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Kyoto University
Error Bounds of P-matrix Linear Complementarity Problems\footnote{This work is partly supported by a Grant-in-Aid from Japan Society for the Promotion of Science.}

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1 Introduction

The linear complementarity problem is to find a vector $x \in \mathbb{R}^n$ such that
\[ Mx + q \geq 0, \quad x \geq 0, \quad x^T (Mx + q) = 0, \]
or to show that no such vector exists, where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. We denote this problem by $\text{LCP}(M, q)$. A matrix $M$ is called a P-matrix if
\[ \max_{1 \leq i \leq n} x_i (Mx)_i > 0 \quad \text{for all} \quad x \neq 0. \]
It is well-known that $M$ is a P-matrix if and only if the $\text{LCP}(M, q)$ has a unique solution for any $q \in \mathbb{R}^n$ \cite{6}. Recall the following definitions for an $n \times n$ matrix.

$M$ is called an M-matrix, if $M^{-1} \geq 0$ and $M_{ij} \leq 0$ ($i \neq j$) for $i, j = 1, 2, \ldots, n$.

$M$ is called an H-matrix, if its comparison matrix is an M-matrix.

It is known that an H-matrix with positive diagonals is a P-matrix. Moreover, if $M$ is a P-matrix, then there is a neighborhood $\mathcal{M}$ of $M$, such that all matrices in $\mathcal{M}$ are P-matrices. Hence, we can define a solution function $x(A, b) : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}_+^n$, where $x(A, b)$ is the solution of $\text{LCP}(A, b)$ and $\mathbb{R}_+^n = \{x \in \mathbb{R}^n | x \geq 0\}$.

It is easy to verify that $x^*$ solves the $\text{LCP}(M, q)$ if and only if $x^*$ solves
\[ r(x) := \min(x, Mx + q) = 0, \]
where the min operator denotes the componentwise minimum of two vectors. The function $r$ is called the natural residual of the $\text{LCP}(M, q)$, and often used in error analysis.

Error bounds for the $\text{LCP}(M, q)$ have been studied extensively, see \cite{3, 6, 7, 11, 9, 12, 15}.

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2 Global error bounds for P-matrix linear complementarity problems

For $M$ being a P-matrix, Mathias and Pang [11] present the following error bound

$$\|x - x^*\|_{\infty} \leq \frac{1 + \|M\|_{\infty}}{c(M)} \|r(x)\|_{\infty},$$  \hspace{1cm} (2.1)

for any $x \in \mathbb{R}^n$, where

$$c(M) = \min_{\|x\|_{\infty} = 1} \left\{ \max_{1 \leq i \leq n} x_i (Mx)_i \right\}.$$  

This error bound is well known and widely cited. However, the quantity $c(M)$ in (2.1) is not easy to find. For $M$ being an H-matrix with positive diagonals, Mathias and Pang [11] gave a computable lower bound for $c(M)$,

$$c(M) \geq \frac{(\min_i b_i)(\min_i (\tilde{M}^{-1}b)_i)}{(\max_j (\tilde{M}^{-1}b)_j)^2}, =: \tilde{c}(b),$$  \hspace{1cm} (2.2)

for any vector $b > 0$, where $\tilde{M}$ is the comparison matrix of $M$, that is

$$\tilde{M}_{ii} = M_{ii} \quad \tilde{M}_{ij} = -|M_{ij}| \text{ for } i \neq j.$$  

However, finding a large value of $\tilde{c}(b)$ is not easy. For some $b$, $\tilde{c}(b)$ can be very small, and thus the error coefficient

$$\mu(b) := \frac{1 + \|M\|_{\infty}}{\tilde{c}(b)}$$  \hspace{1cm} (2.3)

can be very large.

Interval methods for validation of solution of the LCP($M, q$) have been studied in [1, 14]. When a numerical validation condition for the existence of a solution holds, a numerical error bound is provided. However, the numerical validation condition is not ensured to be held at every point $x$.

