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<th>Title</th>
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</tr>
</thead>
<tbody>
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Kyoto University
Tsallis entropy に関する作用素不等式とトレース不等式

Operator inequalities and trace inequalities derived from Tsallis entropies

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1 Trace inequalities of Tsallis entropy

We define $q$-logarithm function as follows;

$$\ln_q x = \frac{x^{1-q} - 1}{1-q}, \quad (x \geq 0, q \geq 0, q \neq 1).$$

Then we have the following properties;

(1) $\lim_{q \to 1} \ln_q x = \log x.$ (in uniformly)

(2) $\ln_q xy = \ln_q x + \ln_q y + (1-q) \ln_q x \ln_q y.$

(3) $\ln_q x$: concave in $x$ for $q \geq 0$.

Definition 1 (Tsallis entropy) For density operator $\rho$ on a finite dimensional Hilbert space $\mathcal{H}$, Tsallis entropy $S_q(\rho)$ is defined by

$$S_q(\rho) = \frac{\text{Tr}[\rho^q - \rho]}{1-q}, \quad (q \geq 0, q \neq 1).$$
Proposition 1 We have the following properties;

(1) \( \lim_{\Gamma J} S_{\rho}'(p) = -Tr[\rho \log \rho] \).

(2) \( S_{\rho}(\rho_1 \otimes \rho_2) = S_{\rho}(\rho_1) + S_{\rho}(\rho_2) + (1-q)S_{\rho}(\rho_1)S_{\rho}(\rho_2) \).

Lemma 1 \( S_{\rho}(\rho) \leq \ln d \), \( (d = \dim \mathcal{H}) \).

Proof. Since \( \ln x \) is concave, we have

\[
D_q(A|B) = -\sum_{j=1}^{d} a_j \ln_q \frac{b_j}{a_j} \geq -\ln_q \left( \sum_{j=1}^{d} a_j \frac{b_j}{a_j} \right) = 0.
\]

We put \( A = \{a_j\}, \ B = \{u_j\}, \ u_j = \frac{1}{d} (1 \leq j \leq d) \). Then

\[
D_q(A|B) = -d^{q-1}(S_q(A) - \ln_q d) \geq 0.
\]

Thus \( S_q(A) \leq \ln_q d \). q.e.d.

Lemma 2 If \( f \) is a concave function satisfying \( f(0) = f(1) = 0 \), then

\[
|f(t+s) - f(t)| \leq \max\{f(s), f(1-s)\},
\]

where \( s \in [0, 1/2], t \in [0, 1], 0 \leq s + t \leq 1 \).

Proof. We put

\[
r(t) = f(s) - f(t+s) + f(t).
\]

Then

\[
r'(t) = -f'(t+s) + f'(t).
\]

Since \( f' \) is a monotone decreasing function, \( r'(t) \geq 0 \). Thus we have \( r(t) \geq 0 \) by \( r(0) = 0 \). Therefore \( f(t+s) - f(t) \leq f(s) \). We also put

\[
\ell(t) = f(t+s) - f(t) + f(1-s).
\]

Then

\[
\ell'(t) = f'(t+s) - f'(t).
\]

Since \( f' \) is a monotone decreasing function, \( \ell'(t) \leq 0 \). Thus we have \( \ell(t) \geq 0 \) by \( \ell(1-s) = 0 \). Therefore \( -f(1-s) \leq f(t+s) - f(t) \). Thus we have the result. q.e.d.
Lemma 3 If $|u - v| \leq 1/2$, then $|\eta_q(u) - \eta_q(v)| \leq \eta_q(|u - v|)$, where $\eta_q(x) = \frac{x^q - x}{1 - q}$, $q \in [0, 2]$, $u, v \in [0, 1]$.

Proof. Since $\eta_q$ is a concave function with $\eta_q(0) = \eta_q'(0)$, we have

$$|\eta_q(t + s) - \eta_q(t)| \leq \max\{\eta_q(s), \eta_q(1 - s)\}$$

for $s \in [0, \frac{1}{2}]$ and $t \in [0, 1]$ satisfying $0 \leq t + s \leq 1$ by Lemma 2.

Since $\eta_q(x)$ is a monotone increasing function on $[0, q^{1/(1-q)}]$ and $q^{1/(1-q)} \leq \frac{1}{2}$ for $q \in (0, 2)$,

$$\max\{\eta_q(s), \eta_q(1 - s)\} = \eta_q(s).$$

Thus we have the result by letting $u = t + s$ and $v = t$.

q.e.d.

