数理解析研究所講究録 1458 巻 2005 年 89-96

REVERSE INEQUALITIES ASSOCIATED WITH TSALLIS RELATIVE OPERATOR ENTROPY VIA GENERALIZED KANTOROVICH CONSTANT

Takayuki Furuta 古田孝之 (東京理大 理)

§1. Introduction

A capital letter means an operator on a Hilbert space H. An operator X is said to be *strictly positive* (denoted by $X>0$) if X is positive definite and invertible. For two strictly positive operators A, B and $p\in[0,1]$, p-power mean $A\natural_p B$ is defined by

$$A\natural_p B=A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^pA^{\frac{1}{2}}$$

and we remark that $A\natural_p B=A^{1-p}B^p$ if A commutes with B.

Very recently, Tsallis relative operator entropy $T_p(A|B)$ in Yanagi-Kuriyama-Furuichi [17] is defined by

$$T_p(A|B)=\frac{A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^pA^{\frac{1}{2}}-A}{p}\quad \text{for } p\in(0,1] \tag{1.1}$$

and $T_p(A|B)$ can be written by using the notion of $A\natural_p B$ as follows:

$$T_p(A|B)=\frac{A\natural_p B-A}{p}\quad \text{for } p\in(0,1]. \tag{1.1'}$$

The relative operator entropy $\hat{S}(A|B)$ in [3] is defined by

$$\hat{S}(A|B)=A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \tag{1.2}$$

as an extension of [15].

On the other hand, the genaralized Kantrovich constant $K(p)$ is defined by

$$K(p)=\frac{(h^p-h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p \tag{1.3}$$

for any real number p and $h>1$. Also $S(p)$ is defined by

$$S(p)=\frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}} \tag{1.4}$$

for any real number p. In particular $S(1)=\dfrac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ is said to be *the Specht ratio* and $S(1)>1$ is well known.

Theorem A. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities hold:*

$$(Ax, x)^p \geq (A^p x, x) \geq K(p)(Ax, x)^p \quad \text{for any } 1 \geq p > 0. \tag{1.5}$$

$$S(1)\Delta_x(A) \geq (Ax, x) \geq \Delta_x(A). \tag{1.6}$$

$$K(p) \in (0, 1] \text{ for } p \in [0, 1]. \tag{1.7}$$

$$K(0) = K(1) = 1. \tag{1.8}$$

$$S(1) = e^{K'(1)} = e^{-K'(0)}. \tag{1.9}$$

where the determinant $\Delta_x(A)$ for strictly positive operator A at a unit vector x is defined by $\Delta_x(A) = \exp\langle((\log A)x, x)\rangle$ and (1.6) is shown in [4].

(1.8) and (1.9) of Theorem A are shown in [8, Proposition 1] and (1.7) is shown in [9].

§2 Two reverse inequalities involving Tsallis relative operator entropy $T_p(A|B)$ via generalized Kantorovich constant $K(p)$

At first we shall state the following two reverse inequalities involving Tsallis relative operator entropy $T_p(A|B)$ via generalized Kantorovich constant $K(p)$.

Theorem 2.1. *Let A and B be strictly positive operators such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$, $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0, 1]$. Let Φ be normalized positive linear map on $B(H)$. Then the following inequalities hold:*

$$\left(\frac{1 - K(p)}{p}\right)\Phi(A)\sharp_p\Phi(B) + \Phi(T_p(A|B)) \geq T_p(\Phi(A)|\Phi(B)) \geq \Phi(T_p(A|B)) \tag{2.1}$$

and

$$F(p)\Phi(A) + \Phi(T_p(A|B)) \geq T_p(\Phi(A)|\Phi(B)) \geq \Phi(T_p(A|B)) \tag{2.2}$$

where $K(p)$ is the generalized Kantorovich constant defined in (1.3) *and*

$$F(p) = \frac{m^p}{p}\left(\frac{h^p - h}{h - 1}\right)\left(1 - K(p)^{\frac{1}{p-1}}\right) \geq 0.$$

Remark 2.1. We remark that the second inequality of (2.1) of Theorem 2.1 is shown in [6] along [3] and the first one of (2.1) is a reverse one of the second one and also the second inequality of (2.2) is as the same as the second one in (2.1) and the first one of (2.2) is a reverse one of the second one. We shall give simple proofs of (2.1) and (2.2) including its reverse inequality, respectively, via generalized Kantorovich constant $K(p)$ in (1.3).

