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Trace formulae and principal functions of Hilbert space operators

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This paper is the results of [13], [14] and [15]. Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. About the trace formula, we have the following:

**Theorem 1 (M. Krein, 1953).** Let $A$ be a self-adjoint operator on $\mathcal{H}$ and $K$ be a trace class self-adjoint operator on $\mathcal{H}$. Then there exists a unique function $\delta(t)$ such that

$$
\text{Tr} \left( p(A + K) - p(A) \right) = \int p'(t)\delta(t)dt,
$$

where $p$ is a polynomial.

Let $\mathcal{C}_1$ be the trace class and $A$ be the set of all Laurent polynomials; $P(r,z) = \sum_{k=-N}^{N} p_k(r)z^k$. Let $J(P,Q)$ be the Jacobian of $P, Q$.

**Theorem 2 (Carey-Pincus [8], Helton-Howe [19]).** Let $T = X + iY$ be an operator on $\mathcal{H}$ with trace class self-commutator ($[T^*,T] \in \mathcal{C}_1$). Then there exists a function $g(x, y)$ such that

$$
\text{Tr} \left( [p(X,Y),q(X,Y)] \right) = \frac{1}{2\pi i} \int \int J(p,q)(x,y)g(x,y)dx\,dy,
$$

where $p$ and $q$ are polynomials of two variables.

Functions $\delta(t)$ and $g(x, y)$ in Theorems 1 and 2 are called the phase shift of the perturbation problem $A \to A + K$, and the (Cartesian) principal function of $T$, respectively. Let $T$ be hyponormal and satisfy $[T^*,T] \in \mathcal{C}_1$. For operators $A$ and $K$ of Theorem 1, let $A = TT^*$ and $K = T^*T - TT^* (= [T^*,T] \in \mathcal{C}_1)$. Then Theorem 1 is

$$
\text{Tr} \left( p(T^*T) - p(TT^*) \right) = \int p'(t)\delta(t)dt.
$$
And
\[ \delta(t) = \frac{1}{2\pi} \int_{0}^{2\pi} g(\sqrt{t} \cos \theta, \sqrt{t} \sin \theta) d\theta \quad \text{a.e. t > 0}. \]

Let \( \mathcal{A} \) be the linear space of all Laurent polynomials \( \mathcal{P}(r, z) \) with polynomial coefficients such that \( \mathcal{P}(r, z) = \sum_{k=-N}^{N} p_k(r) z^k \), where \( N \) is a non-negative integer and every \( p_k(r) \) is a polynomial of one variable. For \( T = U|T| \) with unitary \( U \), put \( \mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|) U^k \).

For the polar decomposition \( T = U|T| \), we have the following:

**Theorem 3** ([8],[11],[25]). Let \( T = U|T| \) be semi-hyponormal operator satisfying \( [|T|, U] \in \mathcal{C}_1 \) with unitary \( U \). Then there exists a function \( g_T \) such that, for \( \mathcal{P}, \mathcal{Q} \in \mathcal{A}, \)
\[ \text{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr d\theta. \]

**Definition 1.** \( T \) is \( p \)-hyponormal if \( (T^*T)^p \geq (TT^*)^p \). Especially, \( T \) is called hyponormal and semi-hyponormal if \( p = 1 \) and \( p = 1/2 \), respectively. It holds
\[ \text{hyponormal} \implies \text{semi-hyponormal} \implies \text{p-hyponormal}. \]

(1) If \( T = U|T| \) is semi-hyponormal, then \( S = U|T|^{1/2} \) is hyponormal.

(2) If \( T = U|T| \) is semi-hyponormal with \( 1 \notin \sigma(U) \), then \( R = L^{-1}(U) + i|T| \) is hyponormal, where \( L^{-1}(U) = i(U + 1)(U - 1)^{-1} \).

(3) If \( T = U|T| \) is \( p \)-hyponormal, then \( T_t = |T|^t|U|^1-t \) is \( q \)-hyponormal (Aluthge transformation), where \( q = \min\{p + t, p + 1 - t, 1\} \). Hence, if \( T \) is semi-hyponormal, then \( T_t \) is hyponormal.