Lemma 4 Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of Hermitian matrix $A$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be eigenvalues of Hermitian matrix $B$. Then we have the following;

$$\text{Tr}[|A - B|] \geq \sum_{i=1}^{n} |\lambda_i - \mu_i|.$$

Theorem 1 (Generalized Fannes's inequality) For two density operators $\rho_1, \rho_2$ on $\mathcal{H}$ and $q \in [0, 2]$, if $\|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)}$, then

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1),$$

where $d = \dim \mathcal{H}$ and $\|A\|_1 = \text{Tr}[|A|]$.

Proof. Let $\lambda_1^{(i)} \geq \cdots \geq \lambda_n^{(i)}$ be eigenvalues of $\rho_i$.

We set

$$\epsilon = \sum_{j=1}^{d} \epsilon_j, \quad \epsilon_j = |\lambda_j^{(1)} - \lambda_j^{(2)}|.$$ 

From Lemma 2,

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^{d} \eta_q(\lambda_j^{(1)}) - \eta_q(\lambda_j^{(2)}) \leq \sum_{j=1}^{d} \eta_q(\epsilon_j).$$

By $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$ and Lemma 1, we have
\begin{align*}
\sum_{j=1}^{d} \eta_{q}(\epsilon_{j}) &= - \sum_{j=1}^{d} \frac{\epsilon_{j}^{q}}{\epsilon} \ln_{q} \epsilon_{j} = \epsilon \{- \sum_{j=1}^{d} \frac{\epsilon_{j}^{q}}{\epsilon} \ln_{q} \left( \frac{\epsilon_{j}}{\epsilon} \right) \} \\
&= \epsilon \{- \sum_{j=1}^{d} \frac{\epsilon_{j}^{q}}{\epsilon} \ln_{q} \epsilon_{j} \} - \sum_{j=1}^{d} \frac{\epsilon_{j}^{q} \ln_{q} \epsilon_{j}}{\epsilon} + \sum_{j=1}^{d} \frac{\epsilon_{j}^{q}}{\epsilon} \ln_{q} \epsilon_{j} \} \\
&= \epsilon_{q} \sum_{j=1}^{d} \eta_{q}(\frac{\epsilon_{j}}{\epsilon}) + \eta_{q}(\epsilon) \leq \epsilon^{q} \ln_{q} d + \eta_{q}(\epsilon).
\end{align*}

Therefore we have

\[ |S_{q}(\rho_{1}) - S_{q}(\rho_{2})| \leq \epsilon^{q} \ln_{q} d + \eta_{q}(\epsilon). \]

From Lemma 3, we have \( \|\rho_{1} - \rho_{2}\| \geq \epsilon \). And \( \eta_{q}(x) \) is monotone increase on \( x \in [0, q^{1/(1-q)}] \). In addition, \( x^{\prime} \) is monotone increase on \( x \in [0, 2] \). Thus we have theorem.

\text{q.e.d.}

Since \( \lim_{q \to 1} q^{1/(1-q)} = 1/e \), we have

**Corollary 1 (Fannes's inequality)** For two density operators \( \rho_{1}, \rho_{2} \) on \( \mathbb{H} \), if \( \|\rho_{1} - \rho_{2}\| \leq 1/e \), then

\[ |S_{1}(\rho_{1}) - S_{1}(\rho_{2})| \leq \|\rho_{1} - \rho_{2}\| \log d + \eta_{1}(\|\rho_{1} - \rho_{2}\|), \]

where \( S_{1}(\rho) = -\text{Tr}[\rho \log \rho] \), \( \eta_{1}(x) = -x \log x \).

## 2 Operator inequalities of Tsallis relative operator entropy

We change the notation \( (\lambda = 1 - q) \). That is, for \( \lambda \in (0, 1] \),

\[ \ln_{\lambda} x = \frac{x^{\lambda} - 1}{\lambda}. \]

**Definition 2 (Tsallis relative operator entropy)** For \( A > 0, B > 0, \lambda \in (0, 1] \), Tsallis relative operator entropy \( T_{\lambda}(A|B) \) is defined by

\[ T_{\lambda}(A|B) = A^{1/2} \ln_{\lambda}(A^{-1/2}BA^{-1/2})A^{1/2}. \]
Proposition 2 we have the following properties;

1. $\lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2}.

