds, and that the upper bound derived by the mean matching is a good approximation for the TRUE value. For the
option value, the TRUE value is appropriately sandwiched in between the bounds, and the lower bound is a good approximation for the TRUE one. From Figure 3, the mean matching provides more accurate approximations for the option value. From these observations, we employ the two methods below for valuing American put options.

- Arithmetic Average:
  \[ P_A(t, S_t) = \frac{L(t, S_t) + U(t, S_t)}{2} \]  
  (3.1)
  where \( L(t, S_t) \) and \( U(t, S_t) \) are the lower and upper bound for the option value, respectively.

- Geometric Average:
  \[ P_G(t, S_t) = \sqrt{L(t, S_t) \times U(t, S_t)} \]  
  (3.2)

As described above, the upper bound of the early exercise boundary and the lower bound of the option value are good approximations for the TRUE values. Hence, we also add the lower-bound approximation for the option value in comparisons.

![Graph showing lower and upper bounds for the early exercise boundary](image)

**Figure 4:** Lower and upper bounds for the early exercise boundary \((K = 100, r = 0.05, \sigma = 0.3, \delta = 0)\)

To determine the level \( N \) of the approximation, we make a comparison between \( p(t, S) \) and its PR approximation. Obviously, the exact values of \( p(t, S) \) can be computed by the Black-Scholes formula (2.2). Figure 5 illustrates the relative percentage errors of approximations for \( p(0, S) \) as functions of \( S \). The approximations become better as \( N \) increases and we have sufficient accuracy for \( N \geq 6 \). Hence, we will employ \( N = 8 \) in our numerical experiments.

![Graph showing relative percentage errors of the approximations](image)

**Figure 5:** Relative percentage errors of the approximations for the vanilla European put value \( p(0, S) \) \((T = 1.0, K = 100, r = 0.05, \sigma = 0.3, \delta = 0.05)\)
4 Computational Results

Figure 6(a) (6(b)) shows some relations between the early exercise boundary and the volatility (dividend rate). Also, Figure 7(a) (7(b)) shows some relations between the option values and the volatility (maturity). In order to check the performance of the PR method in detail, we compare them with other approximations for particular cases quoted from numerical experiments in Ait-Sahalia and Carr (1997). Tables 1 and 2 summarize these results, in which we compute three approximation by the PR method with both the arithmetic and geometric averages and the lower-bound approximation named LB-Random. We employ the arithmetic average of the 1000- and 1001-steps binomial value as a benchmark of the TRUE value. For the methods of OS-Random, Carr, and Geske & Johnson, “N-pts” in these tables denote the number of steps of the N-point Richardson extrapolation. For the finite-difference results, the parameters $N$ and $M$ denote the numbers of time and state steps, respectively. See Ait-Sahalia and Carr (1997) for details of their experiments.

![Figure 6](image1.png)

Figure 6: Early exercise boundaries of put options $(K=100, T=1.0, r=0.05)$

![Figure 7](image2.png)

Figure 7: Values of put options $(K=100, t=0, r=0.05, \delta=0.2)$

The PR method performs very well and competes with OS-Random and Carr’s randomization. In addition, the PR method succeeds in the way that modification of OS-Random that always underestimates the TRUE value, because the PR method provides not only much more accurate approximations for valuing put options but also better approximation than OS-Random. In addition, we see from these figures that the PR method is more accurate than LBA and LUBA, which are also the lower-bound and the lower-and-upper-bounds approximations, respectively.

Table 1 shows the impacts of the initial stock price $S$. The PR method with both of arithmetic and geometric average becomes accurate as $S$ increases, because the early exercise premium relatively constitutes a smaller portion of the value for such cases. The fact is very well deserved from the viewpoints
that the PR method can value European option values as accurate as the Black-Scholes formula and that we can decompose American option value into the early exercise premium plus European option value.

Table 2 demonstrates that remaining time impacts on the option values. The PR method with both arithmetic and geometric averages becomes accurate as the remaining time becomes long. For this tendency, we can give the same prospect from Table 1. In addition, from Tables 1 and 2, we can see that the PR method with arithmetic average is accurate enough and is greater than the one with geometric average. Clearly, this reflects the fact that \( P_A(t, S_t) \geq P_G(t, S_t) \) for all \( (t, S_t) \).

From the observations in Figures 3 and 4, it was considered that the lower-bound approximations for the option values would perform well. However, we see from Tables 1 and 2 that the lower-bound approximations are less accurate than other approximations. We also see from other numerical experiments that the randomization method with mean matching performs well if and only if dividend is zero for which the root \( L^* \) can be computed via

\[
L^* = K \left( \frac{r(t_S - 1)}{\lambda} \right)^{1/\lambda}
\]

without using Newton's method. These observations would imply that the accuracy of the lower-bound (or mean-matching) approximation is highly sensitive to the computational accuracy of the root \( L^* \).

5 Conclusion

The previous randomization methods have crucial problems such as (i) difficulty of implementation for Carr's one and (ii) unstable behavior near expiry for Kimura's one. To rectify these faults at the same time, we have employed an interpolation approximation using a pair of lower and upper bounds obtained by Kimura's randomization method. The idea is based on the Theta property of American put options. Our new method, the PR method, refines Kimura's one, removing another fault of underestimation.

The PR method generates accurate approximations when the initial stock price is in the out-of-money or the remaining time to maturity is long. It is straightforward to interpret these properties from the fact that American option value can decomposed into the early exercise premium and the associated European option value, the latter of which constitutes a greater portion of the whole value. However, the PR method still have a tendency of underestimation from the true value, which needs a further revision of the randomization.

Mathematically, the essential of randomization can be interpreted as an inversion of Laplace or Fourier transforms. This interpretation enables us to apply the randomization methods including the PR method to valuing other options, e.g., exotic or path-dependent options such as Asian, lookback, barrier options and so on. This is a future theme of extensive research. Another extension of the randomization method is the case that the stock return jumps accidentally, that is, the stock price process follows not the Brownian motion but a jump-diffusion process such as Lévy processes. This remains as future work, too.

References


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Table 2: A comparison of approximations for \( P(0, S) \) \((T = 3, K = 100, r = 0.06, \sigma = 0.4, \delta = 0.02)\)

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