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# Trace formulae and principal functions of Hilbert space operators

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This paper is the results of [13], [14] and [15]. Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. About the trace formula, we have the following:

**Theorem 1 (M. Krein, 1953).** *Let $A$ be a self-adjoint operator on $\mathcal{H}$ and $K$ be a trace class self-adjoint operator on $\mathcal{H}$. Then there exists a unique function $\delta(t)$ such that*

$$\mathrm{Tr}\Big(p(A+K)-p(A)\Big)=\int p'(t)\delta(t)dt,$$

*where $p$ is a polynomial.*

Let $\mathcal{C}_1$ be the trace class and $\mathcal{A}$ be the set of all Laurent polynomials; $\mathcal{P}(r,z)=\sum_{k=-N}^{N}p_k(r)z^k$. Let $J(\mathcal{P},\mathcal{Q})$ be the Jacobian of $\mathcal{P}$, $\mathcal{Q}$.

**Theorem 2 (Carey-Pincus [8], Helton-Howe [19]).** *Let $T=X+iY$ be an operator on $\mathcal{H}$ with trace class self-commutator ($[T^*,T]\in\mathcal{C}_1$). Then there exists a function $g(x,y)$ such that*

$$\mathrm{Tr}\Big([p(X,Y),q(X,Y)]\Big)=\frac{1}{2\pi i}\int\int J(p,q)(x,y)g(x,y)dx\,dy,$$

*where $p$ and $q$ are polynomials of two variables.*

Functions $\delta(t)$ and $g(x,y)$ in Theorems 1 and 2 are called the phase shift of the perturbation problem $A\to A+K$, and the (Cartesian) principal function of $T$, respectively. Let $T$ be hyponormal and satisfy $[T^*,T]\in\mathcal{C}_1$. For operators $A$ and $K$ of Theorem 1, let $A=TT^*$ and $K=T^*T-TT^*(=[T^*,T]\in\mathcal{C}_1)$. Then Theorem 1 is

$$\mathrm{Tr}\Big(p(T^*T)-p(TT^*)\Big)=\int p'(t)\delta(t)dt.$$

And

$$\delta(t) = \frac{1}{2\pi}\int_0^{2\pi} g(\sqrt{t}\cos\theta, \sqrt{t}\sin\theta)d\theta \quad \text{a.e. t} > 0.$$

Let $\mathcal{A}$ be the linear space of all Laurent polynomials $\mathcal{P}(r,z)$ with polynomial coefficients such that $\mathcal{P}(r,z) = \sum_{k=-N}^{N} p_k(r)z^k$, where $N$ is a non-negative integer and every $p_k(r)$ is a polynomial of one variable. For $T = U|T|$ with unitary $U$, put $\mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|)U^k$.

For the polar decomposition $T = U|T|$, we have the following:

**Theorem 3** ([8],[11],[25]). *Let $T = U|T|$ be semi-hyponormal operator satisfying $[|T|, U] \in \mathcal{C}_1$ with unitary $U$. Then there exists a function $g_T$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P},\mathcal{Q})(r, e^{i\theta})e^{i\theta} g_T(e^{i\theta}, r)drd\theta.$$

**Definition 1.** $T$ is $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$. Especially, $T$ is called hyponormal and semi-hyponormal if $p = 1$ and $p = 1/2$, respectively. It holds

$$\text{hyponormal} \implies \text{semi-hyponormal} \implies p\text{-hyponormal}.$$

**(1)** If $T = U|T|$ is semi-hyponormal, then $S = U|T|^{\frac{1}{2}}$ is hyponormal.

**(2)** If $T = U|T|$ is semi-hyponormal with $1 \notin \sigma(U)$, then $R = L^{-1}(U) + i|T|$ is hyponormal, where $L^{-1}(U) = i(U+1)(U-1)^{-1}$.

**(3)**If $T = U|T|$ is $p$-hyponormal, then $T_t = |T|^t U|T|^{1-t}$ is $q$-hyponormal (Aluthge transformation), where $q = \min\{p+t, p+1-t, 1\}$. Hence, if $T$ is semi-hyponormal, then $T_t$ is hyponormal.

