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Classes of non-normal operators defined by inequalities for operator means

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1 Class A-f and A-f-paranormality

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. Following [12], *class A* is a class of non-normal operators T such that

$$|T^2| \geq |T|^2.$$

It is also shown in [12] that class A includes *p-hyponormal* ($(T^*T)^p \geq (TT^*)^p$ for $p > 0$) and *log-hyponormal* (T is invertible and $\log T^*T \geq \log TT^*$) operators, and is included in the classes of *paranormal* ($\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$) and *normaloid* ($\|T\| = r(T)$ (the spectral radius)) operators. It is shown in [24] that T belongs to class A if and only if

$$\left(|T^*||T|^2|T^*|\right)^{\frac{1}{2}} \geq |T^*|^2,$$

and in [2] that T is paranormal if and only if $T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$ for all $\lambda > 0$, or equivalently,

$$\frac{1}{2}\left(I + \lambda^2|T^*||T|^2|T^*|\right) \geq \lambda|T^*|^2 \text{ for all } \lambda > 0.$$

From these points of view, we introduced generalizations of class A and paranormality in [29].

Definition 1.A ([29]). Let f be a non-negative continuous function on $[0, \infty)$.

(i) $T \in$ *class A-f* $\Longleftrightarrow f(|T^*||T|^2|T^*|) \geq |T^*|^2$.

(ii) T is *A-f-paranormal* $\Longleftrightarrow \lambda T \in$ class A-f for all $\lambda > 0$.

When f is a representing function of an operator connection σ (see [19]), we also call class A-f and A-f-paranormal *class A-σ* and *A-σ-paranormal*, respectively.

In fact, class A and paranormality coincide with class A-$\sharp$ and A-∇-paranormality, respectively, where ∇ and $\sharp$ are the arithmetic and geometric means, that is,

$$A \nabla B = \frac{1}{2}(A+B) \quad \text{and} \quad A \sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Hence we can explain the inclusion relation between class A and the class of paranormal operators shown in [12] in terms of class A-f and A-f-paranormality as follows:

$$\begin{aligned} T \in \text{class A} &\Longleftrightarrow T \in \text{class A-}\sharp && \text{by Definition 1.A} \\ &\Longleftrightarrow T \text{ is A-}\sharp\text{-paranormal} && \text{since } f_\sharp(\lambda^2 t) = (\lambda^2 t)^{\frac{1}{2}} = \lambda t^{\frac{1}{2}} = \lambda f_\sharp(t) \\ &\Longrightarrow T \text{ is A-}\nabla\text{-paranormal} && \text{since } f_\sharp(t) = t^{\frac{1}{2}} \leq \tfrac{1}{2}(1+t) = f_\nabla(t) \\ &\Longleftrightarrow T \text{ is paranormal} && \text{by Definition 1.A.} \end{aligned}$$

Furthermore, in [29], we introduced parametrized generalizations of class A-f and A-f-paranormality.

Definition 1.B ([29]). Let f be a non-negative continuous function on $[0,\infty)$, and $s,t>0$.

(i) $T \in$ *class* $A(s,t)$-$f \Longleftrightarrow f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$.

(ii) T is $A(s,t)$-f-*paranormal* $\Longleftrightarrow \lambda T \in$ class A(s,t)-f for all $\lambda > 0$.

When f is a representing function of an operator connection σ (see [19]), we also call class A(s,t)-f and A(s,t)-f-paranormal *class* $A(s,t)$-σ and $A(s,t)$-σ-*paranormal*, respectively.