In [4], we observed that for every $x, y \in \mathbb{R}^n$,

$$\min(x_i, y_i) - \min(x^*_i, y^*_i) = (1 - d_i)(x_i - x^*_i) + d_i(y_i - y^*_i), \quad i \in N$$  \hspace{1cm} (2.4)

where

$$d_i = \begin{cases} 0 & \text{if } y_i \geq x_i, \ y_i^* \geq x_i^* \\ 1 & \text{if } y_i \leq x_i, \ y_i^* \leq x_i^* \\ \frac{\min(x_i, y_i) - \min(x^*_i, y^*_i) + x_i^* - x_i}{y_i - y_i^* + x_i^* - x_i} & \text{otherwise.} \end{cases}$$
Moreover, we have $d_i \in [0,1]$. Hence putting $y = Mx + q$ and $y^* = Mx^* + q$ in (2.4), we obtain

$$r(x) = (I - D + DM)(x - x^*),$$  \hspace{1cm} (2.5)$$

where $D$ is a diagonal matrix whose diagonal elements are $d = (d_1, d_2, \ldots, d_n) \in [0,1]^n$. It is known that $M$ is a P-matrix if and only if $I - D + DM$ is nonsingular for any diagonal matrix $D = \text{diag}(d)$ with $0 \leq d_i \leq 1$ [10]. This together with (2.5) yields upper and lower error bounds,

$$\frac{\max_{d \in [0,1]^n} ||r(x)||}{\max_{d \in [0,1]^n} ||I - D + DM||} \leq ||x - x^*|| \leq \max_{d \in [0,1]^n} ||(I - D + DM)^{-1}|| ||r(x)||. \hspace{1cm} (2.6)$$

Moreover, it is not difficult to verify that if $M$ is a P-matrix and $D = \text{diag}(d)$ with $d \in [0,1]^n$, we have

$$\max_{1 \leq i \leq n} x_i((I - D + DM)x)_i > 0, \text{ for all } x \neq 0,$$

that is, $(I - D + DM)$ is a P-matrix. Therefore, computation of rigorous error bounds can be turned into $|| \cdot ||$ optimization problems over a P-matrix interval set, which is related to linear P-matrix interval systems.

The linear interval system has been studied intensively and some highly efficient numerical methods have been developed, see [13, 14] for references. In the rest part of this section, we give some simple upper bounds for

$$\max_{d \in [0,1]^n} ||(I - D + DM)^{-1}||.$$

**Theorem 2.1** [4] Suppose that $M$ is an $H$-matrix with positive diagonals. Then we have

$$\max_{d \in [0,1]^n} ||(I - D + DM)^{-1}|| \leq ||\tilde{M}^{-1}\max(\Lambda, I)||. \hspace{1cm} (2.7)$$

**Remark 1.** Since $\tilde{M}^{-1}\max(\Lambda, I) \geq 0$, we have

$$||\tilde{M}^{-1}\max(\Lambda, I)||_\infty = ||\tilde{M}^{-1}\max(\Lambda, I)e||_\infty$$

and

$$||\tilde{M}^{-1}\max(\Lambda, I)||_1 = ||(e^T\tilde{M}^{-1}\max(\Lambda, I))^T||_\infty.$$

The upper error bound in (2.7) with $|| \cdot ||_\infty$ or $|| \cdot ||_1$ can be computed by solving a linear system of equations $\min(\Lambda^{-1}, I)\tilde{M}x = e$ or $\tilde{M}^T\min(\Lambda^{-1}, I)x = e$. 

Theorem 2.2 [4] Suppose that $M$ is an $M$-matrix. Let $V = \{v \mid M^Tv \leq e, v \geq 0\}$ and $f(v) = \max_{1 \leq i \leq n} (e + v - M^Tv)_i$. Then we have

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_1 = \max_{v \in V} f(v). \quad (2.8)$$

Theorem 2.3 [4] If $M$ is a $P$-matrix, then for any $x \in R^n$, the following inequalities hold.

$$\frac{1}{1 + \|M\|_\infty} \|r(x)\|_\infty \quad (\text{Mathias-Pang [11]})$$

$$\leq \frac{1}{\max(1, \|M\|_\infty)} \|r(x)\|_\infty \quad (\text{Cottle-Pang-Stone [6]})$$

$$= \frac{1}{\max_{d \in [0,1]^n} \|I - D + DM\|_\infty} \|r(x)\|_\infty$$

$$\leq \|x - x^*\|_\infty$$

$$\leq \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty$$

$$\leq \max(1, \|M\|_\infty) \|r(x)\|_\infty$$

$$= \frac{1}{\max_{d \in [0,1]^n} \|I - D + DM\|_\infty} \|r(x)\|_\infty$$

$$\leq \frac{1 + \|M\|_\infty}{c(M)} \|r(x)\|_\infty \quad (\text{Mathias-Pang [11]).}$$