Spectral mapping theorem holds for these transformations; i.e.,
if $T = U|T|$ be semi-hyponormal, then
(1) $\sigma(S) = \{\sqrt{r}e^{i\theta} \;:\; re^{i\theta} \in \sigma(T)\}$;
(2) $\sigma(R) = \{L^{-1}(e^{i\theta}) + ir \;:\; re^{i\theta} \in \sigma(T)\}$;
(3) $\sigma(T_t) = \sigma(T)$.

Functions $g$ and $g_T$ of Theorems 2 and 3 are called *the principal functions* of $T$ related to the Cartesian decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively. For a hyponormal operator $T = X + iY$, the principal function of $T$ is defined by the mosaic 0 □ $B(x,y)$ □ $I$ as follows;

$$g(x,y) = \mathrm{Tr}(B(x,y)).$$

For a semi-hyponormal operator $T = U|T|$, the principal function of $T$ is defined by the mosaic $0 \,\square\, B^P(e^{i\theta}, r) \,\square\, I$ as follows;

$$g_T(e^{i\theta}, r) = \mathrm{Tr}(B^P(e^{i\theta}, r)).$$

For an operator $T = U|T|$ let $T_t = |T|^t U |T|^{1-t}$ $(0 < t < 1)$ be the Aluthge transformation of $T$. Let $g_T$ and $g_{T_t}$ be the principal functions of $T$ and $T_t = |T|^t U |T|^{1-t}$ $(0 < t < 1)$, respectively. Then we have following results ([12]):

(1) If $T$ is invertible $p$-hyponormal with $[|T|, U] \in \mathcal{C}_1$, then $g_T = g_{T_t}$.

(2) If $T$ is hyponormal with $[|T|, U] \in \mathcal{C}_1$, then $g(x, y) = g_T(e^{i\theta}, r)$ for $x + iy = re^{i\theta}$.

The following results are important.

**Theorem 4** ([12]). *If a positive invertible operator $A$ and an operator $D$ satisfy $[A, D] \in \mathcal{C}_1$, then, for any real number $\alpha$, we have*

$$[A^\alpha, D] \in \mathcal{C}_1.$$

**Theorem 5** ([12]). *If $T$ is an invertible operator such that $[T^*, T] \in \mathcal{C}_1$, then $[\tilde{T}^*, \tilde{T}] \in \mathcal{C}_1$, where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ (Aluthge transform of $T$).*

For an invertible operator $T = U|T|$, $[T^*, T] \in \mathcal{C}_1$ if and only if $[|T|, U] \in \mathcal{C}_1$, because

$$[|T|^2, U] = [T^*, T]U \quad \text{and} \quad [T^*, T] = |T|[|T|, U]U^* + [|T|, U]|T|U^*.$$

## 1. Trace formula of $p$-hyponormal operators II

Let $\mathbf{T} = \{e^{i\theta} | 0 \,\square\, \theta < 2\pi\}, \Sigma$ be the set of all Borel sets in $\mathbf{T}$ and $m$ be a measure on the measure space $(\mathbf{T}, \Sigma)$ such that $dm(\theta) = \frac{1}{2\pi} d\theta$. Then we have

**Theorem 3'.** *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in \mathcal{C}_1$. Then there exists a summable function $g_T$ such that, for $\mathcal{P}(r, z), \mathcal{Q}(r, z) \in \mathcal{A}$, it holds*

$$\mathrm{Tr}\Big([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]\Big) = \int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

We denote by $C^\infty(\mathbf{R})$ the set of all smooth functions on $\mathbf{R}$ and by $C_c^\infty(\mathbf{R})$ the set of all functions in $C^\infty(\mathbf{R})$ with compact support. We denote by $\mathcal{B}$ the linear space of all Laurent polynomials $\phi(r, z)$ such that $\phi(r, z) = \sum_{k=-N}^{N} f_k(r) z^k$, where every $f_k \in C^\infty(\mathbf{R})$. For $T = U|T|$ with unitary $U$, put $\phi(|T|, U) = \sum_{k=-N}^{N} f_k(|T|) U^k$ for $\phi \in \mathcal{B}$.

