数理解析研究所講究録 1459 巻 2005 年 51-56

NON-DEGENERATE BILINEAR FORMS AND FIBER FUNCTORS

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ABSTRACT. We shall review a categorical approach to the basic quantum group $SL_q(2,\mathbb{C})$ or $SU_q(2)$ based on the preprint S. Yamagami, Fiber functors on Temperley-Lieb categories, arXiv:math.QA/0405517.

1. CLASSICAL TANNAKA DUALITY

The notion of tensor category is traced back to the celebrated work of T. Tannaka on a duality theory of compact groups.

Given a compact group G, let

$$\mathcal{R}ep(G) = \text{the category of finite-dimensional unitary representations of } G,$$

which is referred to as the **Tannaka dual** of G and provides a typical example of tensor categories:

- Given two objects (representations) V, W, another object $V \otimes W$ is associated functorially so that $(U \otimes V) \otimes W = U \otimes (V \otimes W)$.
- There is a special object I, the trivial representation of G, which satisfies $I \otimes V = V \otimes I$ for any V.
- We have the operation of taking constragradient representation $V \to V^*$, which is categorically characterized by the existence of morphisms $\epsilon : V \otimes V^* \to I$ and $\delta : I \to V^* \otimes V$ such that the compositions

$$V \xrightarrow{1_V \otimes \delta} V \otimes V^* \otimes V \xrightarrow{\epsilon \otimes 1_V} V,$$
$$V^* \xrightarrow{\delta \otimes 1_{V^*}} V^* \otimes V \otimes V^* \xrightarrow{1_{V^*} \otimes \epsilon} V^*$$

  are identities.

The Tannaka dual has the special ferature that it is realized as a subcategory of $\mathcal{V}ec$, the category of finite-dimensional vector spaces, or $\mathcal{H}ilb$, the category of finite-dimensional Hilbert spaces.

The celebrated **Tannaka duality** states that the group G itself is recovered by looking at the categorical information on representations. Here are some important generalizations to quantum groups.

Unitary version (Woronowicz, 1988):

Compact quantum groups

$\Longleftrightarrow$ rigid C*-tensor categories $\subset \mathcal{H}ilb$.

Algebraic version (Ulbrich, 1990):

Algebraic quantum groups

$\Longleftrightarrow$ rigid tensor categories $\subset \mathcal{V}ec$.

Typical examples are $SU_q(2)$ ($q \in \mathbb{R}^\times$) and $SL_q(2, \mathbb{C})$ ($q \in \mathbb{C}^\times$).

2. Fiber Functors

We shall here split the relevant information in Tannaka duality into two parts. Given an abstract tensor category $\mathcal{T}$, a **fiber functor** on $\mathcal{T}$ is, by definition, a faithful tensor fucntor $F : \mathcal{T} \to \mathcal{V}ec$: Each object X, which is conisdered to an abstract label, produces a finite-dimensional vector space $F(X)$ in such a way that $F(X \otimes Y) = F(X) \otimes F(Y)$. In other words, a fiber functor is a kind of representation of an abstract tensor category $\mathcal{T}$ in terms of the concrete tensor category $\mathcal{V}ec$ or $\mathcal{H}ilb$.

Viewing quantum groups this way, we are naturally lead to the problem of their representation theoretical classifications: Again we have two stages.

- Classify abstract tensor categories.
- Classify fiber functors on a tensor category up to equivalences.

Here 'equivalences' are with respect to a natural equivalence, say $\{\varphi_X\}$, preserving tensor products.

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi_X} & F'(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F'(f)} \\ F(Y) & \xrightarrow[\varphi_Y]{} & F'(Y) \end{array}, \qquad \varphi_{X \otimes Y} = \varphi_X \otimes \varphi_Y.$$

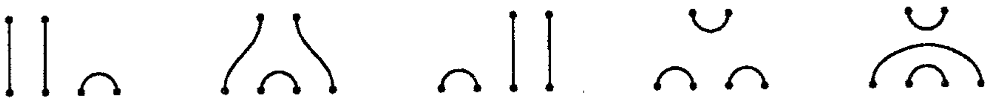
3. Temperley-Lieb Categories

Generally classification is a difficult problem for tensor categories because it consists of determining moduli for non-linear equations. We shall here restrict ourselves to the simple but fundamental case of Temperley-Lieb category $\mathcal{TL}_d$ ($d \in C^\times$ being a complex parameter), which is the linearization of the monoidal category of Kauffman's monoids and turns out to be the representation category of the quantum group $SL_q(2,\mathbb{C})$.