Theorem 2.4 [4] If $M$ is an $H$-matrix with positive diagonals, then for any $x, b \in R^n$, $b > 0$, the following inequalities hold.

$$\|x - x^*\|_\infty$$

$$\leq \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty$$

$$\leq \|\tilde{M}^{-1} \max(\Lambda, I)\|_\infty \|r(x)\|_\infty$$

$$\leq (\mu(b) - \|\tilde{M}^{-1} \min(\Lambda, I)\|_\infty) \|r(x)\|_\infty$$

$$\leq \mu(b) \|r(x)\|_\infty \quad (\text{Mathias-Pang [11]).}$$

In addition, if $M$ is an $M$-matrix, then for any $x \in R^n$, the following inequalities hold.

$$\|x - x^*\|_\infty$$

$$\leq \|M^{-1} \max(\Lambda, I)\|_\infty \|r(x)\|_\infty$$

$$\leq (\frac{1 + \|M\|_\infty}{c(M)} - \|M^{-1} \min(\Lambda, I)\|) \|r(x)\|_\infty$$

$$\leq \frac{1 + \|M\|_\infty}{c(M)} \|r(x)\|_\infty \quad (\text{Mathias-Pang [11]).}$$
Applying Theorem 2.1, we obtain the following relative error bounds.

**Corollary 2.1** [4] Suppose $M$ is an $H$-matrix with positive diagonals. For any $x \in \mathbb{R}^n$, we have

$$\frac{||r(x)||}{(1+||M||)||\tilde{M}^{-1}\max(A,I)||\|(-q)_{+}\|} \leq \frac{||x-x^*||}{||x^*||} \leq \frac{||M||||\tilde{M}^{-1}\max(A,I)||\|r(x)||}{||(-q)_{+}||}.$$

**3 Perturbation bounds of P-matrix linear complementarity problems**

In [6], Cottle, Pang and Stone introduced the following Lemma which has been widely applied in perturbation bounds based on the fundamental quantity associated with a P-matrix,

$$c(M) = \min_{\|x\|_\infty=1} \max_{1 \leq i \leq n} \{x_i(Mx)_i\}.$$

**Lemma 3.1** [6] Let $M \in \mathbb{R}^{n \times n}$ be a P-matrix. The following statements hold:

(i) for any two vectors $q$ and $p$ in $\mathbb{R}^n$,

$$\|x(M,q) - x(M,p)\|_\infty \leq \frac{1}{c(M)} \|q - p\|_\infty$$

(ii) for each vector $q \in \mathbb{R}^n$, there exist a neighborhood $\mathcal{U}$ of the pair $(M,q)$ and a constant $c_0 > 0$ such that for any $(A,b), (B,p) \in \mathcal{U}$, $A, B$ are P-matrices and

$$\|x(A,b) - x(B,p)\|_\infty \leq c_0(\|A-B\|_\infty + \|b - p\|_\infty).$$

Lemma 3.1 shows that when $M$ is a P-matrix, for each $q$, $x(A,b)$ is a locally Lipschitzian function of $(A,b)$ in a neighborhood of $(M,q)$, and $x(M,b)$ is a globally Lipschitzian function of $b$. This property plays a very important rule in the study of the LCP and mathematical programs with LCP constraints [8]. However, the constant $c(M)$ is difficult to compute, and $c_0$ is not specified. It is hard to use this lemma for verifying accuracy of a computed solution of the LCP when the data $(M,q)$ contain errors.

For $M$ being a P-matrix, we [5] introduce the following constant

$$\beta(M) = \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}D\|.$$

In the follows, we compare $\beta(M)$ with $c(M)^{-1}$ in $\| \cdot \|_\infty$ and give a simple version of $\beta(M)$ for $M$ being an M-matrix, a symmetric positive definite matrix, and positive definite matrix.
Theorem 3.1 [5] Let $M$ be a $P$-matrix. Then

$$\beta_{\infty}(M) := \max_{d \in [0,1]^{n}} \|(I - D + DM)^{-1}D\|_{\infty} \leq \frac{1}{c(M)}.$$  

It is known that an $H$-matrix with positive diagonals is a $P$-matrix, and a positive definite matrix is a $P$-matrix [6]. Now, we consider the two subclasses of $P$-matrix.