In [6], Carey and Pincus proved a more general version of Theorem 3. It requires complicate calculations. Using polynomial approximation, we improve Theorem 3 in the following form.

**Theorem 1.1.** *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $\phi, \psi \in \mathcal{B}$, it holds*

$$\mathrm{Tr}\Big( [\phi(|T|, U), \psi(|T|, U)] \Big) = \int\int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

For the proof of Theorem 1.1, it needs Theorems 1.2 - 1.5.

**Theorem 1.2.** *Let $A, \{B_j\}_{j=1,\cdots,n}$ be operators such that $[A, B_j] \in \mathcal{C}_1$ and $||B_j|| \square r$ for all $j$ $(j = 1, 2, ..., n)$. Then*

$$||[A, B_1 B_2 \cdots B_n]||_1 \square n r^{n-1} \max_j ||[A, B_j]||_1.$$

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in \mathcal{C}_1$. Then we have

$$[U, e^{it|T|}] = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [U, |T|^n].$$

By Theorem 1.2,

$$||[U, e^{it|T|}]||_1 \square \sum_{n=1}^{\infty} \frac{|t|^n n ||T||^{n-1}}{n!} ||[U, |T|]||_1 \square |t| \cdot ||[U, |T|]||_1 \, e^{|t| \cdot ||T||}.$$

**Definition 1.1.** Under the assumption above, we define a constant $c_T$ of an operator $T = U|T|$ satisfying $[U, |T|] \in \mathcal{C}_1$ by

$$c_T = \max_{|t| \square 1} ||[U, e^{it|T|}]||_1.$$

Proof of the next theorem is based on an idea of the proof of [19, Lemma 3.2].

**Theorem 1.3.** *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $f \in \mathcal{S}$ and an integer $n$, it holds*

$$||[U^n, f(|T|)]||_1 \square |n| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_T(|t| + 1) |\hat{f}(t)| dt,$$

*where $c_T$ is the constant of Definition* 1.1.

**Theorem 1.4.** *Let $F$ be a compact set of $\mathbf{R}$ and $f \in C^{\infty}(\mathbf{R})$. Then there exist a function $f_1 \in C_c^{\infty}(\mathbf{R})$, a sequence $\{p_n\}$ of polynomials and a sequence $\{\gamma_n\}$ in $C_c^{\infty}(\mathbf{R})$ such that*

$$f(x) = f_1(x), \quad p_n(x) = \gamma_n(x) \ \ for \ \ x \in F,$$

$$\sup_{y\in F}|f_1(y)-\gamma_n(y)|\to 0 \qquad (n\to\infty),$$

$$\sup_{y\in F}|f_1^{(1)}(y)-\gamma_n^{(1)}(y)|\to 0 \qquad (n\to\infty),$$

$$\sup_{t\in \mathbf{R}}|\hat{f}_1(t)-\hat{\gamma}_n(t)|\to 0 \qquad (n\to\infty),$$

*and*

$$\sup_{t\in \mathbf{R}}|t|^3\cdot|\hat{f}_1(t)-\hat{\gamma}_n(t)|\to 0 \qquad (n\to\infty).$$

**Theorem 1.5.** *Let $T=U|T|$ be the polar decomposition. Assume that $U$ is unitary and $[U,|T|]\in\mathcal{C}_1$. Then, for $f,g\in\mathcal{S}$ and integers $m,n$, it holds*

$$|||[f(|T|)U^n, g(|T|)U^m]|||$$
$$\Box\ |n|\cdot|||f|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{g}(t)|dt$$
$$+|m|\cdot|||g|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{f}(t)|dt$$

*where $c_T$ is the constant of Definition* 1.1 *and* $|||h|||=\sup_{x\in\sigma(|T|)}|h(x)|$.

Next we apply this result to $p$-hyponormal operators $(0<p<1/2)$.

**Definition 1.2.** Let $T=U|T|$ be $p$-hyponormal with unitary $U$. Put $S=U|T|^{2p}$. Then $S$ is semi-hyponormal with unitary $U$. Hence there exists the Pincus principal function $g_S$ of $S$ and we define the principal function $g_T$ of $T$ by

$$g_T(e^{i\theta},r)=g_S(e^{i\theta},r^{\frac{1}{2p}})$$

(see [11, Definition 3]).

