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An extension of Kantorovich inequality to $n$-operators

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ABSTRACT

In this report, we shall extend Kantorovich inequality. This is an estimate by using the geometric mean of $n$-operators which have been defined by Ando-Li-Mathias in [1]. As a related result, we obtain a reverse inequality of arithmetic-geometric means one of $n$-operators via Kantorovich constant. Moreover, we give a formula of geometric mean of $n$-touples of 2-by-2 matrices with a trace condition, and we shall obtain more precise results of extended Kantorovich inequality in case 2-by-2 matrices case.

This is based on the following preprint:

1. INTRODUCTION

In what follows a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator $T$ is said to be positive if $(Tx,x) \geq 0$ holds for all $x \in \mathcal{H}$. For an operator $T$ such that $0 < mI < T < MI$, the following inequality is called "Kantorovich inequality" [6, 7]:

\[
(Tx,x)(T^{-1}x,x) \leq \frac{(m+M)^2}{4mM} \quad \text{for } \|x\| = 1.
\]

We call the constant $\frac{(m+M)^2}{4mM}$ Kantorovich constant. (1.1) is closely related to properties of convex functions, and many authors have given many results and comments [3, 5, 9, 10, 12]. It is well known that (1.1) is equivalent to the following form by replacing $x$ with $\frac{T^{\frac{1}{2}}x}{\|T^{\frac{1}{2}}x\|}$ in (1.1):

\[
(T^2x,x) \leq \frac{(m+M)^2}{4mM}(Tx,x)^2 \quad \text{for } \|x\| = 1.
\]

For positive invertible operators $A$ and $B$, the geometric mean $A\#B$ of $A$ and $B$ is defined as follows [8]:

\[A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.\]

$A\#B$ is an extension of the geometric mean $\sqrt{ab}$ of positive numbers $a$ and $b$. It is well known that Kantorovich inequality is equivalent to the following inequality [2]: Let
A and B be positive invertible operators whose spectrums are contained in \([m, M]\) with \(0 < m < M\). Then

\[
(1.2) \quad \langle Ax, x \rangle \langle Bx, x \rangle \leq \frac{(m + M)^2}{4mM} \langle A^\# Bx, x \rangle^2 \quad \text{for } x \in \mathcal{H}.
\]

In this report, we call it “Kantorovich inequality of 2-operators.”

Very recently, as an extension of \(A^\# B\), the geometric mean \(G(A_1, A_2, \cdots, A_n)\) of \(n\)-touples of positive invertible operators \(A_i\) have been defined by T. Ando, C.-K. Li and R. Mathias [1] as follows:

**Definition 1** (Geometric mean of \(n\)-operators [1]). Let \(A_i\) be positive invertible operators for \(i = 1, 2, \cdots, n\). Then the geometric mean \(G(A_1, A_2, \cdots, A_n)\) is defined by induction as follows:

(i) \(G(A_1, A_2) = A_1^\# A_2\).

(ii) Assume that the geometric mean of any \(n - 1\)-tuple of operators is defined. Let

\[
G((A_j)_{j \neq i}) = G(A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n),
\]

and let sequences \(\{A_i^{(r)}\}_{r=0}^\infty\) be \(A_i^{(0)} = A_i\) and \(A_i^{(r)} = G((A_j^{(r-1)})_{j \neq i})\). If there exists \(\lim_{r \to \infty} A_i^{(r)}\), and it does not depend on \(i\), then we define the geometric mean of \(n\)-operators as

\[
\lim_{r \to \infty} A_i^{(r)} = G(A_1, A_2, \cdots, A_n).
\]

In [1], it has been shown that for any positive invertible operators \(A_i\) for \(i = 1, 2, \cdots, n\), there exists \(\lim_{r \to \infty} A_i^{(r)}\) and

\[
\lim_{r \to \infty} A_i^{(r)} = G(A_1, A_2, \cdots, A_n),
\]

uniformly. In fact, they have shown it for \(n\)-matrices in [1]. But by their proof, we can understand that the result can be extended to Hilbert space operators.

