

STABILITY AND RIGIDITY OF CERTAIN SPECIAL LAGRANGIAN CONES

阪市大・理学研究科 大仁田 義裕 (Yoshihiro Ohnita)
Department of Mathematics,
Osaka City University

ABSTRACT. In this article we explain some results from the theory of special Lagrangian submanifolds with conical singularities recently developed by D. D. Joyce, and we provide new examples of stable and rigid special Lagrangian cones in higher dimensions.

INTRODUCTION

Recently Dominic Joyce has provided the profound theory of special Lagrangian submanifolds with isolated conical singularities and their moduli spaces in a series of his papers. The notion of the stability-index of special Lagrangian cones was introduced by Joyce. It is related to the deformation of special Lagrangian submanifolds with isolated conical singularities and the regularity of special Lagrangian integral currents. In this article we explain some results from the theory of special Lagrangian submanifolds with conical singularities developed by Joyce, and we shall provide new examples of stable and rigid special Lagrangian cones in higher dimensions.

In November 2004, Mark Haskins has visited Kyushu University and Tokyo Metropolitan University. The author was able to have nice discussion with him about this subject there. The author would like to thank Dr. Mark Haskins for his valuable suggestion about a problem on the existence of stable special Lagrangian cones in higher dimension.

1. SPECIAL LAGRANGIAN SUBMANIFOLDS OF CALABI-YAU MANIFOLDS

Special Lagrangian submanifolds have two aspects of *Lagrangian submanifolds* in symplectic geometry and *calibrated submanifolds* in Riemannian geometry. Calibrated submanifolds are minimal submanifolds in the sense that the mean curvature vector fields vanish. More strongly they are homologically volume minimizing submanifolds, and hence stable minimal submanifolds.

In Euclidean space $\mathbf{R}^{2m} \cong \mathbf{C}^m$, we recall the notion of special Lagrangian submanifolds. The natural group action of $SU(m) \subset U(m)$

preserves the standard Kähler form (symplectic form) defined as

$$\omega := \sqrt{-1} \sum_{i=1}^m dz^i \wedge d\bar{z}^i$$

and the standard complex volume form defined as

$$\Omega := dz^1 \wedge \cdots \wedge dz^m.$$

We decompose Ω into real and imaginary parts as

$$\Omega = \operatorname{Re}(\Omega) + \sqrt{-1}\operatorname{Im}(\Omega).$$

Then $\operatorname{Re}(\Omega)$ and $\operatorname{Im}(\Omega)$ are parallel real n -forms on \mathbf{C}^m .

The calibrated submanifolds by $\operatorname{Re}(\Omega)$ are characterized by the condition that the restrictions of ω and $\operatorname{Im}(\Omega)$ to the submanifold vanish. The *special Lagrangian submanifolds* in \mathbf{C}^m are defined as such submanifolds. Harvey and Lawson showed that minimal Lagrangian submanifolds in \mathbf{C}^m are special Lagrangian submanifolds.

In general, suppose that (M, g) is a Riemannian manifold with holonomy group contained in $SU(m)$, and such Riemannian manifolds become *Calabi-Yau* Kähler manifolds of complex dimension m . Then the parallel Kähler form ω and the parallel complex volume form Ω are defined on the whole M , and $\operatorname{Re}(\Omega)$ defines a calibration on M . The calibrated submanifolds by $\operatorname{Re}(\Omega)$ are characterized by the condition that the pull-backs of ω and $\operatorname{Im}(\Omega)$ to the submanifold vanish.

Definition 1.1. A m -dimensional submanifold L in a Calabi-Yau manifold is called a special Lagrangian submanifold if the pull-backs of both ω and $\operatorname{Im}(\Omega)$ to L vanish.