We remark that class A(s,t)-$\sharp_{\frac{t}{s+t}}$ and A(s,t)-$\nabla_{\frac{t}{s+t}}$-paranormality, introduced in [8] and [26], coincide with *class* $A(s,t)$ $((|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t})$ and *absolute-*(s,t)-*paranormality* $(\frac{s}{s+t}I + \frac{t}{s+t}\lambda^{s+t}|T^*|^t|T|^{2s}|T^*|^t \geq \lambda^t|T^*|^t$ for all $\lambda > 0)$, respectively, where

$$A \nabla_\alpha B = (1-\alpha)A + \alpha B \quad \text{and} \quad A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } \alpha \in [0,1].$$

Particularly, it is pointed out in [17] that class A$(\frac{1}{2},\frac{1}{2})$ coincides with the class of *w-hyponormal* $(|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$, where $\tilde{T}$ is the Aluthge transformation of T) operators introduced in [1].

In [29], we showed several properties of these classes introduced above, which are generalizations of the results on class A(s,t) and absolute-(s,t)-paranormal operators shown in [8][15][17][20][24][25][26][28].

Theorem 1.A ([29]). *Let $s_0, t_0 > 0$ and $\{f_{s,t} \mid s \geq s_0,\ t \geq t_0\}$ be a family of non-negative operator monotone functions on $[0,\infty)$ satisfying $f_{s,t}(x^t g(x)^s) = x^t$, where g is a continuous function. If T is invertible and $T \in$ class $A(s_0,t_0)$-f_{s_0,t_0}, then $T \in$ class $A(s,t)$-$f_{s,t}$ for all $s > s_0$ and $t > t_0$.*

Theorem 1.B ([29]). *Let f be a non-negative, continuously differentiable and convex (or concave) function on $[0,\infty)$ satisfying $f(1) \leq 1$ and $0 < f'(1) < 1$, and $p_0 > 0$. If T is invertible and $T \in$ class $A(\theta' p, \theta p)$-f for all $p \in (0, p_0)$, then T is log-hyponormal, where $\theta = f'(1)$ and $\theta + \theta' = 1$.*

Theorem 1.C ([29]). *Let f be a non-negative operator monotone function on $[0,\infty)$, and $s,t \in (0,1]$. If $T \in$ class $A(s,t)$-f and $T \in$ class A, then $T^n \in$ class $A(\frac{s}{n}, \frac{t}{n})$-$f$ for every positive integer n.*

Proposition 1.D ([29]). *Let f be a non-negative operator monotone function on $[0,\infty)$, and $s,t \in (0,1]$. If $T \in$ class $A(s,t)$-f, then $T|_{\mathcal{M}} \in$ class $A(s,t)$-f, where $T|_{\mathcal{M}}$ is the restriction of T onto an invariant subspace $\mathcal{M}$.*

Theorem 1.E ([29]). *Let f and g be non-negative continuous increasing functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ and $g(0) = 0$, and $s,t > 0$. If $T \in$ class $A(s,t)$-f, then the following hold, where $T = U|T|$ is the polar decomposition and $\tilde{T}_{s,t} = |T|^s U |T|^t$:*

(i) *$\tilde{T}_{s,t}$ is f-hyponormal if $f \circ g^{-1}$ is operator monotone and $x^t \geq (f \circ g^{-1})(x^s)$.*

(ii) *$\tilde{T}_{s,t}$ is g-hyponormal if $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^t) \geq x^s$.*

2 Furuta inequality and its generalizations

The following result is essential for the study of class A(s,t) operators.

Theorem F (Furuta inequality [9]).

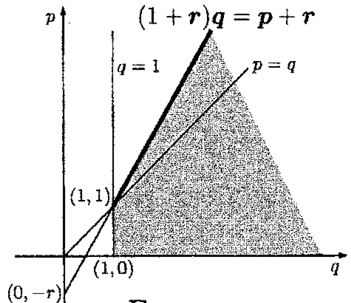
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

We remark that Theorem F yields Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" when we put $r = 0$ in (i) or (ii) stated above. Other proofs are given in [5][18] and also an elementary one-page proof in [10]. It is shown in [21] that the domain of p, q and r is the best possible in Theorem F.

The chaotic order defined by $\log A \geq \log B$ for $A, B > 0$ is weaker than the usual order since $\log t$ is operator monotone. The following extension of a result in [3] can be obtained as an application of Theorem F. Other proofs are given in [7][22], and the best possibility is shown in [27].