The geometric mean defined above has the following properties in [1]:

(P1) Consistency with scalars. If \(A_i\) commute with each other, then

\[
G(A_1, A_2, \cdots, A_n) = (A_1 A_2 \cdots A_n)^{\frac{1}{n}}.
\]

(P2) Joint homogeneity. For positive numbers \(s_i\),

\[
G(s_1 A_1, s_2 A_2, \cdots, s_n A_n) = (s_1 s_2 \cdots s_n)^{\frac{1}{n}} G(A_1, A_2, \cdots, A_n).
\]

(P3) Permutation invariance. For any permutation \(\pi(A_1, A_2, \cdots, A_n)\) of \((A_1, A_2, \cdots, A_n)\),

\[
G(\pi(A_1, A_2, \cdots, A_n)) = G(A_1, A_2, \cdots, A_n).
\]

(P4) Monotonicity. If \(A_i \geq B_i > 0\), then \(G(A_1, A_2, \cdots, A_n) \geq G(B_1, B_2, \cdots, B_n)\).

(P5) Continuity above. For each \(i\), if \(\{A_{i,k}\}_{k=1}^\infty\) are monotonic decreasing sequences converging to \(A_i\) as \(k \to \infty\), respectively, then

\[
\lim_{k \to \infty} G(A_{1,k}, A_{2,k}, \cdots, A_{n,k}) = G(A_1, A_2, \cdots, A_n)\}.
\]

(P6) Congruence invariance. For an invertible operator \(S\),

\[
G(S^* A_1 S, S^* A_2 S, \cdots, S^* A_n S) = S^* G(A_1, A_2, \cdots, A_n) S.
\]
Joint concavity. The map \((A_1, A_2, \cdots, A_n) \mapsto G(A_1, A_2, \cdots, A_n)\) is jointly concave, i.e., for \(0 < \lambda < 1\),
\[
G(\lambda A_1 + (1 - \lambda) B_1, \lambda A_2 + (1 - \lambda) B_2, \cdots, \lambda A_n + (1 - \lambda) B_n)
\geq \lambda G(A_1, A_2, \cdots, A_n) + (1 - \lambda) G(B_1, B_2, \cdots, B_n).
\]

Self-duality. \(G(A_1, A_2, \cdots, A_n) = G(A_1^{-1}, A_2^{-1}, \cdots, A_n^{-1})^{-1}\).

Determinant identity. For positive invertible matrices \(A_i\),
\[
det(A_1, A_2, \cdots, A_n) = (\det A_1 \cdot \det A_2 \cdots \det A_n)^{\frac{1}{n}}.
\]

Moreover, \(G(A_1, A_2, \cdots, A_n)\) satisfies the arithmetic-geometric means inequality:
\[
G(A_1, A_2, \cdots, A_n) \leq \frac{A_1 + A_2 + \cdots + A_n}{n}.
\]

For positive numbers \(a_i\), as a reverse inequality of arithmetic-geometric means one, it is known the following inequality [11]: For positive numbers \(a_i\) with \(0 < m < a_i < M\),
\[
\frac{a_1 + a_2 + \cdots + a_n}{n} \leq S_h \sqrt[n]{a_1 a_2 \cdots a_n},
\]
holds, where \(h = \frac{m}{M} > 1\) and \(S_h = \frac{(h - 1)h^{\frac{1}{h-1}}}{e \log h}\). We call \(S_h\) the Specht’s ratio, and there are a lot of properties of Kantorovich constant and Specht’s ratio in [3, 4, 5]. We remark that Specht’s ratio in (1.3) is the optimal constant.

In this report, we shall give an extension of Kantorovich inequality of 2-operators to one of \(n\)-operators via geometric mean by Ando-Li-Mathias. As a related result of it, we shall discuss on an extension of (1.3). These results are estimates via Kantorovich constant. Next, we shall show more precise estimations of them under some cases.

2. MAIN RESULTS

**Theorem 2.1.** Let \(A_i\) be positive operators for \(i = 1, 2, \cdots, n\) satisfying \(0 < mI \leq A_i \leq MI\) with \(m < M\). Then
\[
\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \left\{\frac{(m + M)^2}{4mM}\right\}^{\frac{n-1}{2}} G(A_1, A_2, \cdots, A_n).
\]

**Theorem 2.2.** Let \(A_i\) be positive operators for \(i = 1, 2, \cdots, n\) satisfying \(0 < mI \leq A_i \leq MI\) with \(0 < m < M\). Then
\[
\langle A_1 x, x \rangle \langle A_2 x, x \rangle \cdots \langle A_n x, x \rangle \leq \left\{\frac{(m + M)^2}{4mM}\right\}^{\frac{n(n-1)}{2}} \langle G(A_1, A_2, \cdots, A_n) x, x \rangle^n
\]
holds for all \(x \in \mathcal{H}\).

