数理解析研究所講究録 1461 巻 2005 年 96-110

Approximation Algorithms for Optimization Problems Related to the Edge Dominating Set

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1 Introduction

Let $\mathbb{Z}_+$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the sets of nonnegative integers, rational numbers and real numbers, respectively. Moreover, let $G=(V,E)$ be a simple undirected graph. We say that an edge $e=(u,v)$ *dominates* edges incident to u or v, and define an *edge dominating set* (EDS) to be a set F of edges such that each edge in E is dominated by at least one edge in F. Given a cost vector $w\in\mathbb{Q}_+^E$ together with G, the EDS problem asks to find an EDS with the minimum cost. This problem is one of the fundamental covering problems such as the well-known vertex cover problem and has some useful applications [2, 18]. The problem with a cost vector w with $w(e)=1$ for all $e\in E$ is called the cardinality case; otherwise the problem is called the cost case. The cardinality case is NP-hard even for some restricted classes of graphs such as planar or bipartite graphs of maximum degree 3 [10, 18]. Moreover, it is proven that the cardinality case is hard to approximate within any constant factor smaller than 7/6 unless P=NP [4]. In addition to these hardness, some polynomially solvable cases are also found for the cardinality case [10, 13, 17].

For the cost case, the problem is approximable within factor of $2r$ if there is an r-approximation algorithm for the minimum cost vertex cover problem [3], where currently $r\leq 2$ is known. Furthermore, Carr et al. [3] presented a 2.1-approximation algorithm. Their algorithm constructs an instance of the *minimum cost edge cover problem* from the original instance and finds an optimal edge cover for the resulting instance. A key property for this method is that an edge cover in the resulting instance is also an EDS for the original instance and that its cost is at most 2.1 times of the minimum cost of an EDS in the original instance. The property is proved based on a relation between the fractional edge dominating set polyhedron and the edge cover polyhedron. The former is a polyhedron containing all incidence vectors of EDSs, which may not be the convex hull of these vectors. In contrast the edge cover polyhedron is the convex hull of all incidence vectors of edge covers, which is shown to be an integer polyhedron [16]. Afterwards, Fujito and Nagamochi [6] gave a 2-approximation algorithm by using a refined EDS polyhedron. Moreover, Könemann et al. proposed 3-approximation algorithms for the problem of finding a minimum cost EDS which forms a tree or a tour [11].

In this paper, we discuss the approximability of the following four problems related to the EDS problem.

The (b,c)-*edge dominating set* ((b,c)-EDS) problem This is a capacitated version of the EDS problem. We are given a graph $G=(V,E)$, a demand vector $b\in\mathbb{Z}_+^E$, a capacity vector $c\in\mathbb{Z}_+^E$ and a cost vector $w\in\mathbb{Q}_+^E$. A set F of edges in G is called a (b,c)-EDS if each $e\in E$ is adjacent to at least $b(e)$ edges in F, where we allow F to contain at most $c(e)$ multiple copies of edge e. The problem asks to find a minimum cost (b,c)-EDS. If $b(e)=1$ for all $e\in E$, this problem is equivalent to the EDS problem.

The *EDS problem in hypergraphs* (HEDS problem) This is an extension of the EDS to a hypergraph. We are given a hypergraph $H = (V, E)$, and a cost vector $w \in \mathbb{Q}_+^E$. The problem asks to find a minimum cost hyperedge set F such that each hyperedge $e \in E$ is either contained in F or adjacent to a hyperedge in F.

The *(d, c)-edge cover with degree constraints over subsets* This is an extension of the edge cover. We are given a graph $G = (V, E)$, a cost vector $w \in \mathbb{Q}_+^E$, a family $\mathcal{S} \subseteq 2^V$ of vertex sets, a demand vector $b \in \mathbb{Z}_+^{\mathcal{S}}$, and a capacity vector $c \in \mathbb{Z}_+^E$. The problem asks to find a minimum cost edge set F such that the sum of degrees in graph (V, F) over $S \in \mathcal{S}$ is at least $b(S)$, where F can contain at most $c(e)$ multiple copies of edge e.

The *(b, c)-edge packing* problem This is a packing problem. As in the (b, c)-EDS problem, we are given $G = (V, E)$, $b \in \mathbb{Z}_+^E$, $c \in \mathbb{Z}_+^E$ and $w \in \mathbb{Q}_+^E$. A set F of edges in G is called a (b, c)-edge packing if each $e \in E$ is adjacent to at most $b(e)$ edges in F, where we allow F to contain at most $c(e)$ multiple copies of edge e. The problem asks to find a maximum cost (b, c)-edge packing.

The first two problems are proposed and analyzed by O. Parekh in [15]. The third problem is shown in [7] and the fourth problem is proposed in [8]. In this paper, we introduce these results and give more detailed analysis.

The paper is organized as follows. Section 2 introduces notations used in this paper. Sections 3, 6, 4 and 5 describe the above four problems and propose the approximation algorithms for them, respectively.