Theorem C ([6][11]). *Let $A, B > 0$. The following are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p \geq 0$ *and* $r \geq 0$.

A lot of related studies to Theorem F and Theorem C have been done. Among others, we here introduce the following result.

Theorem 2.A ([13] et al.). *Let $A, B > 0$ and $\alpha_0, \beta_0 > 0$. If*

$$(B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0} \quad or \quad A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}, \tag{2.1}$$

then for each real number δ,

$$B^{\frac{-\beta}{2}} (B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} B^{\frac{-\beta}{2}} \quad and \quad A^{\frac{-\alpha}{2}} (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{-\delta+\alpha}{\alpha+\beta}} A^{\frac{-\alpha}{2}} \tag{2.2}$$

is increasing and decreasing, respectively, for $\alpha \geq \max\{\alpha_0, \delta\}$ and $\beta \geq \max\{\beta_0, -\delta\}$.

The "order-like" relations between $A, B \geq 0$ defined by the inequalities in (2.1) for some fixed $\alpha_0, \beta_0 > 0$ are weaker than the usual and chaotic orders by Theorem F and Theorem C. For $A, B > 0$, the inequalities in (2.1) are mutually equivalent and each function in (2.2) is the inverse of the other since

$$S^{\frac{1}{2}} (S^{\frac{-1}{2}} T S^{\frac{-1}{2}})^{\alpha} S^{\frac{1}{2}} = S \,\sharp_{\alpha}\, T = T \,\sharp_{1-\alpha}\, S = T^{\frac{1}{2}} (T^{\frac{-1}{2}} S T^{\frac{-1}{2}})^{1-\alpha} T^{\frac{1}{2}}$$

for $S, T > 0$ and $\alpha \in [0, 1]$. Hence Theorem 2.A can be summarized as follows: for each $p, a > 0$ and $\delta \in [-a, p]$,

$$\begin{aligned} (B^{\frac{a}{2}} A B^{\frac{a}{2}})^{\frac{a}{p+a}} \geq B^a &\Longrightarrow B^{\frac{-r}{2}} (B^{\frac{r}{2}} A B^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} B^{\frac{-r}{2}} \text{ is increasing for } r \geq a, \\ A^a \geq (A^{\frac{a}{2}} B A^{\frac{a}{2}})^{\frac{a}{p+a}} &\Longrightarrow A^{\frac{-r}{2}} (A^{\frac{r}{2}} B A^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} A^{\frac{-r}{2}} \text{ is decreasing for } r \geq a, \end{aligned} \tag{2.3}$$

and it turns out by scrutinizing the proof of Theorem 2.A that (2.3) is still valid even if the hypotheses are weakened to

$$\log (B^{\frac{a}{2}} A B^{\frac{a}{2}})^{\frac{a}{p+a}} \geq \log B^a \quad \text{and} \quad \log A^a \geq \log (A^{\frac{a}{2}} B A^{\frac{a}{2}})^{\frac{a}{p+a}}.$$

The following generalizations of Theorem F, Theorem C and Theorem 2.A are shown in the recent paper [23] by M. Uchiyama. In fact, Theorem 2.B yields Theorem F and Theorem C by putting $\psi_r(x) = x^{\frac{r}{p+r}}$, $\phi_r(x) = x^{\frac{1+r}{p+r}}$, $g(x) = x^p$ and $h(x) = x$. Theorem 2.B also yields (2.3) by putting $\psi_r(x) = x^{\frac{r}{p+r}}$, $\phi_r(x) = x^{\frac{\delta+r}{p+r}}$, $g(x) = x^p$ and $h(x) = x^{\delta}$.