The following theorem is a generalization of Theorem 10 of [11].

**Theorem 1.6.** *Let $T=U|T|$ be an invertible $p$-hyponormal operator. If $|T|^{2p}-U|T|^{2p}U^*\in\mathcal{C}_1$, then for $\mathcal{P}(r,z),\mathcal{Q}(r,z)\in\mathcal{A}$ it holds*

$$\mathrm{Tr}\Big([\mathcal{P}(|T|,U),\mathcal{Q}(|T|,U)]\Big)=\int\int J(\mathcal{P},\mathcal{Q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drdm(\theta).$$

## 2. Trace formulae associated with the polar decomposition of operators

Let $\mathcal{S}(\mathbf{R}^2)$ be the Schwartz space of rapidly decreasing functions at infinity. For $T=X+iY$, let $\mathcal{E}$ and $\mathcal{F}$ be the spectral measures of self-adjoint operators $X$ and $Y$, respectively. We define $\tau$ on $\mathcal{S}(\mathbf{R}^2)$ by

$$(*)\qquad \tau(\phi)=\int\int\phi(x,y)d\mathcal{E}(x)d\mathcal{F}(y)\quad(\phi\in\mathcal{S}(\mathbf{R}^2)).$$

By a standard argument, we have

$$\int\int e^{itX}e^{isY}\hat{\phi}(t,s)dtds = \int\int \phi(x,y)d\mathcal{E}(x)d\mathcal{F}(y),$$

where

$$\hat{\phi}(t,s) = \frac{1}{2\pi}\int\int e^{-i(tx+sy)}\phi(x,y)dxdy$$

is the Fourier transform of the function $\phi$ (see, for example, [22, p.237]).

Put $\nu(E) = \int\int_E \hat{\phi}(t,s)dtds$ for a measurable set $E \subset \mathbf{R}^2$. Since $\hat{\phi}(t,s) \in \mathcal{S}(\mathbf{R}^2)$, we have

$$\int\int (1+|t|)(1+|s|)|\hat{\phi}(t,s)|dtds < \infty.$$

Following Carey-Pincus [8], put $G(x,y) = \int\int e^{itx+isy}d\nu(t,s)$ and define

$$G(X,Y) = \int\int G(x,y)d\mathcal{E}(x)d\mathcal{F}(y).$$

Then

$$\tau(\phi) = \int\int e^{itX}e^{isY}\nu(t,s)dtds = G(X,Y).$$

Note here that we have $\tau(\psi) = \tau(\phi)$ for any smooth function $\psi(x,y)$ which coincides with $\phi(x,y)$ on $\mathrm{supp}(\tau)$.

The map $\tau : \mathcal{S}(\mathbf{R}^2) \to B(\mathcal{H})$ has the following properties [22, Chapter X, §2];

(1) $\tau$ is linear, continuous and $\mathrm{supp}\,(\tau) \subseteq \sigma(X) \times \sigma(Y)$,
(2) $\tau(1) = I, \tau(p+q) = p(X) + q(Y)$ for polynomials $p$ and $q$ of one varaible of $x$ and $y$, respectively.
(3) $\tau(\phi)\tau(\psi) - \tau(\phi\psi) \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,
(4) $\tau(\phi)^* - \tau(\bar{\phi}) \in \mathcal{C}_1$.

By (3) we have an important property $[\tau(\phi), \tau(\psi)] \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$.

The following theorem is a basis.

**Theorem 6 (Carey-Pincus,** [8, Theorem 5.1]**).** *Let $T = X + iY$ be an operator with $[T^*, T] \in \mathcal{C}_1$. Let $\mathcal{E}, \mathcal{F}$ be the spectral measures of $X$ and $Y$, respectively and $\tau$ be given by $(*)$. Then there exists a summable function $g$ such that, for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,*

$$\mathrm{Tr}\Big([\tau(\phi), \tau(\psi)]\Big) = \frac{1}{2\pi i}\int\int J(\phi,\psi)(x,y)g(x,y)dxdy.$$

*Moreover, if $T$ is hyponormal, then $g \geq 0$ and $g(x,y) = 0$ for $x + iy \notin \sigma(T)$.*

In this section, the main theorem is Theorem 2.7. We prepare some results.

