<table>
<thead>
<tr>
<th>Title</th>
<th>Operator inequalities and trace inequalities derived from Tsallis entropies (Recent Developments in Linear Operator Theory and its Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Furuichi, Shigeru; Yanagi, Kenjiro; Kuriyama, Ken</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1458: 97-105</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47897">http://hdl.handle.net/2433/47897</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Tsallis entropy に関する作用素不等式とトレース不等式

Operator inequalities and trace inequalities derived from Tsallis entropies

古市 茂 (Shigeru Furuichi)
山口東京理科大
(Tokyo University of Science in Yamaguchi)
柳 研二郎 (Kenjiro Yanagi)
山口大・工
(Department of Applied Science, Yamaguchi University)
栗山 慎 (Ken Kuriyama)
山口大・工
(Department of Applied Science, Yamaguchi University)

1 Trace inequalities of Tsallis entropy

We define $q$-logarithm function as follows;
\[
\ln_q x = \frac{x^{1-q} - 1}{1-q}, \quad (x \geq 0, q \geq 0, q \neq 1).
\]

Then we have the following properties;

(1) $\lim_{q \to 1} \ln_q x = \log x$. (in uniformly)

(2) $\ln_q xy = \ln_q x + \ln_q y + (1-q) \ln_q x \ln_q y$.

(3) $\ln_q x$: concave in $x$ for $q \geq 0$.

Definition 1 (Tsallis entropy) For density operator $\rho$ on a finite dimensional Hilbert space $\mathcal{H}$, Tsallis entropy $S_q(\rho)$ is defined by
\[
S_q(\rho) = \frac{\text{Tr}[\rho^q - \rho]}{1-q}, \quad (q \geq 0, q \neq 1).
\]
Proposition 1 We have the following properties;

(1) \( \lim_{\Gamma J} \underline{\setminus} J S'_{1}(p) = -Tr[p \log p] \).

(2) \( S_{q}(\rho_{1} \otimes \rho_{2}) = S_{q}(\rho_{1}) + S_{q}(\rho_{2}) + (1 - q)S_{q}(\rho_{1})S_{q}(\rho_{2}) \).

Lemma 1 \( S_{q}(\rho) \leq \ln_{q} d, \quad (d = \dim \mathcal{H}) \).

Proof. Since \( \ln_{q} x \) is concave, we have

\[
D_{q}(A|B) = -\sum_{j=1}^{d}a_{j}\ln_{q}\frac{b_{j}}{a_{j}} \geq -\ln_{q}(\sum_{j=1}^{d}a_{j}\frac{b_{j}}{a_{j}}) = 0.
\]

We put \( A = \{a_{j}\}, \; B = \{u_{j}\}, \; u_{j} = \frac{1}{d} (1 \leq j \leq d) \). Then

\[
D_{q}(A|B) = -d^{q-1}(S_{q}(A) - \ln_{q} d) \geq 0.
\]

Thus \( S_{q}(A) \leq \ln_{q} d \). q.e.d.

Lemma 2 If \( f \) is a concave function satisfying \( f(0) = f(1) = 0 \), then

\[
|f(t+s) - f(t)| \leq \max\{f(s), f(1 - s)\},
\]

where \( s \in [0, 1/2], \; t \in [0, 1], \; 0 \leq s + t \leq 1 \).

Proof. We put

\[
r(t) = f(s) - f(t+s) + f(t).
\]

Then

\[
r'(t) = -f'(t+s) + f'(t).
\]

Since \( f' \) is a monotone decreasing function, \( r'(t) \geq 0 \). Thus we have \( r(t) \geq 0 \)

by \( r(0) = 0 \). Therefore \( f(t+s) - f(t) \leq f(s) \). We also put

\[
\ell(t) = f(t+s) - f(t) + f(1 - s).
\]

Then

\[
\ell'(t) = f'(t+s) - f'(t).
\]

Since \( f' \) is a monotone decreasing function, \( \ell'(t) \leq 0 \). Thus we have \( \ell(t) \geq 0 \)

by \( \ell(1 - s) = 0 \). Therefore \( -f(1 - s) \leq f(t+s) - f(t) \). Thus we have the result.

q.e.d.
Lemma 3 If $|u - v| \leq 1/2$, then $|\eta_q(u) - \eta_q(v)| \leq \eta_q(|u - v|)$, where $\eta_q(x) = \frac{x^q - x}{1 - q}$, $q \in [0, 2]$, $u, v \in [0, 1]$.