**Remark.** In [1], the following inequality has been already shown: For positive invertible operators \(A_i\),
\[
\langle G(A_1, A_2, \cdots, A_n) x, x \rangle^n \leq \langle A_1 x, x \rangle \langle A_2 x, x \rangle \cdots \langle A_n x, x \rangle.
\]
Hence Theorem 2.2 is a reverse inequality of the above one.
For positive invertible operators $A$ and $B$, as a kind of distance between $A$ and $B$, the following $R(A, B)$ is defined in [1]:

$$R(A, B) = \max\{r(A^{-1}B), r(B^{-1}A)\},$$

where $r(T)$ means the spectral radius of $T$. Especially, the following inequality holds:

$$R(A^{(1)}_i, A^{(1)}_k) = R(G((A_j)_{j\neq i}), G((A_j)_{j\neq k})) \leq R(A_i, A_k)^{\frac{1}{n-1}}. \quad (2.1)$$

To prove the above theorems, we shall show the following lemma:

**Lemma 2.3.** Let $A_i$ be positive invertible operators for $i = 1, 2, \cdots, n$, and $h = \max_{i,j} R(A_i, A_j)$. Then

$$\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \left(\frac{1+h}{2\sqrt{h}}\right)^{n-1} G(A_1, A_2, \cdots, A_n).$$

**Proof.** Here we shall introduce the proof of the cases $n = 2$ and 3. The complete proof is obtained in [Y].

In case $n = 2$. Let $X = A^{\frac{1}{2}}BA^{\frac{1}{2}}$, and

$$X = \int \lambda dE_\lambda$$

be the spectral decomposition of $X$. Since $h = R(A, B)$, then we have $\frac{1}{h} \leq \lambda \leq h$ and

$$\frac{1+X}{2} = \int \frac{1+\lambda}{2} dE_\lambda = \int \frac{1+\lambda}{2\sqrt{\lambda}} \sqrt{\lambda} dE_\lambda \leq \int \frac{1+h}{2\sqrt{h}} \sqrt{\lambda} dE_\lambda = \frac{1+h}{2\sqrt{h}} X^{\frac{1}{2}}.$$

Hence we have

$$\frac{1 + A^{\frac{1}{2}}BA^{\frac{1}{2}}}{2} \leq \frac{1+h}{2\sqrt{h}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}.$$

Multiplying $A^{\frac{1}{2}}$ to both sides of this inequality we have

$$\frac{A+B}{2} \leq \frac{1+h}{2\sqrt{h}} A^{\frac{1}{2}}B = \frac{1+h}{2\sqrt{h}} G(A, B).$$

Next we shall prove the case $n = 3$. For a nonnegative integer $r$, we define $A_r, B_r, C_r, h_r$ and $K_r$ as follows:

$$A_0 = A \text{ and } A_r = G(B_{r-1}, C_{r-1}),$$

$$B_0 = B \text{ and } B_r = G(C_{r-1}, A_{r-1}),$$

$$C_0 = C \text{ and } C_r = G(A_{r-1}, B_{r-1}),$$

$$h_0 = h \text{ and } h_r = \max\{R(A_r, B_r), R(B_r, C_r), R(C_r, A_r)\},$$

$$K_r = \frac{1+h_r}{2\sqrt{h_r}}. \quad (2.2)$$
Then by the case \( n = 2 \), we have

\[
\frac{A + B + C}{3} = \frac{1}{3} \left( \frac{A_0 + B_0}{2} + \frac{B_0 + C_0}{2} + \frac{C_0 + A_0}{2} \right) \\
\leq \frac{1}{3} (K_0G(A_0, B_0) + K_0G(B_0, C_0) + K_0G(C_0, A_0)) \\
= K_0 \frac{A_1 + B_1 + C_1}{3} \\
\leq K_0K_1 \frac{A_2 + B_2 + C_2}{3} \\
\vdots \\
\leq K_0K_1 \cdots K_r \frac{A_{r+1} + B_{r+1} + C_{r+1}}{3}
\]