2 Preliminaries

We denote by $\theta_k \in \mathbb{Q}_+$ the k-th harmonic number $\sum_{i=1}^k \frac{1}{i}$. Let $G = (V, E)$ denote a simple undirected graph with a vertex set V and an edge set E. An edge $e = (u, v) \in E$ in G is defined as a pair of distinct vertices u and v. Let $H = (V, E)$ denote a hypergraph, where an edge is defined by a set of two or more vertices and an edge in H may be called a hyperedge. For a vertex v, $\delta(v)$ denotes the set of edges incident to v. For an edge e, $\delta(e)$ denotes the set of edges incident to vertices contained in e, i.e., $\delta(e) = \{e' \in E \mid e \cap e' \neq \phi\}$. For a subset $S \subseteq V$, $\delta(S)$ denotes the set of edges $e = (u, v)$ with $u \in S$ and $v \in V - S$, and $E[S]$ denotes the set of edges contained in S, i.e., $E[S] = \{e \in E \mid e \subseteq S\}$. Let x be an $|E|$-dimensional nonnegative real vector, i.e., $x \in \mathbb{R}_+^E$. We indicate the entry in x corresponding to an edge e by $x(e)$. For a subset F of E, we denote $x(F) = \sum_{e \in F} x(e)$. For an edge set F such that each edge $e' \in F$ corresponds to an edge $e \in E$, $x\langle F\rangle \in \mathbb{R}_+^F$ denotes a projection of x to F, i.e., $x\langle F\rangle(e') = x(e)$ for all $e' \in F$.

3 (b, c)-EDS problem

3.1 (b, c)-EDS, (d, c)-edge cover and polytopes

For a graph $G = (V, E)$, a demand vector $b \in \mathbb{Z}_+^E$, a capacity vector $c \in \mathbb{Z}_+^E$ and a cost vector $w \in \mathbb{Q}_+^E$, an integer program of the (b, c)-EDS is given as

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & x(e) \leq c(e) & \text{for each } e \in E, \\ & x(\delta(e)) \geq b(e) & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{1}$$

A vector $x \in \mathbb{Z}_+^E$ satisfying (1) is called a (b,c)-EDS.

Let us define a polytope EDS(G,b,c) as the set of vectors $x \in \mathbb{R}_+^E$ such that

$$\begin{array}{lll} \text{(a)} & 0 \leq x(e) \leq c(e) & \text{for each } e \in E, \\ \text{(b)} & x(\delta(e)) \geq b(e) & \text{for each } e \in E. \end{array}$$

This is the feasible region of an LP relaxation of problem (1). Thus the cost of an optimal solution in EDS(G,b,c) is a lower bound on the minimum cost of a given instance (G,b,c,w).

We now review some results on the (d,c)-*edge cover problem*, which is another important covering problem. This problem consists of a simple undirected graph $G = (V,E)$, a demand vector $d \in \mathbb{Z}_+^V$ defined on V, a capacity vector $c \in \mathbb{Z}_+^E$ and a cost vector $w \in \mathbb{Q}_+^E$. An integer vector $x \in \mathbb{Z}_+^E$ is called a (d,c)-*edge cover* if $x(\delta(v)) \geq d(v)$ for each $v \in V$ and $x(e) \leq c(e)$ for each $e \in E$. The objective of the (d,c)-edge cover problem is to find a minimum cost (d,c)-edge cover, which is formulated as

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & x(e) \leq c(e) & \text{for each } e \in E, \\ & x(\delta(v)) \geq d(v) & \text{for each } v \in V, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{2}$$

There exists a polynomial time algorithm for this problem [14]. Furthermore, it is known [16] that this problem has an equivalent linear program formulation, where the convex hull of all feasible solutions is characterized by the following set of inequalities:

$$\begin{array}{lll} \text{(c)} & 0 \leq x(e) \leq c(e) & \text{for each } e \in E, \\ \text{(d)} & x(\delta(v)) \geq d(v) & \text{for each } v \in V, \\ \text{(e)} & x(E[U]) + x(\delta(U)) - x(F) \geq \left\lceil \frac{d(U)-c(F)}{2} \right\rceil & \text{for each } U \subseteq V,\ F \subseteq \delta(U) \text{ with odd } d(U)-c(F). \end{array}$$

Let EC(G,d,c) denote the polytope represented by these inequalities.

3.2 Approximation algorithm

Given an instance (G,b,c,w) of the (b,c)-EDS problem, we first construct an instance of the (d,c)-edge cover and then computes an optimal solution for it as an approximate solution to the input instance. Formal description is given in Algorithm 1. A parameter f is given to the algorithm.

If the input instance is infeasible, then there exists an edge $e \in E$ with $c(\delta(e)) < b(e)$. Then, the LP relaxation to be solved in Step 1 is also infeasible. Hence DOMINATE(f) stops in Step 1 at that time.

We first show that $\bar{x}$ is a (b,c)-EDS. For an edge $e = (u,v) \in E$, let us suppose $x^*(\delta(u) - E') \geq x^*(\delta(v) - E')$. Then,

$$\bar{x}(\delta(u) - E') \geq d_{x^*}(u) \geq b(e) - c(\delta(e) \cap E').$$

The above first inequality holds since $\bar{x}\langle E - E' \rangle$ is a d_{x^*}-edge cover, and the second one holds by the definition of d_{x^*}. Since $\bar{x}(\delta(e) \cap E') = c(\delta(e) \cap E')$, it holds

$$\bar{x}(\delta(e)) \geq \bar{x}(\delta(u) - E') + \bar{x}(\delta(e) \cap E') \geq b(e).$$

We can easily check that $0 \leq \bar{x}(e) \leq c(e)$ also holds. Hence, $\bar{x}$ is a (b,c)-EDS and algorithm DOMINATE(f) outputs a feasible solution.