Proof. Since $\eta_q$ is a concave function with $\eta_q(0) = \eta_q(0)$, we have

$$|\eta_q(t + s) - \eta_q(t)| \leq \max\{\eta_q(s), \eta_q(1 - s)\}$$

for $s \in [0, \frac{1}{2}]$ and $t \in [0, 1]$ satisfying $0 \leq t + s \leq 1$ by Lemma 2. Since $\eta_q(x)$ is a monotone increasing function on $[0, q^{1/(1-q)}]$ and $q^{1/(1-q)} \leq \frac{1}{2}$ for $q \in (0, 2)$,

$$\max\{\eta_q(s), \eta_q(1 - s)\} = \eta_q(s).$$

Thus we have the result by letting $u = t + s$ and $v = t$. q.e.d.

Lemma 4 Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of Hermitian matrix $A$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be eigenvalues of Hermitian matrix $B$. Then we have the following:

$$\text{Tr}[\|A - B\|] \geq \sum_{i=1}^{n} |\lambda_i - \mu_i|.$$

Theorem 1 (Generalized Fannes's inequality) For two density operators $\rho_1, \rho_2$ on $\mathcal{H}$ and $q \in [0, 2]$, if $\|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)}$, then

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \|\rho_1 - \rho_2\|^2_q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1),$$

where $d = \dim \mathcal{H}$ and $\|A\|_1 = \text{Tr}[|A|]$.

Proof. Let $\lambda_1^{(i)} \geq \cdots \geq \lambda_n^{(i)}$ be eigenvalues of $\rho_i$.

We set

$$\epsilon = \sum_{j=1}^{d} \epsilon_j, \quad \epsilon_j = |\lambda_j^{(1)} - \lambda_j^{(2)}|.$$ 

From Lemma 2,

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^{d} |\eta_q(\lambda_j^{(1)}) - \eta_q(\lambda_j^{(2)})| \leq \sum_{j=1}^{d} \eta_q(\epsilon_j).$$

By $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$ and Lemma 1, we have
\[
\sum_{j=1}^{d} \eta_\eta(\epsilon_j) = - \sum_{j=1}^{d} e_j^q \ln_q e_j = \epsilon\{- \sum_{j=1}^{d} \frac{e_j^q}{\epsilon} \ln_q (\frac{e_j}{\epsilon})\}
\]
\[
= \epsilon\{- \sum_{j=1}^{d} \frac{e_j^q}{\epsilon} \ln_q e_j - \sum_{j=1}^{d} \frac{e_j^q}{\epsilon} (\frac{e_j}{\epsilon})^{1-q} \ln_q e_j\}
\]
\[
= \epsilon_q \sum_{j=1}^{d} \eta_q(\frac{e_j}{\epsilon}) + \eta_q(\epsilon) \leq \epsilon^q \ln_q d + \eta_q(\epsilon).
\]

Therefore we have

\[
|S_q(\rho_1) - S_q(\rho_2)| \leq \epsilon^q \ln_q d + \eta_q(\epsilon).
\]

From Lemma 3, we have \(\|\rho_1 - \rho_2\|_1 \geq \epsilon\). And \(\eta_\eta(x)\) is monotone increase on \(x \in [0, q^{1/(1-q)}]\). In addition, \(x^\eta\) is monotone increase on \(x \in [0, 2]\). Thus we have theorem.

q.e.d.

Since \(\lim_{q \to 1} q^{1/(1-q)} = 1/e\), we have

**Corollary 1 (Fannes's inequality)** For two density operators \(\rho_1, \rho_2\) on \(\mathcal{H}\), if \(\|\rho_1 - \rho_2\|_1 \leq 1/e\), then

\[
|S_1(\rho_1) - S_1(\rho_2)| \leq \|\rho_1 - \rho_2\|_1 \log d + \eta_1(\|\rho_1 - \rho_2\|_1),
\]

where \(S_1(\rho) = -\text{Tr}[\rho \log \rho], \ \eta_1(x) = -x \log x\).

## 2 Operator inequalities of Tsallis relative operator entropy

We change the notation \((\lambda = 1 - q)\). That is, for \(\lambda \in (0, 1]\).

\[
\ln_\lambda x = \frac{x^\lambda - 1}{\lambda}.
\]

**Definition 2 (Tsallis relative operator entropy)** For \(A > 0, B > 0, \lambda \in (0, 1]\), Tsallis relative operator entropy \(T_\lambda(A|B)\) is defined by

\[
T_\lambda(A|B) = A^{1/2} \ln_\lambda (A^{-1/2}BA^{-1/2})A^{1/2}.
\]
Proposition 2 we have the following properties:

1. \( \lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2} \).