Since

\[
\lim_{r \to \infty} A_r = G(A, B, C),
\]

we have

\[
\lim_{r \to \infty} \frac{A_{r+1} + B_{r+1} + C_{r+1}}{3} = G(A, B, C).
\]

So we have only to prove the following inequality:

\[
\lim_{r \to \infty} K_0K_1 \cdots K_r \leq K_0^2.
\]

By (2.1), we have

\[
1 \leq h_r \leq h_{r-1}^{\frac{1}{r}} \leq \cdots \leq h_0^{\left(\frac{1}{r}\right)}.
\]

Since

\[
\frac{1}{2} \left( \frac{1}{x} + x \right) \leq \frac{1}{2} \left( \frac{1}{y^\alpha} + y^\alpha \right) \leq \left\{ \frac{1}{2} \left( \frac{1}{y} + y \right) \right\}^\alpha
\]

holds for \( 1 \leq x \leq y^\alpha \) and \( \alpha \in (0, 1] \), we have

\[
K_r = \frac{1 + h_r}{2\sqrt{h_r}} = \frac{1}{2} \left( \frac{1}{\sqrt{h_0} + \sqrt{h_r}} \right) \leq \left\{ \frac{1}{2} \left( \frac{1}{\sqrt{h_0} + \sqrt{h_0}} \right) \right\}^{\left(\frac{r}{2}\right)} = K_0^{\left(\frac{1}{r}\right)}.
\]

Therefore we obtain

\[
K_0K_1 \cdots K_r \leq K_0^{1 + \frac{1}{2} + \cdots + \left(\frac{1}{r}\right)} \to K_0^2 \quad \text{as } r \to \infty.
\]

Hence we have

\[
\frac{A + B + C}{3} \leq \left( \frac{1 + h}{2\sqrt{h}} \right)^2 G(A, B, C).
\]

This completes the proof.

\[
\square
\]

**Proof of Theorem 2.1.** By putting \( h = \frac{M}{m} \) in Lemma 2.3. we obtain Theorem 2.1. \( \square \)
Proof of Theorem 2.2. By using Theorem 2.1 and arithmetic-geometric means inequality, we have

\[
\prod_{i=1}^{n} \langle A_i x, x \rangle^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} \langle A_i x, x \rangle \\
= \left( \frac{1}{n} \sum_{i=1}^{n} A_i x, x \right) \\
\leq \left\{ \frac{(m+M)^2}{4mM} \right\}^{\frac{n-1}{2}} \langle G(A_1, A_2, \cdots, A_n) x, x \rangle.
\]

This completes the proof. □

3. MORE PRECISE ESTIMATIONS

In this section, we shall give more precise estimations than the results shown in section 2 under some cases.

Theorem 3.1. Let \(A, B, C\) be positive operators whose spectrums are contained in \([m, M]\) with \(0 < m < M\). Then

\[
\frac{A + B + C}{3} \leq \frac{h^2 - 1}{2h \log h} G(A, B, C),
\]

where \(h = \frac{M}{m} > 1\).

Proof. As in the proof of Lemma 2.3, we have

\[
\frac{A + B + C}{3} \leq K_0 K_1 \cdots K_r \frac{A_{r+1} + B_{r+1} + C_{r+1}}{3},
\]

where

\[
K_r = \frac{h_r + 1}{2\sqrt{h_r}} \quad \text{and} \quad h_r = \max\{R(A_r, B_r), R(B_r, C_r), R(C_r, A_r)\}.
\]

By (2.1), \(1 \leq h_r \leq h_{r-1}^{\frac{1}{r}} \leq \cdots \leq h_{\frac{1}{2}}^{\frac{1}{2^{r-1}}}, \) and we obtain

\[
K_r = \frac{1}{2} \left( \frac{1}{h_r^{\frac{1}{2}}} + h_r^{\frac{1}{2}} \right) \leq \frac{1}{2} \left( \frac{1}{h^{\frac{1}{2^{r+1}}}} + h^{\frac{1}{2^{r+1}}} \right) = \frac{h^{\frac{1}{2}} + 1}{2h^{\frac{1}{2^{r+1}}}}.
\]