Spectral mapping theorem holds for these transformations; i.e., if \( T = U|T| \) be semi-hyponormal, then
(1) \( \sigma(S) = \{ \sqrt{r} e^{i\theta} : re^{i\theta} \in \sigma(T) \} \);
(2) \( \sigma(R) = \{ L^{-1}(e^{i\theta}) + ir : re^{i\theta} \in \sigma(T) \} \);
(3) \( \sigma(T_t) = \sigma(T) \).

Functions \( g \) and \( g_T \) of Theorems 2 and 3 are called the principal functions of \( T \) related to the Cartesian decomposition \( T = X + iY \) and the polar decomposition \( T = U|T| \), respectively. For a hyponormal operator \( T = X + iY \), the principal function of \( T \) is defined by the mosaic \( 0 \square B(x, y) \square I \) as follows;
\[ g(x, y) = \text{Tr}(B(x, y)). \]
For a semi-hyponormal operator $T = U|T|$, the principal function of $T$ is defined by the mosaic $0 \sqcup B^p(e^{i\theta}, r) \sqcup I$ as follows;

$$g_T(e^{i\theta}, r) = \text{Tr}(B^p(e^{i\theta}, r)).$$

For an operator $T = U|T|$ let $T_t = |T|^t U|T|^{1-t}$ $(0 < t < 1)$ be the Aluthge transformation of $T$. Let $g_T$ and $g_{T_t}$ be the principal functions of $T$ and $T_t = |T|^t U|T|^{1-t}$ $(0 < t < 1)$, respectively. Then we have following results ([12]):

1. If $T$ is invertible $p$-hyponormal with $|[T], U| \in C_1$, then $g_T = g_{T_t}$.

The following results are important.

**Theorem 4** ([12]). If a positive invertible operator $A$ and an operator $D$ satisfy $[A, D] \in C_1$, then, for any real number $\alpha$, we have

$$[A^{\alpha}, D] \in C_1.$$

**Theorem 5** ([12]). If $T$ is an invertible operator such that $[T^*, T] \in C_1$, then $[\tilde{T}^*, \tilde{T}] \in C_1$, where $\tilde{T} = |T|^{1/2} U|T|^{1/2}$ (Aluthge transform of $T$).

For an invertible operator $T = U|T|$, $[T^*, T] \in C_1$ if and only if $|[T], U| \in C_1$, because


1. **Trace formula of $p$-hyponormal operators II**

Let $T = \{e^{i\theta}|0 \sqcup \theta < 2\pi\}, \Sigma$ be the set of all Borel sets in $T$ and $m$ be a measure on the measure space $(T, \Sigma)$ such that $dm(\theta) = \frac{1}{2\pi}d\theta$. Then we have

**Theorem 3'**. Let $T \in B(H)$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of $T$. Assume that $U$ is unitary and $|[U, |T]| \in C_1$. Then there exists a summable function $g_T$ such that, for $P(r, z), Q(r, z) \in C_1$, it holds

$$\text{Tr}
\left[\left[P(|T|, U), Q(|T|, U)\right]\right] = \int \int J(P, Q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

We denote by $C^\infty(R)$ the set of all smooth functions on $R$ and by $C^\infty_c(R)$ the set of all functions in $C^\infty(R)$ with compact support. We denote by $B$ the linear space of all Laurent polynomials $\phi(r, z)$ such that $\phi(r, z) = \sum_{k=-N}^{N} f_k(r)z^k$, where every $f_k \in C^\infty(R)$.

For $T = U|T|$ with unitary $U$, put $\phi(|T|, U) = \sum_{k=-N}^{N} f_k(|T|)U^k$ for $\phi \in B$. 


In [6], Carey and Pincus proved a more general version of Theorem 3. It requires complicate calculations. Using polynomial approximation, we improve Theorem 3 in the following form.

**Theorem 1.1.** Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in C_1$. Then, for $\phi, \psi \in B$, it holds

$$\text{Tr} \left( \phi(|T|, U), \psi(|T|, U) \right) = \int \int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

For the proof of Theorem 1.1, it needs Theorems 1.2 - 1.5.