2. \( T_\lambda(\alpha A|\alpha B) = \alpha T_\lambda(A|B), \alpha \in \mathbb{R}^+ \).

3. If \( B \leq C \) then \( T_\lambda(A|B) \leq T_\lambda(A|C) \).

4. \( T_\lambda(A_1 + A_2|B_1 + B_2) \geq T_\lambda(A_1|B_1) + T_\lambda(A_2|B_2) \).

5. \( T_\lambda(\alpha A_1 + \beta A_2|\alpha B_1 + \beta B_2) \geq \alpha T_\lambda(A_1|B_1) + \beta T_\lambda(A_2|B_2) \).

6. \( T_\lambda(UAU^*|UBU^*) = UT_\lambda(A|B)U^* \).

7. \( \Phi(T_\lambda(A|B)) \leq T_\lambda(\Phi(A)|\Phi(B)) \), where \( U \) is an unital positive linear map.

Remark 1 Same properties are shown for a more general case by Fujii et al.

Solderarity As \( B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} \) for operator monotone \( f \).

Since
\[
\frac{x^{-\lambda} - 1}{-\lambda} \leq \log x \leq \frac{x^\lambda - 1}{\lambda}
\]
for \( x > 0, \lambda > 0 \), we have the following.

Proposition 3 For \( A > 0, B > 0, \lambda \in (0, 1] \), we have the following;
\[
T_{-\lambda}(A|B) \leq S(A|B) \leq T_\lambda(A|B).
\]

Since
\[
1 - \frac{1}{x} \leq \ln_\lambda x \leq x - 1
\]
for \( x > 0, 0 < \lambda \leq 1 \), we have the following.

Proposition 4 For \( A > 0, B > 0, \lambda \in (0, 1] \), we have the following;
\[
A - AB^{-1}A \leq T_\lambda(A|B) \leq B - A.
\]

Since
\[
x^\lambda(1 - \frac{1}{\alpha x}) + \ln_\lambda \frac{1}{\alpha} \leq \ln_\lambda x \leq \frac{x}{\alpha} - 1 - x^\lambda \ln_\lambda \frac{1}{\alpha}
\]
for \( \alpha > 0, x > 0, 0 < \lambda \leq 1 \), we have the following.
Theorem 2 For $A > 0, B > 0, \alpha > 0, \lambda \in (0, 1]$, we have the following:

$$A \#_{\lambda} B - \frac{1}{\alpha} A \#_{\lambda - 1} B + (\ln_{\lambda} \frac{1}{\alpha}) A \leq T_{\lambda}(A|B) \leq \frac{1}{\alpha} B - A - (\ln_{\lambda} \frac{1}{\alpha}) A \#_{\lambda} B,$$

where $A \#_{\lambda} B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$.

We have the following by taking $\lambda \to 0$, $\alpha = 1$, respectively;

Corollary 2 For $A > 0, B > 0, \alpha > 0$, we have the following;

$$(1 - \log \alpha) A - \frac{1}{\alpha} AB^{-1}A \leq S(A|B) \leq (\log \alpha - 1) A + \frac{1}{\alpha} B.$$

For $A > 0, B > 0$, we have the following;

$$A - AB^{-1}A \leq S(A|B) \leq B - A.$$

Lemma 5 For $X > 0, Y > 0, a \in \mathbb{R}$, we have

$$(X \otimes Y)^{a} = X^{a} \otimes Y^{a}.$$

Theorem 3 For $A_1, A_2, B_1, B_2 > 0, \lambda \in (0, 1]$, we have the following;

$$T_{\lambda}(A_1 \otimes A_2 | B_1 \otimes B_2) = T_{\lambda}(A_1|B_1) \otimes A_2 + A_1 \otimes T_{\lambda}(A_2|B_2) + \lambda T_{\lambda}(A_1|B_1) \otimes T_{\lambda}(A_2|B_2).$$

Proof. From Lemma 5, we have for $X > 0, Y > 0, \lambda \in (0, 1]$,

$$\ln_{\lambda}(X \otimes Y) = (\ln_{\lambda} X) \otimes I + I \otimes (\ln_{\lambda} Y) + \lambda (\ln_{\lambda} X) \otimes (\ln_{\lambda} Y).$$

By putting $X = A_1^{-1/2}B_1A_1^{-1/2}, Y = A_2^{-1/2}B_2A_2^{-1/2}$ and by multiplying $A_1^{1/2} \otimes A_2^{1/2}$ from both sides, we have the theorem. q.e.d.