Theorem 2.B ([23]). *Let* $\{\psi_r \mid r > 0\}$ *and* $\{\phi_r \mid r > 0\}$ *be families of non-negative operator monotone functions satisfying*

$$\psi_r(x^r g(x)) = x^r \quad and \quad \phi_r(x^r g(x)) = x^r h(x),$$

where g *and* h *are non-negative continuous functions. If* $A \geq B \geq 0$ *or if* $A, B > 0$ *and* $\log A \geq \log B$, *then for* $r > 0$,

$$\psi_r(B^{\frac{r}{2}} g(A) B^{\frac{r}{2}}) \geq B^r, \qquad A^r \geq \psi_r(A^{\frac{r}{2}} g(B) A^{\frac{r}{2}}),$$
$$\phi_r(B^{\frac{r}{2}} g(A) B^{\frac{r}{2}}) \geq B^{\frac{r}{2}} h(A) B^{\frac{r}{2}}, \qquad A^{\frac{r}{2}} h(B) A^{\frac{r}{2}} \geq \phi_r(A^{\frac{r}{2}} g(B) A^{\frac{r}{2}}).$$

Theorem 2.C ([23]). *Let* $A, B \geq 0$ *and* $a > 0$, *and let* $\{\psi_r \mid r \geq a\}$ *and* $\{\phi_r \mid r \geq a\}$ *be families of non-negative operator monotone functions satisfying*

$$\psi_r(x^r g(x)) = x^r \quad and \quad \phi_r(x^r g(x)) = x^r h(x),$$

where g *and* h *are non-negative continuous functions. Then the following hold:*

(i) *If* $A^a \, \sigma_{\psi_a} \, B \geq I$, *then* $A^r \, \sigma_{\phi_r} \, B$ *is increasing for* $r \geq a$.

(ii) *If* $A, B > 0$ *and* $A^a \, \sigma_{\psi_a} \, B \leq I$, *then* $A^r \, \sigma_{\phi_r} \, B$ *is decreasing for* $r \geq a$.

Here σ_f *denotes the operator mean whose representing function is* f.

Theorem 2.B and Theorem 2.C play important roles for the study of class $\mathrm{A}(s,t)$-f and $\mathrm{A}(s,t)$-f-paranormal operators. Particularly, the proof of Theorem 1.A is based on Theorem 2.C. In this report, we shall give modifications of Theorem 2.C and Theorem 1.A.

3 Results

The following is a modification of Theorem 2.C.

Theorem 3.1. *Let* $A, B \geq 0$ *and* $a > 0$, *and let* $\{\psi_r \mid r \geq a\}$ *and* $\{\phi_r \mid r \geq a\}$ *be families of non-negative operator monotone functions satisfying*

$$\psi_r(x^r g(x)) = x^r \quad and \quad \phi_r(x^r g(x)) = x^r h(x), \tag{3.1}$$

where g *and* h *are non-negative continuous functions. Then the following hold for* $a \leq s \leq t$:

(i) *If* $\psi_a(B^{\frac{a}{2}} A B^{\frac{a}{2}}) \geq B^a$, *or if* $A, B > 0$ *and* $\log \psi_a(B^{\frac{a}{2}} A B^{\frac{a}{2}}) \geq \log B^a$, *then*

$$B^{\frac{t-s}{2}} \phi_s(B^{\frac{s}{2}} A B^{\frac{s}{2}}) B^{\frac{t-s}{2}} \leq \phi_t(B^{\frac{t}{2}} A B^{\frac{t}{2}}).$$

(ii) *If* $A^a \geq \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$ *and* $\overline{R(A)} \cap N(B) = \{0\}$, *or if* $A, B > 0$ *and* $\log A^a \geq \log \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$, *then*

$$A^{\frac{t-s}{2}} \phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}) A^{\frac{t-s}{2}} \geq P\phi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}})P,$$

where P *is the projection onto* $N(A)^{\perp}$.

The following is a modification of Theorem 1.A.

Theorem 3.2. *Let* $s_0, t_0 > 0$ *and* $\{f_{s,t} \mid s \geq s_0, t \geq t_0\}$ *be a family of non-negative operator monotone functions on* $[0, \infty)$ *satisfying* $f_{s,t}(x^t g(x)^s) = x^t$, *where* g *is a continuous function. If* $T \in$ *class* $A(s_0, t_0)$-f_{s_0,t_0}, *then* $T \in$ *class* $A(s,t)$-$f_{s,t}$ *for all* $s > s_0$ *and* $t > t_0$.