**Theorem 1.2.** Let $A, \{B_j\}_{j=1,\ldots,n}$ be operators such that $[A, B_j] \in C_1$ and $||B_j|| \square r$ for all $j (j = 1, 2, \ldots, n)$. Then

$$||[A, B_1 B_2 \cdots B_n]||_1 \square n^{n-1} \max_j ||[A, B_j]||_1.$$

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in C_1$. Then we have

$$[U, e^{it|T|}] = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [U, |T|^n].$$

By Theorem 1.2,

$$||[U, e^{it|T|}]||_1 \square \sum_{n=1}^{\infty} \frac{|t|^n |T||^{n-1}}{n!} ||[U, |T|]||_1 \square |t| \cdot ||[U, |T|]||_1 e^{|t| \cdot ||T||}.$$

**Definition 1.1.** Under the assumption above, we define a constant $c_T$ of an operator $T = U|T|$ satisfying $[U, |T|] \in C_1$ by

$$c_T = \max_{||t|| \square 1} ||[U, e^{it|T|}]||_1.$$

Proof of the next theorem is based on an idea of the proof of [19, Lemma 3.2].

**Theorem 1.3.** Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of $T$. Assume that $U$ is unitary and $[U, |T|] \in C_1$. Then, for $f \in S$ and an integer $n$, it holds

$$||[U^n, f(|T|)]||_1 \square |n| \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_T(|t| + 1)|\hat{f}(t)| dt,$$

where $c_T$ is the constant of Definition 1.1.

**Theorem 1.4.** Let $F$ be a compact set of $\mathbb{R}$ and $f \in C^\infty(\mathbb{R})$. Then there exist a function $f_1 \in C_c^\infty(\mathbb{R})$, a sequence $\{p_n\}$ of polynomials and a sequence $\{\gamma_n\}$ in $C_c^\infty(\mathbb{R})$ such that

$$f(x) = f_1(x), \quad p_n(x) = \gamma_n(x) \quad \text{for} \quad x \in F,$$
\[ \sup_{y \in F} |f_{1}(y) - \gamma_{n}(y)| \rightarrow 0 \quad (n \rightarrow \infty), \]
\[ \sup_{y \in F} |f_{1}^{(1)}(y) - \gamma_{n}^{(1)}(y)| \rightarrow 0 \quad (n \rightarrow \infty), \]
\[ \sup_{t \in \mathbb{R}} |\hat{f}_{1}(t) - \hat{\gamma}_{n}(t)| \rightarrow 0 \quad (n \rightarrow \infty), \]
and
\[ \sup_{t \in \mathbb{R}} |t|^{3} \cdot |\hat{f}_{1}(t) - \hat{\gamma}_{n}(t)| \rightarrow 0 \quad (n \rightarrow \infty). \]

**Theorem 1.5.** Let \( T = U|T| \) be the polar decomposition. Assume that \( U \) is unitary and \( [U, |T|] \in \mathcal{C}_{1} \). Then, for \( f, g \in \mathcal{S} \) and integers \( m, n \), it holds
\[
|||f(|T|)U^{n}, g(|T|)U^{m}||| \\
\leq |n| \cdot |||f||| \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{T}(|t| + 1)|\hat{g}(t)|dt \\
+ |m| \cdot |||g||| \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{T}(|t| + 1)|\hat{f}(t)|dt,
\]
where \( c_{T} \) is the constant of Definition 1.1 and \( |||h||| = \sup_{x \in \sigma(|T|)} |h(x)| \).

Next we apply this result to \( p \)-hyponormal operators \((0 < p < 1/2)\).

**Definition 1.2.** Let \( T = U|T| \) be \( p \)-hyponormal with unitary \( U \). Put \( S = U|T|^{2p} \). Then \( S \) is semi-hyponormal with unitary \( U \). Hence there exists the Pincus principal function \( g_{S} \) of \( S \) and we define the principal function \( g_{T} \) of \( T \) by
\[ g_{T}(e^{i\theta}, r) = g_{S}(e^{i\theta}, r^{1/2p}) \]
(see [11, Definition 3]).

The following theorem is a generalization of Theorem 10 of [11].