We then analyze the approximation factor of algorithm DOMINATE(f) by establishing a relation between EDS(G,b,c) and EC(G,d_{x^*},c). In the following discussion, we suppose that

Algorithm 1 DOMINATE(f)

Input: An instance (G, b, c, w) of the (b, c)-EDS problem.
Output: A (b, c)-EDS to instance (G, b, c, w) and a real $f > 0$.

Step 1: Compute an optimal solution $x^* \in \mathbb{R}_+^E$ to the linear program that minimizes $\min w^T x$ subject to $x \in \mathrm{EDS}(G, b, c)$. If it is infeasible, outputs "infeasible". Moreover, let $E' := \emptyset$.
Step 2: For each $e \in E$ with $fx^*(e) > c(e)$, let $\bar{x}(e) := c(e)$, $E' := E' \cup \{e\}$ and set $b(e') := \max\{0, b(e') - c(e)\}$ for all $e' \in \delta(e)$.
Step 3: For each edge $e = (u, v) \in E$, let $b'_{x^*}(u, e) := b(e)$ and $b'_{x^*}(v, e) := 0$ if $x^*(\delta(u) - E') \geq x^*(\delta(v) - E')$, and let $b'_{x^*}(u, e) := 0$ and $b'_{x^*}(v, e) := b(e)$ otherwise.
Step 4: For each vertex $v \in V$, let $d_{x^*}(v) := \max_{e \in \delta(v)} b'_{x^*}(v, e)$.
Step 5: Set $\bar{x}\langle E - E'\rangle$ to a minimum cost (d_{x^*}, c)-edge cover for $G' = (V, E - E')$ and $w\langle E - E'\rangle$. Then output $\bar{x}$ as a (b, c)-EDS to (G, b, c, w).

$b(e) \geq 1$ for at least one edge $e \in E$ because, if $b(e) = 0$ for all edges $e \in E$, DOMINATE(f) apparently outputs the optimal solution $\bar{x} = 0^E$. Moreover, we consider the case of $c = +\infty$ at first. In this case, parameter f makes no effect on the choice of E' in the algorithm and it always holds $E' = \emptyset$.

Lemma 1. *Let x be a vector in* $\mathrm{EDS}(G = (V, E), b, +\infty)$ *and $d_x \in \mathbb{Z}_+^V$ be a vector constructed from x by Step 4 of algorithm* DOMINATE(f). *Then vector $2x \in \mathbb{R}_+^E$ satisfies conditions* (c) *and* (d) *for* $\mathrm{EC}(G, d_x, +\infty)$.

Proof: Let $x \in \mathrm{EDS}(G, b, +\infty)$. Then vector $2x$ satisfies condition (c) for $\mathrm{EC}(G, d_x, +\infty)$ because $x \in \mathbb{R}_+^E$ holds by (a) for $\mathrm{EDS}(G, b, +\infty)$. We now show that $2x$ satisfies (d), i.e., $2x(\delta(v)) \geq d_x(v)$ for all $v \in V$. Let v be a vertex in V. Then there is an edge $e = (u, v) \in E$ such that $d_x(v) = b'_x(v, e)$. If $b'_x(v, e) = 0$, then we have $2x(\delta(v)) \geq 0 = d_x(v)$ since $x \in \mathbb{R}_+^E$ holds. Therefore, let us assume $b'_x(v, e) > 0$. Then $b'_x(v, e) = b(e)$ and $x(\delta(v)) \geq x(\delta(u))$ hold. Now $x(\delta(e)) \geq b(e)$ holds by (b) for $\mathrm{EDS}(G, b, +\infty)$, which implies $x(\delta(v)) + x(\delta(u)) = x(\delta(e)) + x(e) \geq b(e) + x(e)$ holds. Then we have

$$2x(\delta(v)) \geq x(\delta(u)) + x(\delta(v)) \geq b(e) + x(e) \geq b(e) = b'_x(v, e) = d_x(v).$$

Therefore, (d) also holds for $2x$. □

If $c = +\infty$, condition (e) for $\mathrm{EC}(G, b, c)$ is equivalent to

$$x(E[U]) + x(\delta(U)) \geq \left\lceil \frac{d(U)}{2} \right\rceil \quad \text{for each } U \subseteq V \text{ with odd } d(U).$$

This is because $d(U) - c(F) = -\infty$ for $F \neq \emptyset$.

Lemma 2. *For a simple undirected graph $G = (V, E)$ and a demand vector $d \in \mathbb{Z}_+^V$, let $\beta = \min_{v \in V, b(v) \neq 0} d(v)$. Then, for any vector $x' \in \mathbb{R}_+^E$ satisfying conditions* (c) *and* (d) *for* $\mathrm{EC}(G, d, +\infty)$, *vector*

$$y = \left(1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1}\right) x' \in \mathbb{R}_+^E$$

satisfies condition (e) *for* $\mathrm{EC}(G, d, +\infty)$.