Proposition 1.1. *Let L be an oriented Lagrangian submanifold of a Calabi-Yau manifold M . Then L is a minimal submanifold in M if and only if L is a special Lagrangian submanifold calibrated by $\operatorname{Re}(e^{\sqrt{-1}\theta}\Omega)$ for some $\theta \in \mathbf{R}$.*

Proof. If L is a minimal submanifold of M , then f is constant. We have $\Omega(e_1, \dots, e_m) = e^{-\sqrt{-1}\theta}$ for some constant $\theta \in \mathbf{R}$. Thus L is a special Lagrangian submanifold calibrated by $\operatorname{Re}(e^{\sqrt{-1}\theta}\Omega)$. \square

Lemma 1.1. *Suppose that L is an oriented Lagrangian submanifold immersed in a Calabi-Yau manifold M . If we define a smooth function f on L with values in unit complex numbers by*

$$f(x) := \Omega(e_1, \dots, e_m), \quad (1.1)$$

where $\{e_1, \dots, e_m\}$ is an orthonormal basis of $T_x L$ compatible with the orientation of L , then the formula

$$df = -f\sqrt{-1}\alpha_H, \quad (1.2)$$

namely

$$d\log(f) = -\sqrt{-1}\alpha_H \quad (1.3)$$

holds. Here H denotes the mean curvature vector field of L . Note that $\log(f)$ is multi-valued. Locally, we have

$$d\psi = -\alpha_H. \quad (1.4)$$

Proof. The second fundamental form α of a Lagrangian submanifold L in a Kähler manifold (M, g, J) is a symmetric bilinear form on TL with values in the normal bundle $T^\perp L$ defined by

$$\alpha(X, Y) := \nabla_X^M Y - \nabla_X^L Y \quad \text{for each } X, Y \in \mathfrak{X}(L) \quad (1.5)$$

and is known to satisfy the equation

$$g(\alpha(X, Y), JZ) = g(\alpha(X, Z), JY) \quad (1.6)$$

for each $X, Y, Z \in T_p L$, that is, $g(\alpha(X, Y), JZ)$ is symmetric with respect to X, Y, Z .

Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame field on L and $\{\theta_1, \dots, \theta_m\}$ be its dual coframe field. Then

$$\Omega = e^{\sqrt{-1}\psi}(\theta^1 + \sqrt{-1}J\theta^1) \wedge \dots \wedge (\theta^m + \sqrt{-1}J\theta^m).$$

Here ψ is a smooth function defined on a neighborhood of L .

The smooth function $f := \Omega(e_1, \dots, e_m)$ is well-defined on L (up to ± 1). Note that $f = e^{\sqrt{-1}\psi}$. Let ∇^L denote the Levi-Civita connection of L with respect to the induced metric from M . Let p be an arbitrary point of L . We may assume that $(\nabla^L e_i)_p = 0$. For arbitrary vector

field X on L , we compute

$$\begin{aligned}
Xf &= X(\Omega(e_1, \dots, e_m)) \\
&= (\nabla_X^M \Omega)(e_1, \dots, e_m) + \sum_{i=1}^m \Omega(e_1, \dots, \nabla_X^M e_i, \dots, e_m) \\
&= \sum_{i=1}^m \Omega(e_1, \dots, \alpha(X, e_i), \dots, e_m) \\
&= \sum_{i=1}^m f(p) (\theta^1 + \sqrt{-1} J \theta^1) \wedge \dots \\
&\quad \wedge (\theta^m + \sqrt{-1} J \theta^m) (e_1, \dots, \alpha(X, e_i), \dots, e_m) \\
&= f(p) \sqrt{-1} \sum_{i=1}^m g(Je_i, \alpha(X, e_i)) \\
&= f(p) \sqrt{-1} \sum_{i=1}^m g(JX, \alpha(e_i, e_i)) \\
&= f(p) \sqrt{-1} g(JX, H) = -f(p) \sqrt{-1} \alpha_H(X).
\end{aligned}$$

□

2. SPECIAL LAGRANGIAN CONES

Let $S^{2m-1}(1)$ denote the unit standard hypersphere of \mathbf{C}^m . Let $\Sigma \subset S^{2m-1}(1)$ be an $(m-1)$ -dimensional smooth submanifold embedded in $S^{2m-1}(1)$. The cone over Σ in \mathbf{C}^m is defined as

$$C = C\Sigma := \{t\sigma \in \mathbf{C}^m \mid t \geq 0, \sigma \in \Sigma\}.$$

Let $\pi : S^{2m-1}(1) \rightarrow \mathbf{C}P^{m-1}$ be the Hopf fibration over the $(m-1)$ -dimensional complex projective space. Then we know that $C\Sigma$ is a special Lagrangian submanifold with an isolated singularity at o if and only if Σ is a minimal Legendrian submanifold with respect to the standard contact structure of $S^{2m-1}(1)$, and equivalently $L = \pi(\Sigma)$ is a minimal Lagrangian submanifold in $\mathbf{C}P^{m-1}$.