Theorem 3.2 [5] Let $M$ be an $H$-matrix with positive diagonals. Then

$$\beta(M) \leq ||\tilde{M}^{-1}||,$$

where $\tilde{M}$ is the comparison matrix of $M$. In particular, if $M$ is an $M$-matrix, then the equality holds.

Theorem 3.3 [5] Let $M$ be a symmetric positive definite matrix. Then

$$\beta_{2}(M) := \max_{d \in [0,1]^{n}} \|(I - D + DM)^{-1}D\|_{2} = ||M^{-1}||_{2}.$$  

In comparison to Lemma 3.1, the following theorem gives sharp perturbation error estimates for the $P$-matrix LCP

Theorem 3.4 [5] Let $M \in R^{n \times n}$ be a $P$-matrix. Then the following statements hold:

(i) For any two vectors $q$ and $p$ in $R^{n}$,

$$\|x(M, q) - x(M, p)\| \leq \beta(M)\|q - p\|.$$  

(ii) Every matrix $A \in \mathcal{M} := \{A \mid \beta(M)\|M - A\| \leq \eta < 1\}$ is a $P$-matrix. Let

$$\alpha(M) = \frac{1}{1 - \eta} \beta(M).$$

Then for any $A, B \in \mathcal{M}$ and $q, p \in R^{n}$

$$\|x(A, q) - x(B, p)\| \leq \alpha(M)^{2}\|(p)_{+}\|\|A - B\| + \alpha(M)\|q - p\|.$$  

From Theorem 3.2 and Theorem 3.3, the Lipschitz constants $\beta(M)$ and $\alpha(M)$ can be estimated by matrix norms, if $M$ is an $H$-matrix with positive diagonals or a symmetric positive definite matrix. In particular, we have the following two corollaries.

Corollary 3.1 [5] Let $M \in R^{n \times n}$ be an $H$-matrix with positive diagonals. Then the following statements hold:
(i) For any two vectors $q$ and $p$ in $\mathbb{R}^n$,
\[
\|x(M, q) - x(M, p)\|_{\infty} \leq \|M^{-1}\|_{\infty} \|q - p\|_{\infty}
\]

(ii) Every matrix $A \in \mathcal{M}_{\infty} := \{A \mid \|M^{-1}\|_{\infty} \|M - A\|_{\infty} \leq \eta < 1\}$ is an H-matrix with positive diagonals. Let
\[
\alpha_{\infty}(M) = \frac{1}{1 - \eta} \|M^{-1}\|_{\infty}.
\]

Then for any $A, B \in \mathcal{M}_{\infty}$ and $q, p \in \mathbb{R}^n$
\[
\|x(A, q) - x(B, p)\|_{\infty} \leq \alpha_{\infty}(M)^2 \|(p)_{+}\|_{\infty} \|A - B\|_{\infty} + \alpha_{\infty}(M) \|q - p\|_{\infty}.
\]

**Corollary 3.2** [5] Let $M \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Then the following statements hold:

(i) For any two vectors $q$ and $p$ in $\mathbb{R}^n$,
\[
\|x(M, q) - x(M, p)\|_{2} \leq \|M^{-1}\|_{2} \|q - p\|_{2}
\]

(ii) Every matrix $A \in \mathcal{M}_{2} := \{A \mid \|M^{-1}\|_{2} \|M - A\|_{2} \leq \eta < 1\}$ is a P-matrix. Let
\[
\alpha_{2}(M) = \frac{1}{1 - \eta} \|M^{-1}\|_{2}.
\]

Then for any $A, B \in \mathcal{M}_{2}$ and $q, p \in \mathbb{R}^n$
\[
\|x(A, q) - x(B, p)\|_{2} \leq \alpha_{2}(M)^2 \|(p)_{+}\|_{2} \|A - B\|_{2} + \alpha_{2}(M) \|q - p\|_{2}.
\]

A matrix $A$ is called positive definite if
\[
x^T A x > 0, \quad 0 \neq x \in \mathbb{R}^n.
\]

Since $x^T A x = x^T \frac{A + A^T}{2} x$, $A$ is positive definite if and only if $\frac{A + A^T}{2}$ is symmetric positive definite. Note that a positive definite matrix is not necessarily symmetric. Such asymmetric matrices frequently appear in the context of the LCP.