4 Proofs

We use the following well-known results in order to give a proof of Theorem 3.1.

Theorem 4.A ([14]). *Let* X *and* A *be bounded linear operators on a Hilbert space* H. *We suppose that* $X \geq 0$ *and* $\|A\| \leq 1$. *If* f *is an operator convex function defined on* $[0, \infty)$ *such that* $f(0) \leq 0$, *then*

$$A^* f(X) A \geq f(A^* X A).$$

Theorem 4.B ([4]). *Let* A *and* B *be bounded linear operators on a Hilbert space* H. *The following statements are equivalent;*

(1) $R(A) \subseteq R(B)$;

(2) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$; *and*

(3) *there exists a bounded linear operator* C *on* H *so that* $A = BC$.

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator C *so that*

(a) $\|C\|^2 = \inf\{\mu \mid AA^* \leq \mu BB^*\}$;

(b) $N(A) = N(C)$; *and*

(c) $R(C) \subseteq \overline{R(B^*)}$.

We consider when the operator C, determined uniquely in Theorem 4.B, satisfies the equality of (c).

Lemma 4.1. *Let* A *and* B *be operators which satisfy (1), (2) and (3) of Theorem 4.B, and* C *be the operator which is given in (3) and determined uniquely by (a), (b) and (c) of Theorem 4.B. Then* $\overline{R(C)} = \overline{R(B^*)}$ *if and only if* $N(A^*) = N(B^*)$.

Proof. $N(C^*) \supseteq N(B)$ by (c) of Theorem 4.B, so that $N(C^*) = N(B) \oplus (N(C^*) \cap \overline{R(B^*)})$. Hence $\overline{R(C)} = \overline{R(B^*)}$ is equivalent to $N(C^*) \cap R(B^*) = \{0\}$, which is equivalent to $N(A^*) \subseteq N(B^*)$ since $N(C^*) \cap R(B^*) = \{B^*x \mid x \in N(A^*)\}$ by (3) of Theorem 4.B. $N(A^*) \supseteq N(B^*)$ follows from (2) of Theorem 4.B, hence the proof of complete. □

Proof of Theorem 3.1. (i-1) In case $\psi_a(B^{\frac{a}{2}}AB^{\frac{a}{2}}) \geq B^a$, it suffices to show that

$$\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}) \geq B^s \Longrightarrow B^{\frac{t-s}{2}}\phi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})B^{\frac{t-s}{2}} \leq \phi_t(B^{\frac{t}{2}}AB^{\frac{t}{2}}) \tag{4.1}$$

holds for $a \leq s \leq t \leq 2s$ since we obtain

$$\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}) \geq B^s \Longrightarrow \psi_t(B^{\frac{t}{2}}AB^{\frac{t}{2}}) \geq B^{\frac{t-s}{2}}\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})B^{\frac{t-s}{2}} \geq B^t$$

by choosing $\{\psi_r\}$ as $\{\phi_r\}$ in (4.1). If $\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}}) \geq B^s$, then there exists a contraction X such that

$$X^* \left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{2s}} = \left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{2s}} X = B^{\frac{t-s}{2}} \tag{4.2}$$

by Löwner-Heinz theorem and Theorem 4.B. Hense we have

$$\begin{aligned}
\phi_t(B^{\frac{t}{2}}AB^{\frac{t}{2}}) &= \phi_t\left(X^* \left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{2s}} B^{\frac{s}{2}}AB^{\frac{s}{2}} \left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{2s}} X\right) && \text{by (4.2)} \\
&\geq X^*\phi_t\left(\left(B^{\frac{s}{2}}AB^{\frac{s}{2}}\right)\left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{s}}\right)X && \text{by Theorem 4.A} \\
&= X^*\phi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{s}} X && \text{by (4.3)} \\
&= B^{\frac{t-s}{2}}\phi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})B^{\frac{t-s}{2}} && \text{by (4.2).}
\end{aligned}$$