**Lemma 2.1.** *Let $A$ be a positive invertible operator and operators $D, E, F$ satisfying $[A,D],[E,D],[F,D] \in \mathcal{C}_1$. Then for any real number $\alpha$, we have*

$$[EA^\alpha F, D] \in \mathcal{C}_1.$$

**Lemma 2.2 (cf.p.158 of** [8]**).** *Let $T = X + iY$ be an invertible operator such that $[T^*, T] \in \mathcal{C}_1$. Let $\psi \in \mathcal{S}(\mathbf{R}^2)$, $D = \tau(\psi)$ and operators $E, F$ satisfy $[E,D],[F,D] \in \mathcal{C}_1$. Then, for $\phi(x,y) = (x^2+y^2)^\alpha$ with a real number $\alpha$,*

$$\mathrm{Tr}\Big([E\tau(\phi)F, D]\Big) = \mathrm{Tr}\Big([E|T|^{2\alpha}F, D]\Big).$$

**Theorem 2.3.** *Let $T = U|T|$ be an invertible operator with $[T^*,T] \in \mathcal{C}_1$ and let $g$ be the principal function assoicated with the Cartesian decomposition of $T = X + iY$. Then there exists a summable function $g_T$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P},\mathcal{Q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drd\theta,$$

*and $g_T(e^{i\theta}, r) = g(x,y)$ almost everywhere $x+iy = re^{i\theta}$ on $\mathbf{C}$.*

An invertible operator $T$ is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ [17].

For the proof of the next result, we need the following two lemmas. For an operator $T$, let $\sigma_{ap}(T)$ and $\sigma_p(T)$ be the approximate point spectrum and the point spectrum of $T$, respectively. The following lemmas are important for the main theorem.

**Lemma 2.4.** *Let $T = U|T|$ be an invertible semi-hyponormal operator with $[|T|,U] \in \mathcal{C}_1$. Then the principal function $g^P$ associated with the polar decomposition $T = U|T|$ of $T$ satisfies $g_T(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(T)$.*

**Lemma 2.5.** *Let $T = U|T|$ be an operator with unitary $U$ and put $S = U(|T|+I)$. If $z \in \partial\sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.*

By the above lemmas, for the next theorem we can give another proof of [11, Theorem 9].

**Theorem 2.6.** *Let $T = U|T|$ be a semi-hyponormal operator with unitary $U$ and $[|T|,U] \in \mathcal{C}_1$. Then there exists a summable function $g_T$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P},\mathcal{Q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drd\theta.$$

**Theorem 2.7.** *Let $T = X+iY = U|T|$ be a semi-hyponormal operator with unitary $U$ and $[|T|,U] \in \mathcal{C}_1$. If $g$ and $g_T$ are the principal function associated with the Cartesian decomposition of $T$ and the summable function in Theorem 2.6, respectively, then*

$$g(x,y) = g_T(e^{i\theta}, r)$$

*almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.*

### 3. Principal functions for high powers of operators

In this section, we denote $g_T$ and $g_T^P$ be the principal functions of the Cartesian decomposition and the polar decomposition of $T$, respectively. C.A. Berger gave the principal functions $g_{T^n}$ of powers $T^n$ of $T$ in terms of $g_T$ and proved that for a sufficiently high $n$, $T^n$ has a non-trivial invariant subspace for a hyponormal operator $T$ ([5]). For a hyponormal operator $T$ with $[T^*,T]\in\mathcal{C}_1$, it holds that

$$g_{T^n}(z)=\sum_{k=1}^{n} g_T(\zeta_k), \quad \text{where } \ \zeta_k^n = z \ (k=1,...,n).$$

More generally, in [4], for a polynomial $p$, it holds

$$g_{p(T)}(z)=\sum_{\zeta}\{\ g_T(\zeta)\ :\ p(\zeta)=z\ \}.$$

**Theorem 3.1.** *Let $T=X+iY=U|T|$ be an operator satisfying the following trace formula:*

$$\mathrm{Tr}([\phi(|T|,U),\psi(|T|,U)])=\frac{1}{2\pi}\int\int J(\phi,\psi)(r,e^{i\theta})e^{i\theta}g_T^P(e^{i\theta},r)drd\theta$$