**Theorem 1.6.** Let \( T = U|T| \) be an invertible \( p \)-hyponormal operator. If \( |T|^{2p} - U|T|^{2p}U^* \in \mathcal{C}_{1} \), then for \( \mathcal{P}(r, z), \mathcal{Q}(r, z) \in \mathcal{A} \) it holds
\[
\text{Tr} \left( [\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)] \right) = \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta} g_{T}(e^{i\theta}, r)drdm(\theta).
\]

2. Trace formulae associated with the polar decomposition of operators

Let \( \mathcal{S}(\mathbb{R}^{2}) \) be the Schwartz space of rapidly decreasing functions at infinity. For \( T = X + iY \), let \( \mathcal{E} \) and \( \mathcal{F} \) be the spectral measures of self-adjoint operators \( X \) and \( Y \), respectively. We define \( \tau \) on \( \mathcal{S}(\mathbb{R}^{2}) \) by
\[
\tau(\phi) = \int \int \phi(x, y)d\mathcal{E}(x)d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbb{R}^{2})).
\]
By a standard argument, we have
\[ \int \int e^{itX}e^{isY} \hat{\phi}(t, s)dtds = \int \int \phi(x, y)d\mathcal{E}(x)d\mathcal{F}(y), \]
where
\[ \hat{\phi}(t, s) = \frac{1}{2\pi} \int \int e^{-i(tx+sy)}\phi(x, y)dxdy \]
is the Fourier transform of the function \( \phi \) (see, for example, [22, p.237]).

Put \( \nu(E) = \int \int \hat{\phi}(t, s)dtds \) for a measurable set \( E \subset \mathbb{R}^2 \). Since \( \hat{\phi}(t, s) \in \mathcal{S}(\mathbb{R}^2) \), we have
\[ \int \int (1+|t|)(1+|s|)|\hat{\phi}(t, s)|dtds < \infty. \]

Following Carey-Pincus [8], put \( G(x, y) = \int \int e^{itx+isy}\nu(t, s)dtds \) and define
\[ G(X, Y) = \int \int G(x, y)d\mathcal{E}(x)d\mathcal{F}(y). \]
Then
\[ \tau(\phi) = \int \int e^{itX}e^{isY}\nu(t, s)dtds = G(X, Y). \]

Note here that we have \( \tau(\psi) = \tau(\phi) \) for any smooth function \( \psi(x, y) \) which coincides with \( \phi(x, y) \) on \( \text{supp}(\tau) \).

The map \( \tau : \mathcal{S}(\mathbb{R}^2) \rightarrow B(\mathcal{H}) \) has the following properties [22, Chapter X, §2];

1. \( \tau \) is linear, continuous and \( \text{supp}(\tau) \subseteq \sigma(X) \times \sigma(Y) \),
2. \( \tau(1) = I, \tau(p + q) = p(X) + q(Y) \) for polynomials \( p \) and \( q \) of one variable of \( x \) and \( y \), respectively.
3. \( \tau(\phi)\tau(\psi) - \tau(\phi\psi) \in \mathcal{C}_1 \) for \( \phi, \psi \in \mathcal{S}(\mathbb{R}^2) \),
4. \( \tau(\phi)^* - \tau(\overline{\phi}) \in \mathcal{C}_1 \).

By (3) we have an important property \( [\tau(\phi), \tau(\psi)] \in \mathcal{C}_1 \) for \( \phi, \psi \in \mathcal{S}(\mathbb{R}^2) \).

The following theorem is a basis.

**Theorem 6 (Carey-Pincus, [8, Theorem 5.1]).** Let \( T = X + iY \) be an operator with \( [T^*, T] \in C_1 \). Let \( \mathcal{E}, \mathcal{F} \) be the spectral measures of \( X \) and \( Y \), respectively and \( \tau \) be given by (*) . Then there exists a summable function \( g \) such that, for \( \phi, \psi \in \mathcal{S}(\mathbb{R}^2) \),
\[ \text{Tr}(\tau(\phi), \tau(\psi)) = \frac{1}{2\pi i} \int \int J(\phi, \psi)(x, y)g(x, y)dxdy. \]

Moreover, if \( T \) is hyponormal, then \( g \geq 0 \) and \( g(x, y) = 0 \) for \( x + iy \notin \sigma(T) \).