Proof: Let U be a subset of V such that $d(U)$ is odd. It suffices to show that (e) holds for $x = y$ and U. If U contains a vertex v such that $d(v) = 0$, then (e) follows inductively from that $y(E[U']) + y(\delta(U')) \geq \lceil d(U')/2 \rceil$ for $U' = U - \{v\}$ since $y(E[U]) + y(\delta(U)) \geq y(E[U']) + y(\delta(U'))$ and $d(U) = d(U')$ hold. Hence we assume without loss of generality that $d(v) \geq \beta$ for all $v \in U$. Moreover, if $|U| = 1$, then (e) is implied by (d) since for $U = \{v\}$, $y(E[U]) + y(\delta(U)) = y(\delta(v)) \geq x'(\delta(v)) \geq d(v) \geq \lceil d(v)/2 \rceil$. We now consider the case of $|U| = 2$. Let $U = \{v_1, v_2\}$. Since $d(U) = d(v_1) + d(v_2)$ is odd, $d(v_1) \neq d(v_2)$ holds, where we assume without loss of generality $d(v_1) > d(v_2)$. Then

$$\left\lceil \frac{d(U)}{2} \right\rceil = \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil \leq d(v_1).$$

We have

$$x'(E[U]) + x'(\delta(U)) \geq x'(\delta(v_1))$$

because $E[U] \cup \delta(U) \supseteq \delta(v_1)$. Since x' satisfies $x'(\delta(v_1)) \geq d(v_1)$ by (d), we have

$$\begin{aligned} y(E[U]) + y(\delta(U)) &\geq x'(E[U]) + x'(\delta(U)) \\ &\geq x'(\delta(v_1)) \geq d(v_1) \geq \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil = \left\lceil \frac{d(U)}{2} \right\rceil. \end{aligned}$$

In what follows, we assume that $|U| \geq 3$ and $d(v) \geq \beta$ for all $v \in U$.

Since $x'(\delta(v)) \geq d(v)$ holds for all $v \in U$ by (d) for $\mathrm{EC}(G, d, +\infty)$, we have

$$2x'(E[U]) + x'(\delta(U)) = \sum_{v \in U} x'(\delta(v)) \geq d(U),$$

for which it holds

$$x'(E[U]) + x'(\delta(U)) \geq \frac{d(U) + x'(\delta(U))}{2} \geq \frac{d(U)}{2}.$$

To show (e), we only have to prove that

$$\frac{\lceil d(U)/2 \rceil}{d(U)/2} = 1 + \frac{1}{d(U)} \leq 1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1},$$

or equivalently

$$d(U) \geq 2\lfloor 3\beta/2 \rfloor + 1. \tag{3}$$

From the assumption, $d(U) \geq 3\beta$ holds. Moreover, since $d(U)$ is odd, $d(U) \geq 3\beta + 1$ if 3β is even. This implies (3). □

Theorem 1. *Let* $\beta = \min_{e \in E, b(e) \neq 0} b(e)$. *Algorithm* DOMINATE($f$) *delivers an approximate solution of a cost within a factor of*

$$\rho = 2\left(1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1}\right) \left(\leq \frac{8}{3}\right)$$

to the $(b, +\infty)$-EDS *problem.*

Proof: Let $\bar{x} \in \mathbb{Z}_+^E$ be a vector obtained by algorithm DOMINATE(f). We have already observed that $\bar{x}$ is a $(b, +\infty)$-EDS to instance $(G, b, +\infty, w)$. We show that $\bar{x}$ is a ρ-approximate solution. We denote by OPT the minimum cost of a $(b, +\infty)$-EDS for $(G, b, +\infty, w)$. Let $x^* \in \mathbb{R}_+^E$ be a vector computed in Step 1 of DOMINATE(f). Since EDS$(G, b, +\infty)$ contains a minimum cost $(b, +\infty)$-EDS, it holds $w^T x^* \leq$ OPT. By Lemma 1, vector $2x^*$ satisfies conditions (c) and (d) for EC$(G, d_{x^*}, +\infty)$. Since $b(e) \geq \beta$ for all $e \in E$ such that $b(e) \neq 0$, we see that $d_{x^*}(v) \geq \beta$ or $d_{x^*}(v) = 0$ holds for each $v \in V$. Therefore, from Lemma 2, we have $\rho x \in$ EC$(G, d_{x^*}, +\infty)$. Since algorithm DOMINATE(f) outputs a solution $\bar{x}$ of the minimum cost over all vectors in EC$(G, d_{x^*}, +\infty)$, we have $w^T \bar{x} \leq \rho w^T x^*$, from which it follows $w^T \bar{x} \leq \rho$OPT, as required. □

In addition, algorithm DOMINATE(f) achieves a better approximation factor in some special cases. We introduce some results (but omit the proofs).

Theorem 2. *For a demand vector $b \in \mathbb{Z}_+^E$ such that $\beta = \min_{e \in E} b(e) \geq 1$, algorithm* DOMINATE($f$) *delivers an approximate solution of a cost within a factor of*

$$\rho = 2\left(1 + \frac{1}{4\beta + 1}\right) \left(\leq \frac{12}{5}\right)$$

to the $(b, +\infty)$-EDS problem.

Theorem 3. DOMINATE(f) *is a 2-approximation algorithm for the $(b, +\infty)$-EDS problem in bipartite graphs.*

Theorem 4. *Suppose that $b(e) = \beta$ for all $e \in E$. Then algorithm* DOMINATE(f) *delivers an approximate solution of a cost within a factor of* 2.1 *if $\beta = 1$ or a factor of* 2 *if $\beta \geq 2$.*