Hence we have

\[
H_0 K_1 \cdots K_r \leq \frac{h + 1}{2h^{\frac{1}{2}}} \cdot \frac{h^{\frac{1}{2}} + 1}{2h^{\frac{1}{4}}} \cdots \frac{h^{\frac{3}{2}} + 1}{2h^{\frac{3}{4}}} \\
= \frac{h + 1}{2h^{\frac{1}{2}}} \cdot \frac{h^{\frac{1}{2}} + 1}{2h^{\frac{1}{4}}} \cdots \frac{h^{\frac{3}{2}} + 1}{2h^{\frac{3}{4}}} \frac{h^{\frac{1}{2^{r}} - 1}}{2h^{\frac{1}{2^{r+1}}}} \\
= \frac{h^2 - 1}{2h^{1 - \frac{1}{2^{r+1}}}} \cdot \frac{h^{\frac{1}{2^{r}} - 1}}{2h^{\frac{1}{2^{r}}}} \\
\rightarrow \frac{h^2 - 1}{2h \log h} \quad \text{as} \ n \rightarrow \infty,
\]

where the limit is given by \(\lim_{n \to \infty} n(h^{\frac{1}{n}} - 1) = \log h\).
This completes the proof. \(\square\)

**Theorem 3.2.** Let \(A, B, C\) be positive invertible operators whose spectrums are contained in \([m, M]\) with \(0 < m < M\). Then

\[
\langle Ax, x \rangle \langle Bx, x \rangle \langle Cx, x \rangle \leq \left( \frac{h^3 - 1}{2h \log h} \right)^3 \langle G(A, B, C)x, x \rangle^3,
\]

where \(h = \frac{M}{m} > 1\).

Theorem 3.2 is easily obtained by the same way to the proof of Theorem 2.2.

**Remark.** In Theorem 3.1, we obtain a more precise constant \(\frac{h^2 - 1}{2h \log h}\) than Theorem 2.1. However, this is not less than the Specht's ratio in (1.3) as follows: First of all, we shall show

\[(3.1) \quad f(h) = (h - 1) \log(h + 1) - (h - 1) \log 2 - h \log(h + 1) \geq 0 \quad \text{for} \quad h \geq 1.
\]

By easy calculation, we have

\[
f'(h) = \log(h + 1) - \log h - \frac{2}{h + 1} + 1 - \log 2
\]

\[
f''(h) = \frac{h - 1}{h(h + 1)^2} \geq 0 \quad \text{for} \quad h \geq 1.
\]

Since \(f'(1) = 0\) and \(f''(h) \geq 0\) holds for \(h \geq 1\), \(f'(h) \geq 0\) for \(h \geq 1\). Then by \(f(1) = 0\) and \(f'(h) \geq 0\) for \(h \geq 1\), we have (3.1).

Next, (3.1) is equivalent to

\[
\frac{h}{h - 1} \log h - 1 \leq \log \left( \frac{h + 1}{2} \right),
\]

i.e.,

\[
\frac{h^{1 \frac{1}{h-1}}}{e} \leq \frac{h + 1}{2h} \quad \text{for} \quad h \geq 1.
\]

Hence we obtain

\[
S_h = \frac{h - 1}{\log h} \cdot \frac{h^{1 \frac{1}{h-1}}}{e} \leq \frac{h - 1}{\log h} \cdot \frac{h + 1}{2h} = \frac{h^2 - 1}{2h \log h}.
\]

The next theorem is a formula of geometric mean of \(n\)-touples of 2-by-2 matrices.

**Theorem 3.3.** Let \(A_i\) be positive 2-by-2 matrices satisfying the following conditions: (i) \(\det A_i = 1\) (ii) \(\text{tr}(A_i^{-1} A_j) = c\) (constant) for \(i, j = 1, 2, \cdots, n\). Then

\[
G(A_1, A_2, \cdots, A_n) = \frac{A_1 + A_2 + \cdots + A_n}{\sqrt{\det(A_1 + A_2 + \cdots + A_n)}}.
\]