In this section, the main theorem is Theorem 2.7. We prepare some results.
Lemma 2.1. Let $A$ be a positive invertible operator and operators $D, E, F$ satisfying $[A, D], [E, D], [F, D] \in \mathcal{C}_1$. Then for any real number $\alpha$, we have

$$[EA^\alpha F, D] \in \mathcal{C}_1.$$ 

Lemma 2.2 (cf.p.158 of [8]). Let $T = X + iY$ be an invertible operator such that $[T^*, T] \in \mathcal{C}_1$. Let $\psi \in \mathcal{S}(\mathbb{R}^2)$, $D = \tau(\psi)$ and operators $E, F$ satisfy $[E, D], [F, D] \in \mathcal{C}_1$. Then, for $\phi(x, y) = (x^2 + y^2)^\alpha$ with a real number $\alpha$,

$$\text{Tr}\left([E\tau(\phi)F, D]\right) = \text{Tr}\left([E|T|^{2\alpha}F, D]\right).$$

Theorem 2.3. Let $T = U|T|$ be an invertible operator with $[T^*, T] \in \mathcal{C}_1$ and let $g$ be the principal function associated with the Cartesian decomposition of $T = X + iY$. Then there exists a summable function $g_T$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$\text{Tr}(\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U))) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drd\theta,$$

and $g_T(e^{i\theta}, r) = g(x, y)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbb{C}$.

An invertible operator $T$ is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ [17].

For the proof of the next result, we need the following two lemmas. For an operator $T$, let $\sigma_{ap}(T)$ and $\sigma_p(T)$ be the approximate point spectrum and the point spectrum of $T$, respectively. The following lemmas are important for the main theorem.

Lemma 2.4. Let $T = U|T|$ be an invertible semi-hyponormal operator with $[|T|, U] \in \mathcal{C}_1$. Then the principal function $g_T^p$ associated with the polar decomposition $T = U|T|$ of $T$ satisfies $g_T(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(T)$.

Lemma 2.5. Let $T = U|T|$ be an operator with unitary $U$ and put $S = U(|T| + I)$. If $z \in \partial\sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.

By the above lemmas, we can give another proof of [11, Theorem 9].

Theorem 2.6. Let $T = U|T|$ be a semi-hyponormal operator with unitary $U$ and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function $g_T$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$\text{Tr}(\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U))) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drd\theta.$$

Theorem 2.7. Let $T = X + iY = U|T|$ be a semi-hyponormal operator with unitary $U$ and $[|T|, U] \in \mathcal{C}_1$. If $g$ and $g_T$ are the principal function associated with the Cartesian decomposition of $T$ and the summable function in Theorem 2.6, respectively, then

$$g(x, y) = g_T(e^{i\theta}, r)$$

almost everywhere $x + iy = re^{i\theta}$ on $\mathbb{C}$.
3. Principal functions for high powers of operators

In this section, we denote $g_T$ and $g_T^P$ be the principal functions of the Cartesian decomposition and the polar decomposition of $T$, respectively. C.A. Berger gave the principal functions $g_T^n$ of powers $T^n$ of $T$ in terms of $g_T$ and proved that for a sufficiently high $n$, $T^n$ has a non-trivial invariant subspace for a hyponormal operator $T$ ([5]). For a hyponormal operator $T$ with $[T^*, T] \in C_1$, it holds that

$$g_T^n(z) = \sum_{k=1}^{n} g_T(\zeta_k), \quad \text{where} \quad \zeta_k^n = z \ (k = 1, ..., n).$$

More generally, in [4], for a polynomial $p$, it holds

$$g_p(T)(z) = \sum_{\zeta} \{ g_T(\zeta) : p(\zeta) = z \}.$$

**Theorem 3.1.** Let $T = X + iY = U|T|$ be an operator satisfying the following trace formula:

$$\text{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi} \int \int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T^P(e^{i\theta}, r) dr d\theta$$

for any Laurent polynomials $\phi$ and $\psi$. Then the principal function $g_T(x, y)$ related to the Cartesian decomposition $T = X + iY$ of $T$ exists and it is given by $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.