Before closing this section, we mention the analysis shown in [15] for the case where c takes finite values for some edges. In this case, we need to set f to be an appropriate value. Let $\beta = \min_{U \subseteq V, F \subseteq \delta(U) - E'} d_x(U) - c(F)$ and ρ be the factor in Theorem 2 (resp., in Theorem 3) if the graph is not bipartite (resp., the graph is bipartite). If $f \geq \rho$, we can prove that $\rho x \langle E - E' \rangle \in$ EC$(G' = (V, E - E'), d_x, c)$, where $x \in$ EDS(G, b, c) and $\rho = 2(1 + 1/\beta)$ (the proof is similar with that of Lemmas 1 and 2). Then, algorithm DOMINATE(f) achieves the approximation factor of f because of the following reasons; The cost of output edges in E' is bounded as

$$w\langle E' \rangle^T \bar{x} \langle E' \rangle \leq w \langle E' \rangle^T c \langle E' \rangle < f w \langle E' \rangle^T x^* \langle E' \rangle .$$

With regard to edges in $E - E'$, it holds

$$w\langle E - E' \rangle^T \bar{x} \langle E - E' \rangle \leq \rho w \langle E - E' \rangle^T x^* \langle E - E' \rangle$$

from the above-mentioned relation. Hence, it holds

$$w^T \bar{x} = w \langle E' \rangle^T \bar{x} \langle E' \rangle + w \langle E - E' \rangle^T \bar{x} \langle E - E' \rangle < f w^T x^* \leq f\mathrm{OPT},$$

where OPT denotes the cost of the optimal solution. Notice that ρ depends on f. As we make f smaller with keeping $f \geq \rho$, we can obtain better approximation factor. Especially, DOMINATE(8/3) is a 8/3-approximation algorithm.

Theorem 5. DOMINATE(8/3) *is a* 8/3*-approximation algorithm for the (b, c)-EDS problem.*

We also obtain the same results described in Theorems 3 and 4.

Theorem 6. DOMINATE(2) *is a* 2*-approximation algorithm for the (b, c)-EDS problem in bipartite graphs.*

4 Hyperedge dominating set problem

We discuss the hypergraph version of the edge dominating set problem. The following result is also shown in [15].

Given a hypergraph $H = (V, E)$ and a cost vector $w \in \mathbb{Q}_+^E$, the problem is formulated as

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & x(\delta(e)) \geq 1 & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{4}$$

An HEDS is defined as a vector $x \in \mathbb{Z}_+^E$ such that $x(\delta(e)) \geq 1$ for each $e \in E$. In addition, we call the LP relaxation of the HEDS problem *the fractional HEDS problem* and its feasible solution a *fractional HEDS*.

To obtain an approximate solution to the HEDS problem, we transform a given instance of the HEDS problem to an instance of the *set cover problem*. The set cover problem is considered as a hypergraph version of the edge cover problem. A hyperedge set $F \subseteq E$ is called a *set cover* of hypergraph $H = (V, E)$ if $\cup_{e \in F} e = V$ and the set cover problem asks to find a minimum cost set cover. Given a hypergraph $H = (V, E)$ and a cost vector $w \in \mathbb{Q}_+^E$, the formulation of the set cover problem is given as follows.

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & x(\delta(v)) \geq 1 & \text{for each } v \in V, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{5}$$

Note that this problem is proven to be NP-hard [9]. Moreover, the LP relaxation of the set cover problem is called the fractional set cover problem and its feasible solution is called a fractional set cover. It is known that a simple greedy algorithm finds an approximate solution for the set cover problem, and that the cost of the solution is bounded in terms of the minimum cost of a fractional set cover, as described in the following theorem.

Theorem 7. [5, 12] *Let $w \in \mathbb{Q}_+^E$ be a given cost vector, $\hat{x}$ be a minimum cost fractional set cover for a hypergraph $H = (V, E)$, and k be the maximum size of a hyperedge in H. Then a set cover whose cost is at most $\theta_k w^T \hat{x}$ can be obtained in polynomial time.*

Since the HEDS problem is a subclass of the set cover problem, the HEDS problem can be reduced to the set cover problem directly. Let $(H = (V, E), w)$ be a given instance of the HEDS problem. Construct a hypergraph $H' = (V', E')$ such that its vertex set V' consists of vertices v'_e corresponding to its edges $e \in E$ and edge set E' consists of $e'_e = \{v'_{e''} \mid e'' \in \delta(e)\}$ corresponding to $\delta(e)$. A component of the cost vector $w'(e'_e)$ is set to be $w(e)$. Then, it is easy to see that a set cover for (H', w') gives an HEDS for (H, w) of same cost and vice versa. Let d be the maximum size of a hyperedge in E', i.e., the maximum size of $\delta(e)$ for all $e \in E$, where $d = O(|V|^k)$ holds for the maximum size k of a hyperedge in H. By Theorem 7, this direct reduction gives a θ_d-approximation algorithm for the HEDS problem. Note that $\theta_d = O(k \log |V|)$.

In algorithm HYPER described in Algorithm 2, an instance of the HEDS problem is transformed into an instance of the set cover problem. To prove that the approximation factor of HYPER is $k\theta_k$ by using Theorem 7, we show that for a vector x^* obtained in Step 1, vector kx^* is a fractional set cover to (H_{x^*}, w').

Lemma 3. *Let $x \in \mathbb{R}^E$ be a fractional* HEDS *for a hypergraph $H = (V, E)$, $H_x = (V', E')$ be a hypergraph obtained in Step 3 of* HYPER *from x and k be the maximum hyperedge size of H. Then vector $kx\langle E' \rangle \in \mathbb{R}^{E'}$ is a fractional set cover for H_x.*

Algorithm 2 HYPER

Input: A hypergraph $H = (V, E)$ and a cost vector $w \in \mathbb{Q}_+^E$.
Output: An HEDS for H.