Example 2.1. In Harvey-Lawson [5] the first example of a special Lagrangian cone with an isolated singularity at o was given as

$$C_{HL}^m := \{(z^1, \dots, z^m) \in \mathbf{C}^m \mid (\sqrt{-1})^{m+1} z_1 \cdots z_m \in \mathbf{R}, |z^1| = \dots = |z^m|\}.$$

Then

$$\Sigma_{HL}^{m-1} := C_{HL}^m \cap S^{2m-1}(1) \subset S^{2m-1}(1)$$

is isometric to an $(m-1)$ -dimensional flat torus T^{m-1} .

Set $C' := C \setminus \{0\}$. Let Δ and Δ_Σ be the Laplacians of (C', g) and (Σ, g_Σ) on functions, respectively. A function u on C' is called a *homogeneous function of order α* on C' if u satisfies $u \circ t = t^\alpha u$ for each $t > 0$. Then such a function can be expressed as $u(r\sigma) = r^\alpha v(\sigma)$ for some function v on Σ . The relationship between Δ and Δ_Σ is given by the formula

$$\Delta u(r\sigma) = r^{\alpha-2}(\Delta_\Sigma v(\sigma) - \alpha(\alpha + m - 2)v(\sigma)).$$

Hence we see that u is harmonic if and only if v is an eigenfunction on Σ with eigenvalue $\alpha(\alpha + m - 2)$.

Assume that $m > 2$. Set

$$\mathcal{D}_\Sigma := \{\alpha \in \mathbf{R} \mid \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\},$$

which is a countable and discrete subset of \mathbf{R} . For each $\alpha \in \mathcal{D}_\Sigma$, denote by $m_\Sigma(\alpha)$ the multiplicity of eigenvalue $\alpha(\alpha + m - 2)$ of Δ_Σ , which is equal to the dimension of vector space of all homogeneous harmonic functions of order α on C' . Then we define a function $N_\Sigma : \mathbf{R} \rightarrow \mathbf{Z}$ as

$$N_\Sigma(\delta) := - \sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha)$$

if $\delta < 0$ and

$$N_\Sigma(\delta) := \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0, \delta]} m_\Sigma(\alpha)$$

if $\delta \geq 0$. N_Σ is monotone increasing and upper semicontinuous.

Definition 2.1. The *stability-index* of the special Lagrangian cone C is defined by

$$\text{s-ind}(C) := N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G.$$

Here G denotes a compact Lie subgroup of $SU(n)$ defined by

$$G := \{a \in SU(n) \mid a(\Sigma) = \Sigma\}.$$

Note that $m_\Sigma(0) = b^0(\Sigma)$, $m_\Sigma(1) \geq 2m$, $m_\Sigma(2) \geq m^2 - 1 - \dim G$. Indeed, the first equality is trivial. The second inequality follows from the famous Tsunero Takahashi's theorem ([15]). The third inequality can be shown as follows : Assume that $M = \mathbf{C}P^n$ is the n -dimensional complex projective space with constant holomorphic sectional curvature 4. Then M is a compact Einstein-Kähler manifold with positive Einstein constant $\kappa = 2n$. Let $V_1(M)$ denote the vector space of all the first eigen-functions of $\mathbf{C}P^n$ with the first eigenvalue $2\kappa = 4n$. Let

$V_\kappa(\Sigma)$ denote the vector space of all eigen-functions of the Laplacian on Σ with eigenvalue $\kappa = 2n$. Then we can define a linear map

$$V_1(M) \ni f \longmapsto f|_L \circ \pi - \frac{\int_L f|_L dx}{\text{Vol}(L)} \in V_\kappa(\Sigma).$$