The equality on the third line of the above formula can be shown by (3.1) as follows:

$$\phi_t\left(x\left(\psi_s(x)\right)^{\frac{t-s}{s}}\right) = \phi_t\left(y^t g(y)\right) = y^{t-s}\phi_s\left(y^s g(y)\right) = \left(\psi_s(x)\right)^{\frac{t-s}{s}}\phi_s(x), \tag{4.3}$$

where $x = y^s g(y)$, or equivalently, $y = (\psi_s(x))^{\frac{1}{s}}$.

(i-2) In case $A, B > 0$ and $\log\psi_a(B^{\frac{a}{2}}AB^{\frac{a}{2}}) \geq \log B^a$, put $A_1 = \psi_a(B^{\frac{a}{2}}AB^{\frac{a}{2}})$, $B_1 = B^a$ and $r_1 = \frac{s}{a} - 1 \geq 0$, then we have

$$\Psi_{r_1}(B_1^{\frac{r_1}{2}}G(A_1)B_1^{\frac{r_1}{2}}) \geq B_1^{r_1}, \tag{4.4}$$

where $G(x) = \psi_a^{-1}(x) = xg(x^{\frac{1}{a}})$ and $\Psi_r(x) = \left(\psi_{a(1+r)}(x)\right)^{\frac{r}{1+r}}$, which satisfy

$$\Psi_r(x^r G(x)) = \left(\psi_{a(1+r)}(x^{1+r}g(x^{\frac{1}{a}}))\right)^{\frac{r}{1+r}} = x^r.$$

(4.4) can be rewritten as $\left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{s-a}{s}} \geq B^{s-a}$, so that

$$\left(\psi_s(B^{\frac{s}{2}}AB^{\frac{s}{2}})\right)^{\frac{t-s}{s}} \geq B^{t-s}$$

holds for $a \le s \le t \le 2s-a$ by Löwner-Heinz theorem. The rest of the proof can be done in the same way as (i-1).

(ii-1) In case $A^a \ge \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$ and $\overline{R(A)} \cap N(B) = \{0\}$, it suffices to show that

$$A^s \ge \psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}) \Longrightarrow A^{\frac{t-s}{2}}\phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}} \ge \phi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}}) \tag{4.5}$$

holds for $a \le s \le t \le 2s$ since we obtain

$$A^s \ge \psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}}) \Longrightarrow \psi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}}) \le A^{\frac{t-s}{2}}\phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}} \le A^t$$

by choosing $\{\psi_r\}$ as $\{\phi_r\}$ in (4.5). If $A^s \ge \psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})$, then there exists a contraction X such that

$$X^*A^{\frac{t-s}{2}} = A^{\frac{t-s}{2}}X = P\left(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})\right)^{\frac{t-s}{2s}}P \tag{4.6}$$

by Löwner-Heinz theorem and Theorem 4.B, where P is the projection onto $N(A)^{\perp}$. Hense we have

$$\begin{aligned} X^*\phi_t(A^{\frac{t}{2}}BA^{\frac{t}{2}})X &\le \phi_t\left(X^*A^{\frac{t-s}{2}}A^{\frac{s}{2}}BA^{\frac{s}{2}}A^{\frac{t-s}{2}}X\right) && \text{by Theorem 4.A} \\ &= \phi_t\left((A^{\frac{s}{2}}BA^{\frac{s}{2}})\left(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})\right)^{\frac{t-s}{s}}\right) && \text{by (4.6)} \\ &= \phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})\left(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})\right)^{\frac{t-s}{s}} && \text{by (4.3)} \\ &= X^*A^{\frac{t-s}{2}}\phi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})A^{\frac{t-s}{2}}X && \text{by (4.6),} \end{aligned}$$

and the proof is complete since $\overline{R(A)} \cap N(B) = \{0\}$ implies $\overline{R(X)} = \overline{R(A)}$ by Lemma 4.1.