$$\mathcal{TL}_d \equiv \mathcal{R}ep(\mathrm{SL}_q(2,\mathbb{C})), \quad \text{with } d = -q - q^{-1}.$$

By definition, **Kauffman's monoids** are isotopy classes of planar strings stretched out between upper and lower boundaries of a strip. Let $K_{m,n}$ be the set of Kauffman's monoids with m and n vertices placed on upper and lower boundaries respectively. Here is a figure for the case $K_{2,4}$.

$K_{2,4}$:

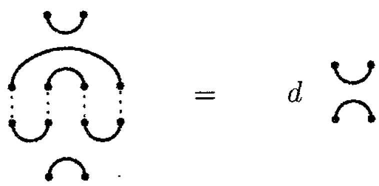


Objects in $\mathcal{TL}_d$ are labeled by the natural numbers $\{0, 1, 2, \dots\}$ with hom-sets given by $\mathcal{TL}_d(n,m) = \mathbb{C}[K_{m,n}]$, the free vector space generated by the set $K_{m,n}$. We regard each n as representing n-th tensor product of the object V labeled by 1:

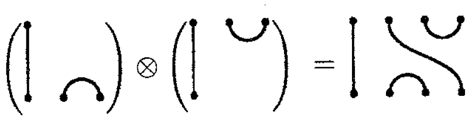
$$n \iff \overbrace{V \otimes \cdots \otimes V}^{n\text{-times}}.$$

The structure of tensor category is then defined as follows:

- The operation of composition is given by the concatenation of monoids with each closed circle replaced by the complex number d.



- Tensor product is defined by the horizontal juxtaposition of base monoids:



Note here that

$$\mathcal{TL}_d \not\cong \mathcal{TL}_{d'} \quad \text{if } d \neq d'.$$

With this geometrical presentation of Temperley-Lieb categories, we have

Theorem

$$\bigsqcup_{d\in\mathbb{C}^\times} (\{\text{fiber functors on } \mathcal{TL}_d\} / \sim)$$
$$\cong$$
$$\{\text{non-degenerate bilinear forms}\} / \sim$$
$$B \sim B' \iff B' = {}^tTBT \quad \text{for some inbertible linear map } T.$$

Basic arcs in Kauffman's monoids are repaced with non-degenerate bilinear forms by fiber functors.

With a choice of linear basis, non-degenerate bilinear forms are represented by invertible matrices and the above classification problem can be deal with by the following result:

Williamson-Wall Theorem: Let

$$\Theta : GL(n,\mathbb{C}) \ni B \mapsto {}^tB^{-1}B \in GL(n,\mathbb{C}).$$

Then

$$GL(n,\mathbb{C})/\sim \quad \cong \quad \Theta(GL(n,\mathbb{C}))/\text{similarity}.$$

An invertible matrix M belongs to $\Theta(GL(n,\mathbb{C}))$ if and only if

(1) $\mu_M(z) = \mu_M(z^{-1})$ for $z \in \mathbb{C}^\times$,
(2) $\mu_M^{(k)}(1)$ is even for even k,
(3) $\mu_M^{(k)}(-1)$ is even for odd k.

Here

$$\mu_M(z) = (\mu_M^{(1)}(z), \mu_M^{(2)}(z), \dots),$$

denotes the multiplicity function of the matrix M: $\mu_M^{(k)}(z)$ is the multiplicity of z-Jordan block of size k in M. Note here that the parameter is related to $\Theta(B)$ by the formula

$$d = \text{trace}(\Theta(B)) = \text{trace}({}^tB^{-1}B).$$

Example: We have the following identifications.

$$GL(2,\mathbb{C})/\sim = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \Big/ (q \leftrightarrow q^{-1}) \bigsqcup \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \longleftrightarrow \mathrm{SL}_{q^{-1}}(2,\mathbb{C}),$$
$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow \text{Woronowicz' Hopf algebra.}$$

More generally, algebraic quantum groups of Dubois-Violette and Launer (1990), together with their representation categories, are classified in this way.