Then its kernel is

$$\begin{aligned} & \{f \in V_1(M) \mid f|_L \text{ is constant}\} \\ & \cong \{V \in \mathfrak{su}(n) \mid \text{the flow of } V \text{ preserves } L\}, \end{aligned}$$

which is isomorphic to the Lie algebra of G . Since $\dim V_1(M) = n^2 - 1$, we obtain $m(2) = \dim V_\kappa(\kappa) \geq \dim \mathfrak{su}(n) - \dim G = n^2 - 1 - \dim G$.

A special Lagrangian cone C is called *stable* if $\text{s-ind}(C) = 0$. A special Lagrangian cone C is called *rigid* if $m_\Sigma(2) = m^2 - 1 - \dim G$. Obviously we have that if C is stable, then C is rigid.

Remark 1. Assume that C is a stable special Lagrangian cone. If $\pi i(\Sigma)$ is an embedded submanifold in $\mathbb{C}P^{n-1}$, then $\pi i(\Sigma)$ is a Hamiltonian stable minimal Lagrangian submanifold in $\mathbb{C}P^{n-1}$. For example, see [1] for Hamiltonian stability problem of Lagrangian submanifolds.

3. SPECIAL LAGRANGIAN SUBMANIFOLDS WITH ISOLATED CONICAL SINGULARITIES

We shall mention the results of Joyce on the deformations of L , or the local structure of moduli spaces around L , and the regularity of special Lagrangian varieties, which are related to stability-index of special Lagrangian submanifolds.

Let \mathcal{M} be the moduli space of compact special Lagrangian submanifolds with isolated conical singularities embedded in M . Mclean showed that if L is smooth (i.e. without singularities), then the moduli space \mathcal{M} is a smooth manifold of dimension $b^1(L)$ around L .

Joyce showed that if L is a special Lagrangian submanifold with isolated conical singularities C_1, \dots, C_k , then the dimension of the obstruction space of L is equal to the sum of stability-indices of special Lagrangian cones C_1, \dots, C_k :

$$\dim \mathcal{O}_L = \sum_{i=1}^k \text{s-ind}(C_i).$$

Next we shall mention the Joyce's regularity results of special Lagrangian integral currents, or special Lagrangian varieties. Geometric measure theory implies the compactness of the space of such objects. Suppose that L is a special Lagrangian integral current and has the multiplicity 1 tangent cone at $x \in \text{supp} L$. Joyce showed that if the

tangent cone of L at x is a rigid special Lagrangian cone, then L has an isolated conical singularity at x .

4. STABILITY-INDEX OF SPECIAL LAGRANGIAN CONES OVER CERTAIN COMPACT SYMMETRIC SPACES

It is really important to study concretely the stability and rigidity of special Lagrangian cones.

Joyce, Marshall proved that C_{HL}^3 is stable and C_{HL}^m is unstable if $m \geq 4$, and C_{HL}^m is rigid if and only if $m \neq 8, 9$.

We shall discuss stability and rigidity of special Lagrangian cones constructed by the Lie theoretic method, which include C_{HL}^m . Let (U, G) be an Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$. Set $\dim(U/G) = 2m$. Let $\langle \cdot, \cdot \rangle_{\mathfrak{u}}$ denote the $\text{Ad}(U)$ -invariant inner product of \mathfrak{u} defined by (-1) -times Killing-Cartan form of \mathfrak{u} . We decompose \mathfrak{g} into the direct sum of the semisimple part \mathfrak{g}_{ss} and the center $\mathfrak{c}(\mathfrak{g})$ as follows : $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{c}(\mathfrak{g})$. There is an element $Z \in \mathfrak{c}(\mathfrak{g})$ such that $\text{ad}Z$ defines the invariant complex structure of (U, G) . Relative to the complex structure the subspace \mathfrak{p} can be identified with a complex Euclidean space \mathbb{C}^m . We take the decomposition of (U, G) into irreducible Hermitian symmetric pairs of compact type :