Step 1: Find a minimum cost solution $x^* \in \mathbb{R}^E$ to the fractional HEDS problem for H and w.
Step 2: Let $V' := \{v \in V \mid x^*(\delta(v)) = \max_{u \in e} x^*(\delta(u)) \text{ for some } e \in E\}$, $E' := \{e \cap V' \mid e \in E\}$, and $w' := w\langle E' \rangle \in \mathbb{Q}_+^{E'}$.
Step 3: Find a set cover $\bar{x}$ for hypergraph $H_{x^*} = (V', E')$ such that $w'^T \bar{x}$ is at most θ_k times the minimum cost of a fractional set cover, and output $\bar{x}\langle E \rangle$ as an HEDS for H.

Proof: Suppose that $v \in V'$ is a vertex in a hyperedge $e \in E'$ such that $x(\delta(v)) \geq x(\delta(u))$ for all $u \in e$. Since $\sum_{u \in e} x(\delta(u)) \geq x(\delta(e)) \geq 1$, we have $x(\delta(v)) \geq 1/k$. Therefore $kx\langle E' \rangle(\delta(v)) \geq 1$, which means that $kx\langle E' \rangle \in \mathbb{R}^{E'}$ is a fractional set cover for H_x. □

Theorem 8. *The algorithm* HYPER *achieves approximation factor of $k\theta_k$ for the* HEDS *problem, where k is the maximum size of a hyperedge.*

Proof: Let $w'(F)$ be the minimum weight of fractional set covers of H_{x^*}. As described in the algorithm, $w'^T \bar{x} \leq \theta_k w'(F)$. Moreover, Lemma 3 indicates $w'(F) \leq k w'^T x^* \langle E' \rangle$. Hence

$$w'^T \bar{x} \leq \theta_k w'(F) \leq k\theta_k w'^T x^* \langle E' \rangle = k\theta_k w^T x^*.$$

Since $w'^T \bar{x}$ is the cost of solution algorithm outputs and $w^T x^*$ is a lower bound of the optimal cost, it completes the proof. □

Note that the approximation factor $k\theta_k = O(k \log k)$ of algorithm HYPER is superior to that of the algorithm obtained from the direct reduction if $k\theta_k < \theta_d$, i.e., H is a dense hypergraph such that $d = \Omega(|V|^k)$.

5 (d, c)-edge cover with degree constraints over subsets

Given an undirected graph $G = (V, E)$, a cost vector $w \in \mathbb{Q}_+^E$, a family $\mathcal{S} \subseteq 2^V$ of subsets of V, a demand vector $d \in \mathbb{Z}_+^{\mathcal{S}}$ and capacity vector $c \in \mathbb{Z}_+^E$, the (d, c)-edge cover with degree constraints over subsets is formulated by

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & \sum_{v \in S} x(\delta(v)) \geq d(S) & \text{for each } S \in \mathcal{S}, \\ & x(e) \leq c(e) & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{6}$$

Note that if $\mathcal{S} = \{\{v\} \mid v \in V\}$, then problem (6) is equivalent to the (d, c)-edge cover problem (2). If $\mathcal{S} = \{\{u, v\} \mid (u, v) \in E\}$, then problem (6) seems similar to the (b, c)-EDS problem, but its first constraint $x(\delta(u)) + x(\delta(v)) \geq b(e)$ on each $e = (u, v) \in E$ is different from the constraint $x(\delta(e)) \geq b(e)$ for the (b, c)-EDS. Let DC$(G, \mathcal{S}, d, c)$ denote the set of all vectors $x \in \mathbb{R}_+^E$ satisfying inequalities in (6), i.e., the relaxation of the covering problem. We show that problem (6) is approximable by algorithm COVER(f) described in Algorithm 3.

Algorithm 3 COVER(f)

Input: A simple undirected graph $G = (V, E)$, a cost vector $w \in \mathbb{Q}_+^E$, a family $\mathcal{S} \subseteq 2^V$ of subsets of V, a demand vector $d \in \mathbb{Z}_+^{\mathcal{S}}$, a capacity vector $c \in \mathbb{Z}_+^E$, and a real $f > 0$.

Output: A vector $x \in \mathbb{Z}_+^E$ feasible to the covering problem (6).

Step 1: Let $E' := \emptyset$. Moreover, find a minimum cost vector x^* over $\mathrm{DC}(G, \mathcal{S}, d, c)$ and let it x^*. If $\mathrm{DC}(G, \mathcal{S}, d, c) = \emptyset$, output "infeasible".

Step 2: For each $e \in E$, $fx^*(e) > c(e)$, then let $\bar{x}(e) := c(e)$, $E' := E' \cup \{e\}$, $d(S) := \max\{0, d(S) - 2c(e)\}$ for each $S \in \mathcal{S}$ with $e \in E[S]$, and $d(S) := \max\{0, d(S) - c(e)\}$ for each $S \in \mathcal{S}$ with $e \in \delta(S)$.

Step 3: For each $S \in \mathcal{S}$, $d'_{x^*}(v, S) := d(S)$ if $x^*(\delta(v) - E') \geq x^*(\delta(u) - E')$ for all $u \in S$ and $d'_{x^*}(v, S) := 0$ otherwise.

Step 4: For each $v \in V$, $\tilde{d}_{x^*}(v) := \max_{S \in \mathcal{S}: v \in S} d'_{x^*}(v, S)$.

