数理解析研究所講究録 1460 巻 2005 年 134-135

Conformal geometry of curves

Jun O'Hara*

Department of Mathematics, Tokyo Metropolitan University

Rémi Langevin

Institut de Mathématiques de Bourgogne, Université de Bourgogne

June 2005

English summary

By a knot we mean an embedding from S^1 into S^3 or $\mathbb{R}^3$, or its image. By a link we mean a disjoint union of knots. We only study 2-component links.

Let K be a knot, and let x, y be a pair of distinct points on K. Let $\Sigma(x, x+dx, y, y+dy)$ be a sphere through the four points $x, x+dx, y, y+dy$. It is uniquely determined unless these four points are concircular. By identifying $\Sigma(x, x+dx, y, y+dy)$ with the complex sphere $\mathbb{C}\cup\{\infty\}$, we can consider four points $x, x+dx, y$, and $y+dy$ as four complex numbers. Let Ω denote the cross ratio of these four complex numbers, and call it the *infinitesimal cross ratio* of a knot K. Although the four complex numbers are not uniquely determined, their cross ratio can be uniquely defined. The infinitesimal cross ratio can be considered as a complex valued 2-form on $K\times K\setminus\Delta$. It is, by definition, invariant under Möbius transformations.

We show that the *energy of knots* $E_\circ^{(2)}$ can be expressed in terms of the infinitesimal cross ratio. An energy of knots is a functional on the space of knots which blows up as a knot degenerates to a singular knot with double points. It was introduced to produce a "canonical embedding" for each knot type as an embedding which gives the minimum value of the energy within its isotopy class. $E_\circ^{(2)}$ is defined by

$$E_\circ^{(2)}(K) = -4 + \iint_{K\times K}\left(\frac{1}{|x-y|^2} - \frac{1}{d_K(x,y)^2}\right)dxdy,$$

where $d_K(x,y)$ denotes the arc-length between x and y.

We then give interpretations of the real and imaginary parts of the infinitesimal cross ratio from a conformal geometric viewpoints, i.e. by using

*supported by JSPS during his stay at Dijon, France from September 2004 to March 2005.

what is invariant under Möbius transformations.

The first interpretation of the real part of the infinitesimal cross ration is given as follows. A pair of ordered distinct points can be considered as an oriented 0-sphere S^0. Let $\mathcal{S}(3,0)$ be a space of oriented S^0 in S^3. It is homeomorphic to the two points configuration space $S^3 \times S^3 \setminus \Delta$. It can also be identified with the total space of the cotangent bundle T^*S^3. Then the real part of the infinitesimal cross ratio is equal to the pull-back of the standard symplectic form of $T^*S^3 \cong S^3 \times S^3 \setminus \Delta$ by an inclusion map $K \times K \setminus \Delta \hookrightarrow S^3 \times S^3 \setminus \Delta$.

The second interpretation of the real part of the infinitesimal cross ration is given as follows. Let us first consider S^3 as the set of the points at infinity of the upper half light cone in the Minkowski space $\mathbb{R}^{4,1}$. Then we can consider $\mathcal{S}(3,0)$ as the oriented Grassmannian manifold of the set of 2-dimensional vector subspace of mixed-type in $\mathbb{R}^{4,1}$. It allows us to endow semi-Riemannian structure with signature $(3,3)$ to $\mathcal{S}(3,0)$. (This fact can be proved in several ways, for example, by using Plücker coordinates, or by considering $\mathcal{S}(3,0)$ as a homogeneous space $SO(4,1)/SO(3) \times SO(1,1)$. But it is convenient to use of pencils of codimension 1 spheres in S^1 for proofs.) As points x and y move along K, the set of pairs (x,y) form a surface in $\mathcal{S}(3,0)$. Then the real part of the infinitesimal cross ratio is equal to the "imaginary" area element of this surface.

Unlike the real part, the imaginary part of the infinitesimal cross ratio cannot have a global interpretation. If we consider S^3 as the boundary of the hyperbolic 4-space, the imaginary part of the infinitesimal cross ratio is locally equal to the "transversal area form" of geodesics in $\mathbb{H}^4$ joining x and y.