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Some Problems in Fourier Analysis and Matrix Theory

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We discuss some problems studied in diverse contexts but with a common theme: the use of Fourier analysis to evaluate norms of some special matrices.

Let $\mathbb{M}_n$ be the space of $n \times n$ matrices. For $A \in \mathbb{M}_n$ let

$$
\|A\| = \sup \{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\},
$$

be the usual operator norm of $A$. Let $A \circ X$ be the entrywise product of two matrices $A$ and $X$ and let

$$
\|A\|_S = \sup \{\|A \circ X\| : \|X\| = 1\}.
$$

This is the norm of the linear map on $\mathbb{M}_n$ defined as $X \mapsto A \circ X$. Since $A \circ X$ is a principal submatrix of $A \otimes X$, we have $\|A \circ X\| \leq \|A \otimes X\| = \|A\| \|X\|$, and hence

$$
\|A\|_S \leq \|A\|.
$$
Let $\lambda_1, \ldots, \lambda_n$ be distinct real numbers and let

$$\delta = \min_{i \neq j} |\lambda_i - \lambda_j|.$$

Let $H$ be the skew-symmetric matrix with entries $h_{rs}$ defined as

$$h_{rs} = \begin{cases} 
1/(&\lambda_r - \lambda_s) & r \neq s \\
0 & r = s.
\end{cases} \quad (1)$$

Motivated by problems arising in number theory, Montgomery and Vaughan [5] proved the following.

**Theorem 1.** The norm of the matrix $H$ is bounded as

$$\|H\| \leq c_1/\delta, \quad (2)$$

where

$$c_1 = \inf \left\{ \|\varphi\|_{L_1} : \varphi \in L_1(\mathbb{R}), \varphi \geq 0, \text{ and } \hat{\varphi}(\xi) = \frac{1}{\xi} \text{ for } |\xi| \geq 1 \right\}. \quad (3)$$

Here $\hat{\varphi}$ stands for the Fourier transform of $\varphi$. Further,

$$c_1 = \pi. \quad (4)$$

A very special case of this theorem is “Hilbert’s inequality”. Let $\lambda_j = j, \ j = 1, 2, \ldots$. Then the (infinite) matrix $H$ defined by (1) is called the Hilbert matrix. Hilbert showed that $H$ defines a bounded
operator on the space $\ell_2$ and $\|H\| < 2\pi$. This was improved upon by Schur who showed that $\|H\| = \pi$. Different proofs of this fact were discovered by others, one using Fourier series by Toeplitz. (Matrices structured as $H$ are now called Toeplitz matrices.)

In particular, this shows that the inequality (2) with $c_1 = \pi$ is sharp (in the sense that if it is to hold for all $n$, then no constant smaller than $\pi$ would work).

Now suppose we have two real $n$-tuples $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ where for all $i$ and $j$ we have $\lambda_i \neq \mu_j$. Let

$$\delta = \min_{i,j} |\lambda_i - \mu_j|.$$  

Let $M$ be the matrix with entries $m_{rs}$ defined as

$$m_{rs} = \frac{1}{\lambda_r - \mu_s}.$$  \hspace{1cm} (5)

Motivated by problems arising in perturbation theory, Bhatia, Davis and McIntosh [1] proved the following.

**Theorem 2.** The norm $\|M\|_S$ is bounded as

$$\|M\|_S \leq \frac{c_2}{\delta},$$  \hspace{1cm} (6)

where

$$c_2 = \inf \left\{ \|\varphi\|_{L_1} : \varphi \in L_1(\mathbb{R}), \varphi(\xi) = \frac{1}{\xi} \text{ for } |\xi| \geq 1 \right\}.$$  \hspace{1cm} (7)
The constant $c_2$ had been evaluated earlier by Sz-Nagy [6] and we have

$$c_2 = \frac{\pi}{2}. \tag{8}$$

Note that the infimum in (7) is over a class of functions larger than the one in (3).

It has been shown by McEachin [4] that the inequality (6) is sharp with $c_2 = \pi/2$, and the extremal value is attained when the points $\{\lambda_i\}$ and $\{\mu_j\}$ are regularly spaced.

The resemblance between the two problems is striking and it is a natural curiosity to ask whether good expressions for the norms $\|M\|$ and $\|H\|_S$ may be found to supplement what is known.

In [1] the authors considered also the case where $\{\lambda_i\}$ and $\{\mu_j\}$ are $n$-tuples of complex numbers with the same restriction as before, viz.,

$$\delta = \min_{i,j} |\lambda_i - \mu_j| > 0.$$

They proved the following.

**Theorem 3.** Let $M$ be the matrix (with complex entries) defined as in (5). Then

$$\|M\|_S \leq c_3/\delta, \tag{9}$$
where

\[ c_3 = \inf \left\{ \| \phi \|_{L_1} : \phi \in L_1(\mathbb{R}^2), \hat{\phi}(\xi_1, \xi_2) = \frac{1}{\xi_1 + i\xi_2} \text{ for } \xi_1^2 + \xi_2^2 \geq 1 \right\}. \]

(10)

An attempt to calculate the constant \( c_3 \) was made by Bhatia, Davis and Koosis [2]. These authors first obtained another characterisation of \( c_3 \). Let \( C \) be the class of all functions \( g \) on \( \mathbb{R} \) that satisfy the following conditions

(i) \( g \) is even,

(ii) \( g(x) = 0 \) for \( |x| \geq 1 \),

(iii) \( \int_{-1}^{1} g(x) dx = 1 \),

(iv) \( \hat{g} \in L_1(\mathbb{R}) \).

The following theorem was proved in [2]

**Theorem 4.**

\[ c_3 = \inf \left\{ \int_{0}^{\infty} |\hat{g}| : g \in C \right\}. \]

(11)

Using this the following estimate was derived in [2]

\[ c_3 \leq \frac{\pi}{2} \int_{0}^{\pi} \frac{\sin t}{t} dt < 2.90901. \]

(12)
The constant $c_2$ occurs in another context called Bohr's inequality. This says that if a function $f$ and its derivative $f'$ satisfy the following conditions

(i) $f \in L_1(\mathbb{R})$, $f' \in L_\infty(\mathbb{R})$,

(ii) $\hat{f}(\xi) = 0$ for $|\xi| \leq \delta$.

Then

$$\|f\|_\infty \leq \frac{c_2}{\delta} \|f'\|_\infty,$$  \hspace{1cm} (13)

and the inequality is sharp.

Attempts have been made to extend this result to functions of several variables. Hörmander and Bernhardsson [3] have shown that if $f$ is a function on $\mathbb{R}^2$ satisfying conditions akin to (i) and (ii) above, then

$$\|f\|_\infty \leq \frac{c_3}{\delta} \|\nabla f\|_\infty.$$  \hspace{1cm} (14)

With this motivation they tried to evaluate $c_3$. Like the authors of [2], they too first prove (11), and then use it more effectively to show that

$$2.903887282 < c_3 < 2.90388728275228.$$  \hspace{1cm} (15)

It would surely be of interest to find the exact value of $c_3$, especially since the formulas (4) and (8) are so attractive.

Some other problems remain open. The estimate (6) has been shown to be sharp by McEachin [4]. The question about (9) does not seem to
have been addressed. The matrix (5) when \( \{\lambda_i\} \) and \( \{\mu_i\} \) are points on the unit circle was considered in [1]. An extremal problem involving Fourier series instead of Fourier transforms as in (7) and (10) arises in this case. This too has not been solved.

References