Step 4: Compute a minimum cost $(\tilde{d}_{x^*}, c)$-edge cover $\bar{x}\langle E - E'\rangle$ for $G' = (V, E - E')$ and $w\langle E - E'\rangle$, and output $\bar{x}$ as a solution to (6).

For each $S \in \mathcal{S}$, the vector $v = \arg\max_{u \in S} x^*(\delta(u) - E')$ satisfies $\sum_{u \in S} \bar{x}(\delta(u) - E') \geq \bar{x}(\delta(v) - E') \geq \tilde{d}_{x^*}(v) \geq d(S) - 2c(E[S] \cap E') - c(\delta(S) \cap E')$. Since $\sum_{u \in S} \bar{x}(\delta(u)) \geq \sum_{u \in S} \bar{x}(\delta(u) - E') + 2\bar{x}(E[S] \cap E') + \bar{x}(\delta(S) \cap E') \geq d(S)$, we can see that $\bar{x}$ is a feasible for problem (6). In which follows, we discuss the approximation factor of COVER(f). It can be derived analogously to that of algorithm DOMINATE(f). First, let us consider the case of $c(e) = +\infty$. Notice that $E' = \emptyset$ for any f in this case.

Lemma 4. *Let $x \in \mathrm{DC}(G, \mathcal{S}, d, +\infty)$ and $h = \max_{S \in \mathcal{S}} |S|$. Then vector hx satisfies conditions* (c) *and* (d) *for $\mathrm{EC}(G, \tilde{d}_x, +\infty)$, where $\tilde{d}_x \in \mathbb{Z}_+^V$ is a vector obtained from x in Step 4 of* COVER(f).

Proof: Since $x \in \mathbb{R}_+^{E-E'}$, vector hx satisfies (a) for $\mathrm{EC}(G, \tilde{d}_x, +\infty)$. We show that hx satisfies (b), i.e., $hx(\delta(v)) \geq \tilde{d}_x(v)$ for each $v \in V$. Let v be a vertex in V. If $\tilde{d}_x(v) = 0$, then $hx(\delta(v)) \geq 0 = \tilde{d}_x(v)$ holds. Then assume $\tilde{d}_x(v) > 0$. There exists a subset $S \in \mathcal{S}$ such that $x(\delta(v)) \geq x(\delta(u))$ holds for all $u \in S$ and $\tilde{d}_x(v) = d'_x(v, S) = d(S)$. From this inequality and the condition $\sum_{u \in S} x(\delta(u)) \geq d(S)$ for $\mathrm{DC}(G, \mathcal{S}, d, +\infty)$, we have

$$hx(\delta(v)) \geq |S| x(\delta(v)) \geq \sum_{u \in S} x(\delta(u)) \geq d(S) = d'_x(v, S) = \tilde{d}_x(v).$$

This implies that hx satisfies (b) for $\mathrm{EC}(G', \tilde{d}_x, +\infty)$. □

Lemmas 2 and 4 indicate the following theorem.

Theorem 9. *Algorithm* COVER(f) *achieves the approximation factor of*

$$h \cdot \left(1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1}\right) \left(\leq \frac{4}{3}h\right)$$

for problem (6) *with $c(e) = +\infty$, where $h = \max\{|S| \mid S \in \mathcal{S}\}$ and $\beta = \min_{S \in \mathcal{S}, b(S) \neq 0} b(S)$.*

Proof: Let $y = h \cdot \left(1 + (2\lfloor 3\beta/2 \rfloor + 1)^{-1}\right) x^*$. By Lemmas 2 and 4, it holds $y \in \mathrm{EC}(G, \tilde{d}_x, +\infty)$, which implies that $w^T y$ is at least the minimum cost over $\mathrm{EC}(G, \tilde{d}_x, +\infty)$. Since the algorithm outputs a vector of minimum cost over $\mathrm{EC}(G, \tilde{d}_x, +\infty)$ and $w^T x^*$ is a lower bound of the optimal cost, the proof is completed. □

For the case where $c(e)$ is finite, we can also derive an approximation factor as in the (b, c)-EDS problem. In the worst case, it achieves the factor of $(4/3)h$.

6 The (b, c)-edge packing problem

6.1 (b, c)-edge packing, (d, c)-matching and polytopes

We now consider the edge packing problem, where $b(e)$ denotes an upper bound on the number of edges dominating e. The objective is to maximize the sum of costs of chosen edges. Formally the problem is described as follows.

$$\begin{array}{lll} \text{maximize} & w^T x & \\ \text{subject to} & x(\delta(e)) \le b(e) & \text{for each } e \in E, \\ & x(e) \le c(e) & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{7}$$

We call a feasible solution of this problem a (b, c)-edge packing, and this problem the (b, c)-*edge packing problem.*

Let $\mathrm{EP}(G, b, c)$ be the set of vectors $x \in \mathbb{R}_+^E$ satisfying

$$\begin{array}{lll} \text{(a)} & 0 \le x(e) \le c(e) & \text{for each } e \in E, \\ \text{(b)} & x(\delta(e)) \le b(e) & \text{for each } e \in E. \end{array}$$

Observe that $\mathrm{EP}(G, b, c)$ represents the feasible region of the linear programming problem obtained from problem (7) by relaxing its integer constraints. Although $\mathrm{EP}(G, b, c)$ contains all feasible solutions of (7), the set of all optimal solutions over the region may not include any integer solutions.