$$(U, G) = (U_1, G_1) \oplus \cdots \oplus (U_s, G_s). \quad (4.1)$$

Set $\dim(U_i/G_i) = 2m_i$ for $i = 1, \dots, s$. Let $\mathfrak{u}_i = \mathfrak{g}_i + \mathfrak{p}_i$ be the canonical decomposition of (U_i, G_i) for each $i = 1, 2, \dots, s$. Assume that there is an element $\eta_i \in \mathfrak{p}_i$ satisfying the condition $(\text{ad}\eta_i)^3 + 4(\text{ad}\eta_i) = 0$. Choose positive numbers $c_1 > 0, \dots, c_s > 0$ with $\sum_{i=1}^s 1/c_i = 1/c$. Put $a_i = 1/\sqrt{2c_i m_i}$ for each $i = 1, \dots, s$. Set $\hat{L}_i = \text{Ad}(G_i)(a_i \eta_i) \subset S^{2m_i-1}(c_i/4) \subset \mathfrak{p}_i$, which is an irreducible symmetric R -space standard embedded in a complex Euclidean space \mathfrak{p}_i .

Set $\eta = a_1 \eta_1 + \cdots + a_s \eta_s \in \mathfrak{p}$. Set $\hat{L} = \text{Ad}(G)(\eta) \subset S^{2m-1}(c/4) \subset \mathfrak{p}$, which is a symmetric R -space standard embedded in a complex Euclidean space $\mathfrak{p} \cong \mathbb{C}^{n+1}$. Note that we have the inclusions

$$\hat{L} = \hat{L}_1 \times \cdots \times \hat{L}_s \subset S^{2m_1-1}(c_1/4) \times \cdots \times S^{2m_s-1}(c_s/4) \subset S^{2m-1}(c/4). \quad (4.2)$$

Note that \hat{L} is a compact H -minimal Lagrangian submanifold embedded in \mathbb{C}^{n+1} (see [3]).

We take an orthogonal decomposition $\mathfrak{c}(\mathfrak{g}) = \mathfrak{c}' \oplus \{Z\}_{\mathbb{R}}$ of $\mathfrak{c}(\mathfrak{g})$. Let $\mathfrak{g}' := \mathfrak{g}_{ss} \oplus \mathfrak{c}'$ and G' denote the analytic subgroup of G generated by \mathfrak{g}' . Set $\Sigma = \text{Ad}(G')(\eta) \subset S^{2m-1}(c/4) \subset \mathfrak{p}$. Then Σ is a Legendrian

submanifold in $S^{2m-1}(c/4)$. Moreover Σ is a minimal submanifold in $S^{2m-1}(c/4)$ if and only if $c_i m_i = cm$ for each $i = 1, 2, \dots, s$.

In the case where (U, G) is irreducible, i.e. $s = 1$, by the classification theory of symmetric R -spaces, Σ is one of S^{m-1} , $SU(p)$, $SU(p)/SO(p)$, $SU(2p)/Sp(p)$, E_6/F_4 .

Now we assume that Σ is one of compact irreducible symmetric spaces standardly embedded in $S^{2m-1}(1)$ as minimal Legendrian submanifolds in the above way :

$$\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p) \quad (p \geq 3), \text{ or } E_6/F_4.$$

Note that the rank of these symmetric spaces is equal to $p - 1$, and the rank of E_6/F_4 is equal to 2. Let $C\Sigma$ be the special Lagrangian cone in \mathbf{C}^m over Σ . Then by using results of [14],[1],[2], we can show the following.

Theorem 4.1. (1) $C\Sigma$ are all rigid.

(2) If $\Sigma = SU(3), SU(3)/SO(3), SU(6)/Sp(3)$ ($p = 3$), E_6/F_4 , then $C\Sigma$ is stable, and hence Legendrian stable.

(3) If $\Sigma = SU(p), SU(p)/SO(p), SU(2p)/Sp(p)$, $p \geq 4$, $C\Sigma$ is not stable, in fact not Legendrian stable.