If an operator $T = U|T|$ is invertible and $[|T|, U] \in C_1$, then $[T^*, T] \in C_1$. So we have the following

**Corollary 3.2.** If an invertible operator $T = X + iY = U|T|$ satisfies $[|T|, U] \in C_1$, then $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.

For a relation between $g_T^P$ and $g_T^n$, we need the following Berger's result:

**Theorem 3.3 (Berger, [5, Theorem 4]).** For an operator $T$, if $[T^*, T] \in C_1$, then for a positive integer $n$,

$$g_T^n(x, y) = \sum_{(u+iv)^n = x+iy} g_T(u, v).$$

**Theorem 3.4.** For an operator $T$ with $[T^*, T] \in C_1$, if $\int \int g(x, y) dx dy \neq 0$, then

$$\lim_{n \to \infty} \text{ess sup} |g_T^n| = \infty.$$

Applying Corollary 3.2 and Theorem 3.4 to $T^n$, we have the following.

**Corollary 3.5.** For an operator $T = U|T|$, let $T^n = U_n T^n$ be the polar decomposition of $T^n$ ($n = 1, 2, ...$). If $[|T^n|, U_n] \in C_1$ for every non-negative integer $n$ and
\[
\int \int g_T^P(e^{i\theta}, r)r\,d\theta dr \neq 0, \text{ then }
\lim_{n \to \infty} \text{ess sup} |g_T^P| = \infty.
\]

Let \( T = U|T| \) and \( T^n = U_n |T^n| \) be the polar decompositions of \( T \) and \( T^n \), respectively. Then it holds that if \( [T^*, T] \in \mathcal{C}_1 \), then \( [T^n, T^n] \in \mathcal{C}_1 \) for any positive integer \( n \). On the other hand, in the polar decomposition case, it is not clear whether \([|T|, U] \in \mathcal{C}_1\) implies \([|T^n|, U_n] \in \mathcal{C}_1\) even if \( n = 2 \). If \( T \) is invertible and \([|T|, U] \in \mathcal{C}_1 \), then, for every \( n \), it holds \([|T^n|, U_n] \in \mathcal{C}_1 \) by [11, Theorem 3].

Next we consider operators with cyclic vectors. First we need the following result.

**Theorem 3.6 (Martin and Putinar, Th.X.4.3 [22]).** Let \( g_T \) and \( g_V \) be the principal functions of operators \( T \) and \( V \) such that \([T^*, T], [V^*, V] \in \mathcal{C}_1\), respectively. If there exists an operator \( A \in \mathcal{C}_1 \) such that \( AV = TA \) and \( \ker(A) = \ker(A^*) = \{0\} \), then \( g_T \sqsubset g_V \).

Proof of the following lemma is based on it of [22, Corollary X.4.4].

**Lemma 3.7.** Let \( T \) be an operator such that \([T^*, T] \in \mathcal{C}_1 \) and \( \sigma(T) \) is an infinite set. If \( T \) has a cyclic vector, then \( g_T \not\sqsubset 1 \).

Let \( S \) be an operator having the principal function \( g_S \) related to the Cartesian decomposition \( S = X + iY \). Then \( g_S^*(x, y) = -g_S(x, -y) \). Hence, as a corollary of Lemma 3.7, we have the following.

**Lemma 3.8.** Let \( T \) be an operator such that \([T^*, T] \in \mathcal{C}_1 \) and \( \sigma(T) \) is an infinite set. If \( T^* \) has a cyclic vector, then \(-1 \not\sqsubset g_T \).

**Theorem 3.9.** Let \( T \) be an operator such that \([T^*, T] \in \mathcal{C}_1 \) and \( \sigma(T) \) is an infinite set. If \( \int \int g_T(x, y) dx dy \neq 0 \), then, for a sufficiently high \( n \), \( T^n \) has a non-trivial invariant subspace.

**References**

1. A. Aluthge, On \( p \)-hyponormal operators for \( 0 < p < 1 \), Integr. Equat. Oper. Th. 13(1990), 307-315.