In [1], the formula of geometric mean of \(2\)-touples of 2-by-2 matrices has been shown, and Theorem 3.3 is an extension of it. To prove the result, we prepare the following lemma:
Lemma 3.4. Let $A_i$ be positive 2-by-2 matrices with $\det A_i = 1$ for $i = 1, 2, \cdots, n$. Then
\[
\det(A_1 + A_2 + \cdots + A_n) = n + \sum_{1 \leq i < j \leq n} \text{tr}(A_i^{-1}A_j).
\]
Especially, if $\text{tr}(A_i^{-1}A_j) = c$ (constant) for $i, j = 1, 2, \cdots, n$, then
\[
(3.2) \quad \det(A_1 + A_2 + \cdots + A_n) = n + \frac{n(n-1)}{2}c.
\]
Proof. Here, we shall introduce the proof in cases $n = 2$ and $3$. Let
\[
A = \begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 \\ b_2 & d_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a_3 & b_3 \\ b_3 & d_3 \end{pmatrix}.
\]
In case $n = 2$. Since $\det A = \det B = 1$, we have
\[
\det(A + B) = (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)^2
= (a_1d_1 - b_1^2) + (a_2d_2 - b_2^2) + (a_1d_2 + a_2d_1 - 2b_1b_2)
= 2 + \text{tr}(A^{-1}B).
\]
Next we shall show the case $n = 3$. By the case $n = 2$ and $\det C = 1$, we have
\[
\det(A + B + C)
= (a_1 + a_2 + a_3)(d_1 + d_2 + d_3) - (b_1 + b_2 + b_3)^2
= (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)^2
\quad + d_3(a_1 + a_2) + a_3(d_1 + d_2) - 2b_3(b_1 + B_2) + a_3d_3 - b_3^2
= 2 + \text{tr}(A^{-1}B) + (a_1d_3 + a_3d_1 - 2b_1b_3) + (a_3d_3 + a_3d_2 - 2b_2b_3) + 1
= 3 + \text{tr}(A^{-1}B) + \text{tr}(A^{-1}C) + \text{tr}(B^{-1}C).
\]
It completes the proof. $\square$

Proof of Theorem 3.3. Here we will prove it the case $n = 3$. The case $n = 2$ have been proven in [1].

Let $A_r$, $B_r$ and $C_r$ be the geometric means which have been introduced in (2.2). Firstly, we will prove that they can be written as the following form:
\[
A_r = \alpha_r A + \beta_r B + \beta_r C \quad \text{(3.3)}
B_r = \beta_r A + \alpha_r B + \beta_r C
C_r = \beta_r A + \beta_r B + \alpha_r C
\]
In case $r = 1$, by the case $n = 2$ and (3.2) in Lemma 3.4, we have
\[
A_1 = G(B, C) = \frac{B + C}{\sqrt{\det(B + C)}} = \frac{B + C}{\sqrt{2 + c}},
B_1 = G(C, A) = \frac{C + A}{\sqrt{\det(C + A)}} = \frac{C + A}{\sqrt{2 + c}},
C_1 = G(A, B) = \frac{A + B}{\sqrt{\det(A + B)}} = \frac{A + B}{\sqrt{2 + c}}.
\]
Hence we have only to set $\alpha_1$ and $\beta_1$ as follows:

$$\alpha_1 = 0 \text{ and } \beta_1 = \frac{1}{\sqrt{2+c}}.$$

Assume that $A_{r-1}$, $B_{r-1}$, $C_{r-1}$ can be written as the following form:

$$A_{r-1} = \alpha_{r-1}A + \beta_{r-1}B + \beta_{r-1}C,$$

$$B_{r-1} = \beta_{r-1}A + \alpha_{r-1}B + \beta_{r-1}C,$$

$$C_{r-1} = \beta_{r-1}A + \beta_{r-1}B + \alpha_{r-1}C.$$  \hspace{1cm} (3.4)

By the case $n = 2$, we have

$$A_r = \frac{B_{r-1} + C_{r-1}}{\sqrt{\det(B_{r-1} + C_{r-1})}},$$

$$B_r = \frac{C_{r-1} + A_{r-1}}{\sqrt{\det(C_{r-1} + A_{r-1})}},$$

$$C_r = \frac{A_{r-1} + B_{r-1}}{\sqrt{\det(A_{r-1} + B_{r-1})}}.$$

(3.5)