Corollary 3 For $A_1, A_2, B_1, B_2 > 0$, we have

$$S(A_1 \otimes A_2 | B_1 \otimes B_2) = S(A_1|B_1) \otimes A_2 + A_1 \otimes S(A_2|B_2).$$
Since we put $B_1 = B_2 = I, A_i = \rho_i$, we have the following;

**Corollary 4 (pseudo additivity)** For $\rho_1, \rho_2$, we have

$$S_\lambda(\rho_1 \otimes \rho_2) = S_\lambda(\rho_1) + S_\lambda(\rho_2) + \lambda S_\lambda(\rho_1) S_\lambda(\rho_2).$$

**Corollary 5** From Theorem 3 we have the following inequalities;

(1) For $\lambda \in (0, 1]$ and $0 < A_i \leq B_i (i = 1, 2)$, we have

(a) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq \lambda T_\lambda(A_1 | B_1) \otimes T_\lambda(A_2 | B_2)$.

(b) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1 | B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2 | B_2)$.

(2) For $\lambda \in (0, 1]$ and $0 < B_i \leq A_i (i = 1, 2)$, we have

(c) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \leq \lambda T_\lambda(A_1 | B_1) \otimes T_\lambda(A_2 | B_2)$.

(d) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1 | B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2 | B_2)$.

## 3 Trace inequalities of Tsallis relative entropy

**Definition 3 (Tsallis relative entropy)** For density operators $\rho, \sigma$, Tsallis relative entropy is defined by

$$D_\lambda(\rho | \sigma) = \frac{Tr[\rho - \rho^{1-\lambda} \sigma^\lambda]}{\lambda}, \ \lambda \in (0, 1].$$

**Theorem 4** $D_\lambda(\rho | \sigma) \leq -Tr[T_\lambda(\rho | \sigma)]$.

**Proof.** We remark that

$$A_\alpha \cdot B = A^{1/2} (A^{-1/2} B A^{-1/2})^\alpha A^{1/2}$$

is $\alpha$ power mean. By Theorem 3.4 in Hiai-Petz [3], we have

$$Tr[e^{A_\alpha \cdot e^B}] \leq Tr[e^{(1-\alpha)A + \alpha B}].$$

for any $\alpha \in [0, 1]$. We put $A = \log \rho, B = \log \sigma$.

$$Tr[\rho_\alpha \cdot \sigma] \leq Tr[\log \rho^{1-\alpha} + \log \sigma^\alpha].$$

We apply Golden-Thompson inequality

$$Tr[e^{A+H}] \leq Tr[e^A e^H]$$
for any Hermitian operators $A, B$. Then we have

$$
Tr[e^{\log \rho^{1-\alpha} + \log \sigma^\alpha}] \leq Tr[e^{\log \rho^{1-\alpha}} e^{\log \sigma^\alpha}] = Tr[\rho^{1-\alpha} \sigma^\alpha].
$$

Thus we have

$$
Tr[\rho^{1/2} (\rho^{-1/2} \sigma \rho^{-1/2})^\alpha \rho^{1/2}] \leq Tr[\rho^{1-\alpha} \sigma^\alpha].
$$

q.e.d.

Corollary 6 (Hiai-Petz) $Tr[\rho (\log \rho - \log \sigma)] \leq -Tr[\rho \log (\rho^{-1/2} \sigma \rho^{-1/2})]$. 

Definition 4 (Tsallis relative entropy) For positive operators $A, B$ and $0 < \lambda \leq 1$, we define

$$
D_\lambda(A\|B) = \frac{Tr[|A - A^{1-\lambda} B^\lambda|]}{\lambda}.
$$

Theorem 5 (Generalized Bogoliubov inequality) For positive operators $A, B$ and $0 < \lambda \leq 1$, we have the following;

$$
D_\lambda(A\|B) \geq \frac{Tr[A] - (Tr[A])^{1-\lambda} (Tr[B])^\lambda}{\lambda}.
$$

Proof. It follows by the application of the Holder's inequality:

$$
|Tr[XY]| \leq Tr[|X|^{1/s} Y^{1/t}]^{1/t}
$$

for $1 < s, t < \infty$, $1/s + 1/t = 1$. q.e.d.

Corollary 7 (Peierls-Bogoliubov inequality) For positive operators $A, B$, we have the following;

$$
Tr[A(\log A - \log B)] \geq Tr[A(\log Tr[A] - \log Tr[B])].
$$
References