(ii-2) In case $A, B > 0$ and $\log A^a \ge \log \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$, put $A_1 = A^a$, $B_1 = \psi_a(A^{\frac{a}{2}}BA^{\frac{a}{2}})$ and $r_1 = \frac{s}{a} - 1 \ge 0$, then we have

$$A_1^{r_1} \ge \Psi_{r_1}(A_1^{\frac{r_1}{2}}G(B_1)A_1^{\frac{r_1}{2}}), \tag{4.7}$$

where $G(x)$ and $\Psi_r(x)$ are as defined in (i-2). (4.7) can be rewritten as $A^{s-a} \ge \left(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})\right)^{\frac{s-a}{s}}$, so that

$$A^{t-s} \ge \left(\psi_s(A^{\frac{s}{2}}BA^{\frac{s}{2}})\right)^{\frac{t-s}{s}}$$

holds for $a \le s \le t \le 2s-a$ by Löwner-Heinz theorem. The rest of the proof can be done in the same way as (ii-1). □

We use the following result in order to give a proof of Theorem 3.2.

Theorem 4.C ([16]). *Let A and B be positive operators, and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying $f(x)g(x) = x$. Then the following hold:*

(i) *$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \ge B$ ensures $A - g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \ge A^{\frac{1}{2}}E_BA^{\frac{1}{2}} - g(0)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$.*

(ii) $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ *ensures* $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) - B \geq f(0)E_{B^{\frac{1}{2}}AB^{\frac{1}{2}}} - B^{\frac{1}{2}}E_A B^{\frac{1}{2}}$.

Here E_X *denotes the projection onto* $N(X)$.

Proof of Theorem 3.2. T belongs to class A(s_0, t_0)-f_{s_0,t_0} if and only if

$$f_{s_0,t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0}) \geq |T^*|^{2t_0}.$$

By (i) of Theorem 3.1, we have

$$f_{s_0,t}(|T^*|^t|T|^{2s_0}|T^*|^t) \geq |T^*|^{t-t_0} f_{s_0,t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0})|T^*|^{t-t_0} \geq |T^*|^{2t} \tag{4.8}$$

holds for $t \geq t_0$. Put $f_{s,t}^{\perp}(x) = \frac{x}{f_{s,t}(x)}$, then (4.8) implies

$$|T|^{2s_0} \geq f_{s_0,t}^{\perp}(|T|^{s_0}|T^*|^{2t}|T|^{s_0}) \tag{4.9}$$

by (i) of Theorem 4.C. Since

$$f_{s_0,t}(x) = f_{s,t}(xg(y)^{s-s_0}) = f_{s,t}\left(x f_{s_0,t}^{\perp}(x)^{\frac{s-s_0}{s_0}}\right) \tag{4.10}$$

holds where $x = y^t g(y)^{s_0}$, we have

$$\begin{aligned} f_{s_0,t}(|T^*|^t|T|^{2s_0}|T^*|^t) &= f_{s,t}\left(|T^*|^t|T|^{2s_0}|T^*|^t f_{s_0,t}^{\perp}(|T^*|^t|T|^{2s_0}|T^*|^t)^{\frac{s-s_0}{s_0}}\right) \qquad \text{by (4.10)} \\ &= f_{s,t}\left(|T^*|^t|T|^{s_0} f_{s_0,t}^{\perp}(|T|^{s_0}|T^*|^{2t}|T|^{s_0})^{\frac{s-s_0}{s_0}}|T|^{s_0}|T^*|^t\right) \\ &\leq f_{s,t}(|T^*|^t|T|^{2s}|T^*|^t) \qquad \text{by (4.9) and Löwner-Heinz theorem,} \end{aligned}$$

so that $f_{s,t}(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t}$ holds for $s_0 \leq s \leq 2s_0$. We obtain the desired conclusion by repeating this process. □

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