*for any Laurent polynomials $\phi$ and $\psi$. Then the principal function $g_T(x,y)$ related to the Cartesian decomposition $T=X+iY$ of $T$ exists and it is given by $g_T(x,y)=g_T^P(e^{i\theta},r)$, where $x+iy=re^{i\theta}$.*

If an operator $T=U|T|$ is invertible and $[|T|,U]\in\mathcal{C}_1$, then $[T^*,T]\in\mathcal{C}_1$. So we have the following

**Corollary 3.2.** *If an invertible operator $T=X+iY=U|T|$ satisfies $[|T|,U]\in\mathcal{C}_1$, then $g_T(x,y)=g_T^P(e^{i\theta},r)$, where $x+iy=re^{i\theta}$.*

For a relation between $g_T^P$ and $g_{T^n}^P$, we need the following Berger's result:

**Theorem 3.3 (Berger,** [5, Theorem 4]**).** *For an operator $T$, if $[T^*,T]\in\mathcal{C}_1$, then for a positive integer $n$,*

$$g_{T^n}(x,y)=\sum_{(u+iv)^n=x+iy} g_T(u,v).$$

**Theorem 3.4**. *For an operator $T$ with $[T^*,T]\in\mathcal{C}_1$, if $\int\int g(x,y)dxdy\neq 0$, then*

$$\lim_{n\to\infty}\ \text{ess sup}\,|g_{T^n}|\ =\ \infty.$$

Applying Corollary 3.2 and Theorem 3.4 to $T^n$, we have the following.

**Corollary 3.5.** *For an operator $T=U|T|$, let $T^n=U_n|T^n|$ be the polar decomposition of $T^n$ $(n=1,2,...)$. If $[|T^n|,U_n]\in\mathcal{C}_1$ for every non-negative integer $n$ and*

$\int\int g_T^P(e^{i\theta}, r) r d\theta dr \neq 0$, *then*

$$\lim_{n\to\infty} \text{ess sup} |g_{T^n}^P| = \infty.$$

Let $T = U|T|$ and $T^n = U_n|T^n|$ be the polar decompositions of $T$ and $T^n$, respectively. Then it holds that if $[T^*, T] \in \mathcal{C}_1$, then $[T^{*n}, T^n] \in \mathcal{C}_1$ for any positive integer $n$. On the other hand, in the polar decomposition case, it is not clear whether $[|T|, U] \in \mathcal{C}_1$ impilies $[|T^n|, U_n] \in \mathcal{C}_1$ even if $n = 2$. If $T$ is invertible and $[|T|, U] \in \mathcal{C}_1$, then, for every $n$, it holds $[|T^n|, U_n] \in \mathcal{C}_1$ by [11, Theorem 3].

Next we consider operators with cyclic vectors. First we need the following result.

**Theorem 3.6 (Martin and Putinar, Th.X.4.3** [22]**).** *Let $g_T$ and $g_V$ be the principal functions of operators $T$ and $V$ such that $[T^*, T], [V^*, V] \in \mathcal{C}_1$, respectively. If there exists an operator $A \in \mathcal{C}_1$ such that $AV = TA$ and* $\ker(A) = \ker(A^*) = \{0\}$, *then $g_T \square g_V$.*

Proof of the following lemma is based on it of [22, Corollary X.4.4].

**Lemma 3.7.** *Let $T$ be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If $T$ has a cyclic vector, then $g_T \square 1$.*

Let $S$ be an operator having the principal function $g_S$ related to the Cartesian decomposition $S = X + iY$. Then $g_{S^*}(x, y) = -g_S(x, -y)$. Hence, as a corollary of Lemma 3.7, we have the following.

**Lemma 3.8.** *Let $T$ be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If $T^*$ has a cyclic vector, then $-1 \square g_T$.*

**Theorem 3.9.** *Let $T$ be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If $\int\int g_T(x, y) dx dy \neq 0$, then, for a sufficiently high $n$, $T^n$ has a non-trivial invariant subspace.*

## References


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