We will show that $\det(A_{r-1} + B_{r-1}) = \det(B_{r-1} + C_{r-1}) = \det(C_{r-1} + A_{r-1})$. Note that by (P9), $\det A_{r-1} = \det B_{r-1} = \det C_{r-1} = 1$ and (3.4), we have

$$\{A_{r-1}\}^{-1} = \alpha_{r-1}A^{-1} + \beta_{r-1}B^{-1} + \beta_{r-1}C^{-1},$$

$$\{B_{r-1}\}^{-1} = \beta_{r-1}A^{-1} + \alpha_{r-1}B^{-1} + \beta_{r-1}C^{-1},$$

$$\{C_{r-1}\}^{-1} = \beta_{r-1}A^{-1} + \beta_{r-1}B^{-1} + \alpha_{r-1}C^{-1}.$$  \hspace{1cm} (3.6)

Since $\text{tr}(A^{-1}B) = \text{tr}(B^{-1}C) = \text{tr}(C^{-1}A) = c$, we have

$$\text{tr}(\{A_{r-1}\}^{-1}B_{r-1}) = \text{tr}(\{\alpha_{r-1}A^{-1} + \beta_{r-1}B^{-1} + \beta_{r-1}C^{-1}\} \{\beta_{r-1}A + \alpha_{r-1}B + \beta_{r-1}C\})$$

$$= c\alpha_{r-1}^2 + (4 + 2c)\alpha_{r-1}\beta_{r-1} + (1 + c)\beta_{r-1}^2,$$

and also we can set

$$\text{tr}(\{A_{r-1}\}^{-1}B_{r-1}) = \text{tr}(\{A_{r-1}\}^{-1}C_{r-1}) = \text{tr}(\{B_{r-1}\}^{-1}C_{r-1}) = c_{r-1}.$$  \hspace{1cm} (3.2)

By Lemma 3.4, we have

$$\det(A_{r-1} + B_{r-1}) = \det(B_{r-1} + C_{r-1}) = \det(C_{r-1} + A_{r-1}) = 2 + c_{r-1}.$$  \hspace{1cm} (3.5)

Hence by (3.5), we obtain

$$A_r = G(B_{r-1}, C_{r-1})$$

$$= \frac{B_{r-1} + C_{r-1}}{\sqrt{\det(B_{r-1} + C_{r-1})}}$$

$$= \frac{2\beta_{r-1}A + (\alpha_{r-1} + \beta_{r-1})B + (\alpha_{r-1} + \beta_{r-1})C}{\sqrt{2 + c_{r-1}}}.$$  \hspace{1cm} (3.7)

Here we set

$$\alpha_r = \frac{2\beta_{r-1}}{\sqrt{2 + c_{r-1}}} \text{ and } \beta_r = \frac{\alpha_{r-1} + \beta_{r-1}}{\sqrt{2 + c_{r-1}}}.$$  \hspace{1cm} (3.8)

Then

$$A_r = \alpha_rA + \beta_rB + \beta_rC.$$
Similarly, we have (3.3).

Next, it has been shown that
\[
\lim_{r \to \infty} A_r = \lim_{r \to \infty} B_r = \lim_{r \to \infty} C_r = G(A, B, C).
\]
Hence by (3.3), we have
\[
\lim_{r \to \infty} \alpha_r = \lim_{r \to \infty} \beta_r = \alpha > 0,
\]
and
\[
G(A, B, C) = \alpha(A + B + C).
\]
Here by (P9), \( \det(G(A, B, C)) = 1 \), and we have
\[
G'(A, B, C) = \frac{A + B + C}{\sqrt{\det(A + B + C)}}.
\]

By Theorem 3.3, we have an extension of Kantorovich inequality of \( n \)-touple of 2-by-2 matrices which is a more precise estimation than Theorem 2.1.

**Theorem 3.5.** Let \( A_i \) be positive 2-by-2 matrices for \( i = 1, 2, \cdots, n \) satisfying \( \det A_i = 1 \), \( \text{tr}(A_i^{-1} A_j) = c \) (constant) and \( 0 < mI \leq A_i \leq MI \) with \( m < M \). Then
\[
\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \left( \frac{m + M}{4mM} \right)^2 G(A_1, A_2, \cdots, A_n).
\]

**Theorem 3.6.** Let \( A_i \) be positive 2-by-2 matrices for \( i = 1, 2, \cdots, n \) satisfying \( \det A_i = 1 \), \( \text{tr}(A_i^{-1} A_j) = c \) (constant) and \( 0 < mI \leq A_i \leq MI \) with \( m < M \). Then
\[
\langle A_1x, x \rangle \langle A_2x, x \rangle \cdots \langle A_nx, x \rangle \leq \left( \frac{(m + M)^2}{4mM} \right)^n \langle G(A_1, A_2, \cdots, A_n)x, x \rangle^n
\]
holds for all \( x \in \mathbb{C}^2 \).