4. Unitary Fiber Functors

A C*-tensor category is, by definition, a tensor category with a compatible C*-structure. It is well-known that the Temperley-Lieb category $\mathcal{TL}_d$ is a C*-tensor category if and only if $d \in \mathbb{R}$ and $|d| \geq 2$. Moreover the C*-structure is unique up to natural equivalences.

We can work out a similar characterization of unitary fiber functors on the Temperley-Lieb C*-category $\mathcal{TL}_d$:

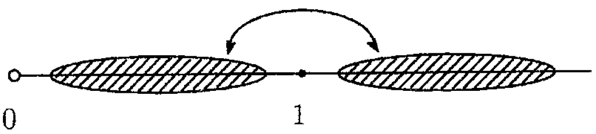
$$\{\text{unitary fiber functors on } \mathcal{TL}_d\} / \sim \cong \left\{\Phi : V \to \overline{V}; \Phi^{-1} = \frac{d}{|d|}\overline{\Phi}\right\} / \sim,$$

where the equivalence relation $\sim$ is defined by $\Phi \sim {}^tU\Phi U$ with U a unitary operator on V.

In terms of an eigenvalue-list of the positive part $|\Phi|$ of Φ, we have the following description:

Theorem:
An equivalence class of unitary fiber functors $\Leftrightarrow$ an unordered sequence $\{h_j > 0\}$ such that

$$\{h_j^{-1}\} = \{h_j\}, \quad \mathrm{tr}(|\Phi|^2) = |d|,$$
$$(d/|d|)^m = 1, \quad m = \dim\ker(|\Phi| - 1).$$



(i) $n = 2$: For each $|d| \geq 2$, $\exists$ a unique

$$\{h, h^{-1}\}, h \geq 1, h^2 + h^{-2} = |d|.$$

(ii) $n = 3$: For $d \geq 3$, a new choice

$$\{h, h^{-1}, 1\}, h \geq 1, h^2 + h^{-2} + 1 = d.$$

(iii) $n = 2k$: For $|d| \geq n$,

$$\{h_j, h_j^{-1}\}, h_j \geq 1, \sum_j (h_j^2 + h_j^{-2}) = |d|.$$

(iv) $n = 2k+1$: For $d \geq n$,

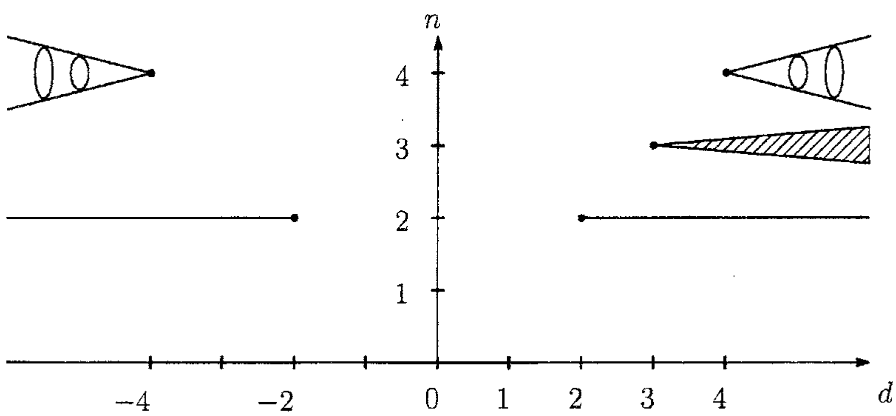
$$\{h_j, h_j^{-1}, 1\}, h_j \geq 1, \sum_j (h_j^2 + h_j^{-2}) = d - 1.$$

Set $t_j = h_j^2 + h_j^{-2} - 2$,

$$\sum_j t_j = |d| - 2k \text{ or } \sum_j t_j = d - 2k - 1.$$

Then the parameter space (moduli) of unitary fiber functors is the $k-1$-dimensional simplex

$$\{(t_1, \ldots, t_k); t_j \geq 0, \sum_j t_j = r\}/S_k.$$



In this way, we have multiparameter families of compact quantum groups, which turns out to be Wang-Banica's universal quantum group of orthogonal type.