We state the following result to prove Theorem 2.1.

Proposition 2.2. *Let $h > 1$ and let $g(p)$ be defined by:*

$$g(p) = \frac{h^p - h}{h-1} + (1-p)\Big(\frac{h^p - 1}{p(h-1)}\Big)^{\frac{p}{p-1}} \quad \textit{for } p \in [0,1].$$

Then the following results hold:

(i) $\quad g(0) = \lim_{p\to 0} g(p) = 0.$

(ii) $\quad g(p) = \frac{h^p - h}{h-1}\Big(1 - K(p)^{\frac{1}{p-1}}\Big) \geq 0 \quad$ *for all* $p \in [0,1]$.

(iii) $\quad g'(0) = \lim_{p\to 0} g'(p) = \log S(1).$

(iv) $\quad \lim_{p\to 0} \frac{g(p)}{p} = \log S(1).$

Also we need the following result to prove Theorem 2.1.

Theorem B. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Let $p \in (0,1)$ and also let Φ be normalized positive linear map on $B(H)$. Then the following inequalities hold:*

(i) $\quad \Phi(A)\natural_p\Phi(B) \geq \Phi(A\natural_p B) \geq K(p)\Phi(A)\natural_p\Phi(B)$

(ii) $\quad \Phi(A)\natural_p\Phi(B) \geq \Phi(A\natural_p B) \geq \Phi(A)\natural_p\Phi(B) - f(p)\Phi(A)$

where $f(p) = m^p\Big[\frac{h^p - h}{h-1} + (1-p)\Big(\frac{h^p - 1}{p(h-1)}\Big)^{\frac{p}{p-1}}\Big]$ *and $K(p)$ is defined in* (1.3).

The right hand side inequalities of (i) and (ii) of Theorem B follow by [14, Corollary 3.5] and the left hand side of (i) is well known [13].

§3 Two results by Furuichi-Yanagi-Kuriyama which are useful to prove our results in §6

Throughout this section, we deal with $n \times n$ matrix. A matrix X is said to be strictly positive definite matrix (denoted by $X > 0$) if X is positide definite and invertible. Let A and B be positive definite matrices. Tsallis relative entropy $D_p(A||B)$ in Furuichi-Yanagi-Kuriyama [5] is defined by

$$D_p(A||B) = \frac{\mathrm{Tr}[A] - \mathrm{Tr}[A^{1-p}B^p]}{p} \quad \text{for } p \in (0,1]. \tag{3.1}$$

Umegaki relative entropy $S(A,B)$ in [16] is defined by

$$S(A,B) = \mathrm{Tr}[A(\log A - \log B)] \quad \text{for } A, B > 0. \tag{3.2}$$

Theorem C. (Genaralized Peierls-Bogoliubov inequality [5]) *Let $A, B > 0$ and also let $p \in (0,1]$. Then the following inequality holds:*

$$D_p(A||B) \geq \frac{\mathrm{Tr}[A] - (\mathrm{Tr}[A])^{1-p}(\mathrm{Tr}[B])^p}{p}. \tag{3.3}$$

Theorem D [5]. *Let $A, B > 0$. The following inequality holds:*

$$-\mathrm{Tr}[\mathrm{T}_p(A|B)] \geq D_p(A||B) \quad \text{for } p \in (0,1]. \tag{3.4}$$

We remark that (3.4) implies $-\mathrm{Tr}[\hat{\mathrm{S}}(A|B)] \geq S(A,B)$ which is well known in [11],[12],[2] and [5].

§4 A result which unifies Theorem C and Theorem D in §3

Also throughout this section, we deal with $n \times n$ matrix. In this section, we shall state the following Proposition E which unifies Theorem C and Theorem D in §3.