To prove above results, we give the following inequality:

**Lemma 3.7.** Let \( A_i \) be positive 2-by-2 matrices satisfying \( \det A_i = 1 \) and \( 0 < mI \leq A_i \leq MI \) with \( m < M \). Then
\[
\det \left( \frac{A_1 + A_2 + \cdots + A_n}{n} \right) \leq \left( \frac{(m + M)^2}{4mM} \right)^2.
\]

**Proof.** For each \( i \), let \( 0 < m_i I \leq A_i \leq M_i I \) and \( M = \max_i M_i \). Note that we have
\[
m_i M_i = 1 \quad \text{and} \quad m = \frac{1}{M} \quad \text{by} \quad \det A_i = 1.
\]
Let \( A_i = \begin{pmatrix} a_i & b_i \\ b_i & d_i \end{pmatrix} \), and let
\[
S = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad T = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \quad \text{and} \quad x = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.
\]
Then we have $0 < mI \leq S \leq MI$ and $0 < mI \leq T \leq MI$ and
\[
\det \left( \frac{A_1 + A_2 + \cdots + A_n}{n} \right) \leq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \left( \frac{d_1 + d_2 + \cdots + d_n}{n} \right)
\]
\[
= \langle Sx, x \rangle \langle Tx, x \rangle
\]
\[
\leq \frac{(m + M)^2}{4mM} \langle S^T \|T^T x, x \rangle^2 \quad \text{by (1.2)}
\]
\[
= \frac{(m + M)^2}{4mM} \left( \frac{\sqrt{a_1 d_1} + \sqrt{a_2 d_2} + \cdots + \sqrt{a_n d_n}}{n} \right)^2.
\]

Here by $M_i \leq M$ and $m = \frac{1}{M}$,
\[
\sqrt{a_i d_i} \leq \frac{a_i + d_i}{2} = \frac{\text{tr} A_i}{2} = \frac{m_i + M_i}{2} \leq \frac{1}{2} \left( \frac{1}{M_i} + M_i \right) = \frac{m + M}{2}.
\]

Therefore we obtain
\[
\det \left( \frac{A_1 + A_2 + \cdots + A_n}{n} \right) \leq \frac{(m + M)^2}{4mM} \left( \frac{\sqrt{a_1 d_1} + \sqrt{a_2 d_2} + \cdots + \sqrt{a_n d_n}}{n} \right)^2
\]
\[
\leq \frac{(m + M)^2}{4mM} \left( \frac{m + M}{2} \right)^2
\]
\[
= \left\{ \frac{(m + M)^2}{4mM} \right\}^2 \quad \text{by } mM = 1.
\]

It completes the proof.

\[
\square
\]

**Proof of Theorem 3.5.** By Theorem 3.3 and Lemma 3.7, we have
\[
\frac{A_1 + A_2 + \cdots + A_n}{n} = \frac{\sqrt{\det (A_1 + A_2 + \cdots + A_n)} G(A_1, A_2, \cdots, A_n)}{n}
\]
\[
= \sqrt{\det \left( \frac{A_1 + A_2 + \cdots + A_n}{n} \right) G(A_1, A_2, \cdots, A_n)}
\]
\[
\leq \frac{(m + M)^2}{4mM} G(A_1, A_2, \cdots, A_n).
\]

\[
\square
\]

**Proof of Theorem 3.6.** It is the same as one of Theorem 2.2.

**Remark.** It is not known whether the constant $\frac{(m + M)^2}{4mM}$ in Theorem 3.5 is optimal or not. But in [5, p. 224, Remark 8.1], it is known that for $0 < m < M$ and $h = \frac{M}{m} > 1$,
\[
S_h \leq \frac{(m + M)^2}{4mM},
\]
i.e., the constant $\frac{(m + M)^2}{4mM}$ in Theorem 3.5 is bigger than one of (1.3).
REFERENCES


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