5. STABILITY-INDEX OF A SPECIAL LAGRANGIAN CONE IN \mathbf{C}^4 OVER A MINIMAL LEGENDRIAN $SU(2)$ -ORBIT

Here we mention about the stability and rigid of an other example in \mathbf{C}^4 .

Let V^3 be the complex vector space of all complex homogeneous polynomials with two variables z_1, z_2 of degree 3. We equip the standard Hermitian inner product such that

$$\{v_k = \frac{1}{\sqrt{k!(3-k)!}} z_1^{3-k} z_2^k \mid k = 0, 1, 2, 3\}$$

is a unitary basis of V^3 , and $V^3 \cong \mathbf{C}^4 \cong \mathbf{R}^8$. We know that V^3 is an irreducible unitary representation of $SU(2)$. Now we consider the orbit of $SU(2)$ through $v = \frac{1}{\sqrt{2}}(v_0 + v_3)$. Then the orbit $\Sigma = \rho_3(SU(2))v \subset S^7(1)$ is a 3-dimensional minimal Legendrian submanifold embedded in $S^7(1)$.

Theorem 5.1. *The special Lagrangian cone C in \mathbf{C}^4 over the orbit $\Sigma = \rho_3(SU(2))(v)$ is not Legendrian stable, and hence not stable, but it is rigid. Its stability-index and Legendrian-index of C are given by*

$$s\text{-ind}(C) = 3 \quad \text{and} \quad l\text{-ind}(C) = 11 (= 8 + 3).$$

Remark 2. We can show that $\pi(\Sigma) = \pi(\rho_3(SU(2))(v))$ is a 3-dimensional compact Hamiltonian stable minimal Lagrangian submanifold embedded in CP^3 .

Question 1. Investigate asymptotically conical special Lagrangian submanifolds in C^n asymptotic to special Lagrangian cones with high symmetry which were discussed in this article ?

Question 2. Can we find compact special Lagrangian submanifolds in Calabi-Yau manifolds with conical singularities which were discussed in this article ?

REFERENCES

- [1] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain minimal Lagrangian submanifolds in complex projective spaces*, Tohoku Math. J. 55 (2003), 583–610.
- [2] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain H-minimal Lagrangian submanifolds and related problems*, RIMS Kokyuroku 1292 (2002), *General Study on Riemannian submanifolds*, 2002, RIMS, Kyoto University, Kyoto Japan, 72–93.
- [3] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain symmetric R-spaces embedded in complex Euclidean spaces*, preprint, Tokyo Metropolitan University, 2002.
- [4] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of parallel Lagrangian submanifolds in complex space forms*, in preparation, a revised version of [3].
- [5] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Mathematica 148 (1982), 47–157.
- [6] M. Haskins, *Special Lagrangian cones*, Amer. J. Math. 126 (2004), 845–871.
- [7] M. Haskins, *The geometric complexity of special Lagrangian T^2 -cones*, Invent. math. 157 (2004), 11–70.
- [8] N. Hitchin, *Lectures on special Lagrangian submanifolds*, Proceedings of the Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, AMS/IP Studies in Advanced Mathematics, Cumrun Vafa and S.-T. Yau, Editors, Volume 23, 2001, 151–182.
- [9] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. I. Regularity*, math.DG/0211294, version 3, 2003.
- [10] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces*, math.DG/0211295, version 3, 2003.
- [11] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications*, math.DG/0303272, version 1, 21 March 2003.
- [12] D. D. Joyce, *The exceptional holonomy groups and calibrated geometry*, arXiv:math.DG/0406011, version 1, 1 Jun 2004.
- [13] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. 6 (1998), 705–747.
- [14] Y. Ohnita, *The first standard minimal immersions of compact irreducible symmetric spaces*, Differential Geometry of Submanifolds, ed. by Katsuei

Kenmotsu, Lecture Notes in Mathematics 1090, Springer-Verlag, 1984, 37-49.

- [15] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan 18 (1966), 380-385.

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUGIMOTO, SUMIYOSHI-KU, OSAKA, 558-8585, JAPAN

E-mail address: ohnita@sci.osaka-cu.ac.jp