Combining the ideas of Mathias and Pang [11] and Corollary 3.2, we present perturbation bounds for the positive definite matrix LCP.

**Theorem 3.5** [5] Let $M \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Then the following statements hold:
(i) For any two vectors \( q \) and \( p \) in \( R^n \),
\[
\|x(M, q) - x(M, p)\|_2 \leq \|((\frac{M + M^T}{2})^{-1})\|_2\|q - p\|_2.
\]

(ii) Every matrix \( A \in \mathcal{M}_2 := \{A \mid \|((\frac{M+M^T}{2})^{-1}\|_2\|M-A\|_2 \leq \eta < 1\} \) is positive definite.

Let
\[
\alpha_2(M) = \frac{1}{1 - \eta} \|((\frac{M+M^T}{2})^{-1}\|_2.
\]

Then for any \( A, B \in \mathcal{M}_2 \) and \( q, p \in R^n \),
\[
\|x(A, q) - x(B, p)\|_2 \leq \alpha_2(M)^2\|(-p)_+\|_2\|A - B\|_2 + \alpha_2(M)\|q - p\|_2.
\]

**Example 3.1** Theorem 3.1 shows that for every \( \text{P} \)-matrix, \( \beta_\infty(M) \leq c(M)^{-1} \). Now we show that \( \beta_\infty(M) \) can be much smaller than \( c(M)^{-1} \) in some case. Consider
\[
M = \begin{pmatrix} 1 & -t \\ 0 & t \end{pmatrix}.
\]

For \( t \geq 1 \), \( M \) is an \( \text{M} \)-matrix. By Theorem 3.2, \( \beta_\infty(M) = \|M^{-1}\|_\infty = 2 \). For \( \bar{x} = (1, t^{-1}) \), we have
\[
c(M) \leq \max_{i \in N} \bar{x}_i(M\bar{x})_i = \frac{1}{t}.
\]

Hence, \( c(M)^{-1} \geq t \to \infty \), as \( t \to \infty \).

Using the results in the last section, we derive relative perturbation bounds expressed in the term of \( \beta(M)\|M\| \).

For the system of linear equations, \( A \) is nonsingular if and only if \( Ax = b \) has a unique solution for any vector \( b \). A system of linear equations is considered to be well-conditioned (ill-conditioned) if small changes in \( A \) or \( b \) result in small (large) changes in the solution \( x \). The condition number of \( A \) is a measure of sensitivity of the solution of \( Ax = b \) for \( A \) being a nonsingular matrix. For the linear complementarity problem, \( M \) is a \( \text{P} \)-matrix if and only if \( \text{LCP}(M, q) \) has a unique solution for any vector \( q \). A linear complementarity problem is considered to be well-conditioned (ill-conditioned) if small changes in \( M \) or \( q \) result in small (large) changes in the solution \( x \). Based on the preceding analysis, we are able to give a perturbation theorem for the \( \text{P} \)-matrix \( \text{LCP} \), and define a measure of sensitivity of the solution of \( \text{LCP}(M, q) \) for \( M \) being a \( \text{P} \)-matrix.
Theorem 3.6 [5] Suppose

\[
\min(x, Mx + q) = 0 \quad M \in \mathbb{R}^{n \times n}, \quad 0 \neq (-q)_+ \in \mathbb{R}^n
\]
\[
\min(y, (M + \Delta M)y + q + \Delta q) = 0 \quad \Delta M \in \mathbb{R}^{n \times n}, \quad \Delta q \in \mathbb{R}^n.
\]

with

\[
\|\Delta M\| \leq \epsilon \|M\|, \quad \|\Delta q\| \leq \epsilon \max(\|(-q)_+\|, \|q\| - \|Mx + q\|).
\]

If \( M \) is a P-matrix and \( \epsilon \beta(M)\|M\| = \eta < 1 \), then \( M + \Delta M \) is a P-matrix and

\[
\frac{\|y - x\|}{\|x\|} \leq \frac{2\epsilon}{1 - \eta} \beta(M)\|M\|.
\]

Theorem 3.6 indicates that \( \beta(M)\|M\| \) is a measure of sensitivity of the solution of the LCP \((M, q)\) for \( M \) being a P-matrix. Application of Theorem 3.6 with Corollary 3.1, Corollary 3.2 and Theorem 3.5 gives \( \beta(M)\|M\| \) in the term of condition number for the H-matrix LCP, symmetric positive definite LCP and positive definite LCP.