2. $T_\lambda(\alpha A|\alpha B) = \alpha T_\lambda(A|B), \alpha \in \mathbb{R}^+.$

3. If $B \leq C,$ then $T_\lambda(A|B) \leq T_\lambda(A|C).$

4. $T_\lambda(A_1 + A_2|B_1 + B_2) \geq T_\lambda(A_1|B_1) + T_\lambda(A_2|B_2).$

5. $T_\lambda(\alpha A_1 + \beta A_2|\alpha B_1 + \beta B_2) \geq \alpha T_\lambda(A_1|B_1) + \beta T_\lambda(A_2|B_2).$

6. $T_\lambda(UAU^*|UBU^*) = UT_\lambda(A|B)U^*.$

7. $\Phi(T_\lambda(A|B)) \leq T_\lambda(\Phi(A)|\Phi(B)),$ where $U$ is an unital positive linear map.

Remark 1 Same properties are shown for a more general case by Fujii et.al.

Solodarity As $B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$ for operator monotone $f.$

Since
\[
\frac{x^{-\lambda} - 1}{-\lambda} \leq \log x \leq \frac{x^{\lambda} - 1}{\lambda}
\]
for $x > 0, \lambda > 0,$ we have the following.

Proposition 3 For $A > 0, B > 0, \lambda \in (0, 1],$ we have the following;

$T_{-\lambda}(A|B) \leq S(A|B) \leq T_\lambda(A|B).$

Since
\[
1 - \frac{1}{x} \leq \ln_\lambda x \leq x - 1
\]
for $x > 0, 0 < \lambda \leq 1,$ we have the following.

Proposition 4 For $A > 0, B > 0, \lambda \in (0, 1],$ we have the following;

$A - AB^{-1}A \leq T_\lambda(A|B) \leq B - A.$

Since
\[
x^{\lambda}(1 - \frac{1}{\alpha x}) + \ln_\lambda \frac{1}{\alpha} \leq \ln_\lambda x \leq \frac{x}{\alpha} - 1 - x^{\lambda}\ln_\lambda \frac{1}{\alpha}
\]
for $\alpha > 0, x > 0, 0 < \lambda \leq 1,$ we have the following.
Theorem 2 For $A > 0, B > 0, \alpha > 0, \lambda \in (0, 1]$, we have the following;
\[ A_{\lambda}B - \frac{1}{\alpha} A_{\lambda - 1}B + (\ln_{\lambda} \frac{1}{\alpha}) A \leq T_{\lambda}(A|B) \leq \frac{1}{\alpha} B - A - (\ln_{\lambda} \frac{1}{\alpha}) A_{\lambda}B, \]
where $A_{\lambda}B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$.

We have the following by taking $\lambda \to 0$, $\alpha = 1$, respectively;

Corollary 2 For $A > 0, B > 0, \alpha > 0$, we have the following;
\[ (1 - \log \alpha) A - \frac{1}{\alpha} AB^{-1}A \leq S(A|B) \leq (\log \alpha - 1) A + \frac{1}{\alpha} B. \]
For $A > 0, B > 0$, we have the following;
\[ A - AB^{-1}A \leq S(A|B) \leq B - A. \]

Lemma 5 For $X > 0, Y > 0, a \in \mathbb{R}$, we have
\[ (X \otimes Y)^{a} = X^{a} \otimes Y^{a}. \]

Theorem 3 For $A_{1}, A_{2}, B_{1}, B_{2} > 0, \lambda \in (0, 1]$, we have the following;
\[ T_{\lambda}(A_{1} \otimes A_{2}|B_{1} \otimes B_{2}) = T_{\lambda}(A_{1}|B_{1}) \otimes A_{2} + A_{1} \otimes T_{\lambda}(A_{2}|B_{2}) + \lambda T_{\lambda}(A_{1}|B_{1}) \otimes T_{\lambda}(A_{2}|B_{2}). \]

Proof. From Lemma 5, we have for $X > 0, Y > 0, \lambda \in (0, 1]$,
\[ \ln_{\lambda}(X \otimes Y) = (\ln_{\lambda} X) \otimes I + I \otimes (\ln_{\lambda} Y) + \lambda (\ln_{\lambda} X) \otimes (\ln_{\lambda} Y). \]
By putting $X = A_{1}^{-1/2}B_{1}A_{1}^{-1/2}, Y = A_{2}^{-1/2}B_{2}A_{2}^{-1/2}$ and by multiplying $A_{1}^{1/2} \otimes A_{1}^{1/2}$ from both sides. we have the theorem. q.e.d.