Proposition E. *Let $A, B > 0$ and also let $p \in (0,1]$. Then the following inequalities hold:*

$$\begin{aligned} \mathrm{Tr}[(1-p)A + pB] &\geq (\mathrm{Tr}[A])^{1-p}(\mathrm{Tr}[B])^p \\ &\geq \mathrm{Tr}[A^{1-p}B^p] \\ &\geq \mathrm{Tr}[A \natural_p B]. \end{aligned} \tag{4.1}$$

Proposition F. *Let $A, B > 0$ and also let $p \in (0,1]$. Then the following inequalities hold:*

$$\begin{aligned} -\mathrm{Tr}[\mathrm{T}_p(A|B)] &\geq D_p(A||B) \\ &\geq \frac{\mathrm{Tr}[A] - (\mathrm{Tr}[A])^{1-p}(\mathrm{Tr}[B])^p}{p} \\ &\geq \mathrm{Tr}[A - B]. \end{aligned} \tag{4.5}$$

Needless to say, the first inequality of (4.5) of Proposition F is just (3.4) of Theorem D and the second one of (4.5) is just (3.3) of Theorem C, and also Proposition E is nothing but another expression form of Proposition F.

Proposition F yields the following result by putting $p \to 0$.

Proposition G. *Let $A, B > 0$. Then the following inequalities hold:*

$$-\mathrm{Tr}[\hat{\mathrm{S}}(A|B)] \geq S(A,B) \tag{4.6}$$

$$\geq \mathrm{Tr}[A(\log \mathrm{Tr}[A] - \log \mathrm{Tr}[B])]$$
$$\geq \mathrm{Tr}[A - B].$$

§5 Related counterexamples to several questions caused by the results in §4

Also throughout this section, we deal with $n \times n$ matrix too. We shall give related counterexamples to several questions caused by the results in §4

Remark 5.1. The following matrix inequality (AG) is quite well known as the matrix version of (4.2) and there are a lot of references (for example, [13],[7]):

$$\text{(AG)} \qquad (1-p)A + pB \geq A\natural_p B \text{ holds for } A, B > 0 \text{ and } p \in (0,1].$$

Suggested by the matrix inequality (AG), the second inequality and the third one on trace inequality (4.1) of Proposition E, we might be apt to suppose that the following matrix inequalities as more exact precise estimation than (AG) : let $A, B > 0$ and $p \in (0,1]$,

$$\text{(AG-1?)} \qquad (1-p)A + pB \geq B^{\frac{p}{2}} A^{1-p} B^{\frac{p}{2}} \geq A\natural_p B$$

and

$$\text{(AG-2?)} \qquad (1-p)A + pB \geq A^{\frac{1-p}{2}} B^p A^{\frac{1-p}{2}} \geq A\natural_p B.$$

But we have the following common counterexample to (AG-1?) and (AG-2?).

Remark 5.2. (i). *If A and B are positive definite matrices and $p \in (0,1]$, then the following inequality holds*:

$$\text{(5.2)} \qquad D_p(A||B) \geq \mathrm{Tr}[A - B].$$

We remark that (5.2) is shown in the proof of [5, (1) of Proposition 2.4] and the second inequality and the third one of (4.5) of Proposition F yield the inequality (5.2), that is, the second inequality and the third one of (4.5) of Proposition F are somewhat more precise estimation than (5.2).

(ii). Also we recall the following result [1, Problem IX.8.12]:

If A and B are strictly positive matrices, then the following inequality holds:

$$\text{(5.3)} \qquad \mathrm{Tr}[A(\log A - \log B)] \geq \mathrm{Tr}[A - B].$$

We remark that the second inequality and the third one of (4.6) of Proposition G imply (5.3) since $S(A, B) = \mathrm{Tr}[A(\log A - \log B)]$, that is, the second inequality and the third one of (4.6) are somewhat more precise estimation than (5.3).

Suggested by (5.3), we might be apt to expect that the following matrix inequality:

$$A^{\frac{1}{2}}(\log A - \log B)A^{\frac{1}{2}} \geq A - B. \tag{5.3-1?}$$

so that it turns out that (5.3-1?) does not hold.

§6. Two trace reverse inequalities associated with $-\mathrm{Tr}[T_p(A|B)]$ and $D_p(A||B)$ via generalized Kantorovich constant $K(p)$

As an application of Theorem 2.1, we shall show the following two trace reverse inequalities associated with $-Tr[T_p(A|B)]$ and $D_p(A||B)$ via generalized Kantorovich constant $K(p)$.

Theorem 6.1.

Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0,1]$. Then the following inequalities hold:

$$\begin{aligned}&\left(\frac{1-K(p)}{p}\right)(\mathrm{Tr}[A])^{1-p}(\mathrm{Tr}[B])^{p} + D_p(A||B)\\ &\geq -\mathrm{Tr}[T_p(A|B)]\\ &\geq D_p(A||B)\end{aligned} \tag{6.1}$$

$$\begin{aligned}&F(p)(\mathrm{Tr}[A]) + D_p(A||B)\\ &\geq -\mathrm{Tr}[T_p(A|B)]\\ &\geq D_p(A||B)\end{aligned} \tag{6.2}$$

where $K(p)$ is the generalized Kantorovich constant defined in (1.3) *and*

$$F(p) = \frac{m^p}{p}\left(\frac{h^p - h}{h-1}\right)\left(1 - K(p)^{\frac{1}{p-1}}\right) \geq 0.$$

Corollary 6.2. [10] *Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequality hold:*

$$\begin{aligned}&\log S(1)\mathrm{Tr}[A] + S(A,B)\\ &\geq -\mathrm{Tr}[\hat{S}(A|B)]\\ &\geq S(A,B)\end{aligned} \tag{6.5}$$

where $S(1)$ is the Specht ratio defined in (1.4) *and the first inequality is the reverse one of the second inequalty.*

The complete paper with proofs will appear elsewhere.

References

[1] R.Bhatia, Matrix Analysis, Springer Verlag, 1997.

[2] N.Bebiano, R.Lemos and J. da Providencia, Inequalities for quantum relative entropy, to appear in Linear Alg. and Its Appl.

[3] J.I.Fujii and E.Kamei, Relative operator entropy in noncommutative information theory, Math. Japon, **34**(1989), 341-348.

[4] J.I.Fujii, S.Izumino and Y.Seo, Determinant for positive operators and Specht's ratio, Sci. Math. **1**(1998), 307-310.

[5] S.Furuichi, K.Yanagi and K.Kuriyama, Fundamental properties of Tsallis relative entropy, J. Math. Phys., **45**(2004), 4868-4877.

[6] S.Furuichi, K.Yanagi and K.Kuriyama, A note on operator inequalities of Tsallis relative operator entropy, preprint.

[7] T.Furuta, Invitation to Linear Operators, Taylor & Francis, London and New York 2001.

[8] T.Furuta, Specht ratio $S(1)$ can be expressed by Kantorovich constant $K(p)$: $S(1) = \exp[K'(1)]$ and its application, Math. Inequal. and Appl.,**6**(2003),521-530.

[9] T.Furuta, Basic properties of the generalized Kantorovich constant $K(p) = \frac{(h^p - h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p$, Acta Sci. Math (Szeged), **70**(2004),319-337.

[10] T.Furuta, Reverse inequalities involving two relative operator entropies and two relative entropies, Linear Alg. and Its Appl., **403**(2005),24-30.

[11] F.Hiai and D.Petz, The proper formula for relative entropy in asymptotics in quantum probability, Comm. math.Phys., **143**(1991),99-114.

[12] F.Hiai and D.Petz, The Golden-Thompson trace inequality is complemented, Linear Alg. and Its Appl.,**181** (1993), 153-185.

[13] F.Kubo and T.Ando, Means of positive linear operators, Math. Ann., **246**(1980), 205-224.

[14] J.Mićić, J.Pečarić and Y.Seo, *Complementary Inequalities to inequalities of Jensen and Ando based on the Mond-Pečarić method,* Linear Alg. and Appl., **318**(2000),87-107.

[15] M.Nakamura and H.Umegaki, A note on entropy for operator algebras, Proc. Japan Acad., **37**(1961), 149-154.

[16] H.Umegaki, Conditional expectation in an operator algebra IV, Kodai Math. Sem. Rep., **14**(1962), 59-85.

[17] K.Yanagi, K.Kuriyama and S.Furuichi, Generalized Shannon inequalities based on Tsallis relative operator entropy, Linear Alg. and Its Appl., **394**(2005), 109-118.

Department of Mathematical Information Science
Tokyo University of Science
1-3 Kagurazaka, Shinjukuku
Tokyo 162-8601
Japan
e-mail: furuta@rs.kagu.tus.ac.jp