Corollary 3.3 [5] Suppose

\[
\min(x, Mx + q) = 0 \quad M \in \mathbb{R}^{n \times n}, \quad 0 \neq (-q)_+ \in \mathbb{R}^n
\]
\[
\min(y, (M + \Delta M)y + q + \Delta q) = 0 \quad \Delta M \in \mathbb{R}^{n \times n}, \quad \Delta q \in \mathbb{R}^n.
\]

(i) If \( M \) is an H-matrix with positive diagonals, \( \epsilon \kappa_{\infty}(\tilde{M}) = \eta < 1 \), and

\[
\|\Delta M\|_{\infty} \leq \epsilon \|\tilde{M}\|_{\infty}, \quad \|\Delta q\|_{\infty} \leq \epsilon \max(\|(-q)_+\|, \|q\|_{\infty} - \|Mx + q\|_{\infty})
\]

then \( M + \Delta M \) is an H-matrix with positive diagonals and

\[
\frac{\|y - x\|_{\infty}}{\|x\|_{\infty}} \leq \frac{2\epsilon}{1 - \eta} \kappa_{\infty}(\tilde{M}).
\]

(ii) If \( M \) is a symmetric positive definite matrix, \( \epsilon \kappa_2(M) = \eta < 1 \), and

\[
\|\Delta M\|_2 \leq \epsilon \|M\|_2, \quad \|\Delta q\|_2 \leq \epsilon \max(\|(-q)_+\|_2, \|q\|_2 - \|Mx + q\|_2),
\]

then \( M + \Delta M \) is a P-matrix and

\[
\frac{\|y - x\|_2}{\|x\|_2} \leq \frac{2\epsilon}{1 - \eta} \kappa_2(M).
\]

(iii) If \( M \) is a positive definite matrix, \( \epsilon \kappa_2(\frac{M + M^T}{2}) = \eta < 1 \), and

\[
\|\Delta M\|_2 \leq \epsilon \frac{\|M + M^T\|_2}{2}, \quad \|\Delta q\|_2 \leq \epsilon \max(\|(-q)_+\|_2, \|q\|_2 - \|Mx + q\|_2) \frac{\|M + M^T\|_2}{2\|M\|_2},
\]

then \( M + \Delta M \) is a positive matrix, and

\[
\frac{\|x - y\|_2}{\|x\|_2} \leq \frac{2\epsilon}{1 - \eta} \kappa_2(\frac{M + M^T}{2}).
\]
Remark 3.1. If $Mx + q = 0$, then (i) of Corollary 3.3 for $M$ being an $M$-matrix and (ii) of Corollary 3.3 reduce to the perturbation bounds for the system of linear equations.

For the $H$-matrix LCP, componentwise perturbation bounds based on the Skeel condition number $||M^{-1}|| \tilde{M} ||_{\infty}$ can be represented as follows.

Theorem 3.7 [5] Suppose

$$\min(x, Mx + q) = 0 \quad M \in \mathbb{R}^{n \times n}, \quad 0 \neq (-q)_+ \in \mathbb{R}^n$$
$$\min(y, (M + \triangle M)y + q + \triangle q) = 0 \quad \triangle M \in \mathbb{R}^{n \times n}, \quad \triangle q \in \mathbb{R}^n.$$  

with

$$|\triangle M| \leq \epsilon |M|, \quad |\triangle q| \leq \epsilon \max((-q)_+, |q| - |Mx + q|). \quad (3.1)$$

If $M$ is an $H$-matrix with positive diagonals and $e \kappa_{\infty}(\tilde{M}) = \eta < 1$, then $M + \triangle M$ is an $H$-matrix with positive diagonals and

$$\frac{||y - x||_{\infty}}{||x||_{\infty}} \leq \frac{2\epsilon}{1 - \eta} ||M^{-1}|| \tilde{M} ||_{\infty}. \quad (3.2)$$

References