Corollary 3 For $A_{1}, A_{2}, B_{1}, B_{2} > 0$, we have
\[ S(A_{1} \otimes A_{2}|B_{1} \otimes B_{2}) = S(A_{1}|B_{1}) \otimes A_{2} + A_{1} \otimes S(A_{2}|B_{2}). \]
Since we put $B_1 = B_2 = I, A_i = \rho_i$, we have the following:

**Corollary 4 (pseudo additivity)** For $\rho_1, \rho_2$, we have

$$S_\lambda(\rho_1 \otimes \rho_2) = S_\lambda(\rho_1) + S_\lambda(\rho_2) + \lambda S_\lambda(\rho_1)S_\lambda(\rho_2).$$

**Corollary 5** From Theorem 3 we have the following inequalities:

1. For $\lambda \in (0, 1]$ and $0 < A_i \leq B_i (i = 1, 2)$, we have
   
   (a) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq \lambda T_\lambda(A_1 | B_1) \otimes T_\lambda(A_2 | B_2)$.
   
   (b) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1 | B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2 | B_2)$.

2. For $\lambda \in (0, 1]$ and $0 < B_i \leq A_i (i = 1, 2)$, we have
   
   (c) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \leq \lambda T_\lambda(A_1 | B_1) \otimes T_\lambda(A_2 | B_2)$.
   
   (d) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1 | B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2 | B_2)$.

### 3 Trace inequalities of Tsallis relative entropy

**Definition 3** (Tsallis relative entropy) For density operators $\rho, \sigma$, Tsallis relative entropy is defined by

$$D_\lambda(\rho|\sigma) = \frac{\text{Tr}[\rho - \rho^{1-\lambda}\sigma^\lambda]}{\lambda}, \quad \lambda \in (0, 1].$$

**Theorem 4** $D_\lambda(\rho|\sigma) \leq -\text{Tr}[T_\lambda(\rho|\sigma)]$.

**Proof.** We remark that

$$A^\#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$$

is $\alpha$ power mean. By Theorem 3.4 in Hiai-Petz [3], we have

$$\text{Tr}[e^{A^\#_\alpha B}] \leq \text{Tr}[e^{(1-\alpha)A + \alpha B}].$$

for any $\alpha \in [0, 1]$. We put $A = \log \rho, B = \log \sigma$.

$$\text{Tr}[\rho^\#_\alpha \sigma] \leq \text{Tr}[e^{(1-\alpha)\log \rho + \alpha \log \sigma}].$$

We apply Golden-Thompson inequality

$$\text{Tr}[e^{A+B}] \leq \text{Tr}[e^A e^B]$$
for any Hermitian operators $A,B$. Then we have

$$Tr[e^{\log \rho^{1-\alpha} + \log \sigma^{\alpha}}] \leq Tr[e^{\log \rho^{1-\alpha}} e^{\log \sigma^{\alpha}}] = Tr[\rho^{1-\alpha} \sigma^{\alpha}].$$

Thus we have

$$Tr[\rho^{1/2}(\rho^{-1/2} \sigma^{-1/2})^{\alpha} \rho^{1/2}] \leq Tr[\rho^{1-\alpha} \sigma^{\alpha}].$$

q.e.d.

Corollary 6 (Hiai-Petz) $Tr[\rho(\log \rho - \log \sigma)] \leq -Tr[\rho \log(\rho^{-1/2} \sigma^{-1/2})].$

Definition 4 (Tsallis relative entropy) *For positive operators $A, B$ and $0 < \lambda \leq 1$, we define*

$$D_{\lambda}(A||B) = \frac{Tr[A - A^{1-\lambda} B^{\lambda}]}{\lambda}.$$

Theorem 5 (Generalized Bogoliubov inequality) *For positive operators $A, B$ and $0 < \lambda \leq 1$, we have the following;*

$$D_{\lambda}(A||B) \geq \frac{Tr[A] - (Tr[A])^{1-\lambda}(Tr[B])^{\lambda}}{\lambda}.$$

Proof. It follows by the application of the Holder's inequality:

$$|Tr[XY]| \leq Tr[|X|^s]^{1/s} Tr[|Y|^t]^{1/t}$$

for $1 < s, t < \infty, \ 1/s + 1/t = 1.$

q.e.d.

Corollary 7 (Peierls-Bogoliubov inequality) *For positive operators $A, B$, we have the following;*

$$Tr[A(\log A - \log B)] \geq Tr[A] (\log Tr[A] - \log Tr[B]).$$
References