To approximate the (b, c)-edge packing problem, we consider the matching problem, which is one of the well-studied problems in the combinatorial optimization. This problem is generalized into the following capacitated d-matching problem.

$$\begin{array}{lll} \text{maximize} & w^T x & \\ \text{subject to} & x(\delta(v)) \le d(v) & \text{for each } v \in E, \\ & x(e) \le c(e) & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E, & \end{array} \tag{8}$$

where $d \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$ are given capacities. In the following, we call the capacitated d-matching problem with capacities d and c the (d, c)-*matching problem* and its feasible solution a (d, c)-*matching*. Note that in the (d, c)-matching problem, the second constraint $x(e) \le c(e)$ is not essential because all instances of the (d, c)-matching problem can be reduced to a special case of (d, c)-matching problem where $c = +\infty$ [16].

It is known that the (d, c)-matching problem can be solved in strongly polynomial time [1, 16]. Let $\mathrm{MA}(G, d, c)$ be the set of vectors $x \in \mathbb{R}_+^E$ such that

$$\begin{array}{lll} \text{(c)} & 0 \le x(e) \le c(e) & \text{for each } e \in E, \\ \text{(d)} & 0 \le x(\delta(v)) \le d(v) & \text{for each } v \in V, \\ \text{(e)} & x(E[U]) + x(F) \le \left\lfloor \frac{d(U) + c(F)}{2} \right\rfloor & \text{for each } U \subseteq V,\ F \subseteq \delta(U) \\ & & \text{with } d(U) + c(F) \text{ odd.} \end{array}$$

MA(G, d, c) is an integer polytope, whose all extreme points are represented by integer vectors [17]. Since every integer vector satisfying conditions (c) and (d) is a (d, c)-matching in G, maximizing $w^T x$ over the polytope MA(G, d, c) is essentially equivalent to solving problem (8). Note that, in the (b, c)-edge packing problem, b is an $|E|$-dimensional vector, while d is defined as a $|V|$-dimensional vector in the (d, c)-matching problem.

6.2 Approximation algorithm

To construct an approximate solution to a given instance (G, b, c, w) of the (b, c)-edge packing problem, we solve an instance (G, d, c, w) of the (d, c)-matching problem. The capacity vector d will be defined so that a (d, c)-matching is also a (b, c)-edge packing in G. The algorithm is described in Algorithm 4.

Algorithm 4 PACK

Input: An instance (G, b, c, w) of the (b, c)-edge packing problem.
Output: A (b, c)-edge packing.

Step 1: For each $e = (u, v) \in E$, $b'(u, e) := \lfloor b(e)/2 \rfloor$ and $b'(v, e) := \lceil b(e)/2 \rceil$.
Step 2: For each $v \in V$, $d(v) := \min_{e \in \delta(v)} b'(v, e)$.
Step 3: Compute a maximum cost (d, c)-matching $\bar{x} \in \mathbb{Z}_+^E$ for the graph G and the cost vector w, and output $\bar{x}$ as a (b, c)-edge packing.

Integer vectors $x \in \mathbb{Z}^E$ satisfying (c) and (d) of MA(G, d, c) are (b, c)-edge packings because $x(\delta(e)) \le x(\delta(u)) + x(\delta(v)) \le d(u) + d(v) \le b(e)$ hold. In the following, we analyze the approximation factor of algorithm PACK.

Lemma 5. *Let $x \in$ EP(G, b, c), and $d \in \mathbb{Z}_+^V$ be a vector obtained in Step 2 of algorithm* PACK. *Vector*

$$x' = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\beta_1}\right) x & \text{if there is an edge with odd } b(e) \\ \frac{1}{2} x & \text{otherwise} \end{cases}$$

satisfies conditions (c) *and* (d) *for* MA(G, d, c), *where* $\beta_1 = \min_{e \in E, b(e) \text{ is odd}} b(e)$.

Proof: Since $x \in$ EP(G, b, c) satisfies $0 \le x(e) \le c(e)$ for each $e \in E$, it is immediate to see that x' satisfies (c) for MA(G, d, c). Then, we show that $x'(\delta(v)) \le d(v)$ holds for each $v \in V$.

Let $v \in V$. There is an edge $e = (u, v) \in E$ such that $d(v) = b'(v, e)$. Note that $x(\delta(v)) \le x(\delta(e)) \le b(e)$ hold by (b) for EDS(G, b, c). If $b'(v, e) = \lceil b(e)/2 \rceil$, then it holds

$$x'(\delta(v)) \le \frac{1}{2} x(\delta(v)) \le \frac{x(\delta(e))}{2} \le \frac{b(e)}{2} \le \left\lceil \frac{b(e)}{2} \right\rceil = d(v),$$

This implies that $x'(\delta(v))$ satisfies (d) in MA(G, d, c),

Consider the other case, $b'(v, e) < \lceil b(e)/2 \rceil$, i.e., $b(e)$ is odd and $b'(v, e) = \lfloor b(e)/2 \rfloor$. Since $x(\delta(v)) \le b(e)$ and $d(v) = b'(v, e) = (b(e) - 1)/2$, we have

$$\frac{d(v)}{x(\delta(v))} \ge \frac{b(e) - 1}{2b(uv)} = \frac{1}{2} - \frac{1}{2b(e)}.$$

From the assumption, $b(e) \geq \beta_1$. Then,

$$\frac{1}{2} - \frac{1}{2b(e)} \geq \frac{1}{2}\left(1 - \frac{1}{\beta_1}\right).$$

Therefore $x'(\delta(v))$ satisfies (d) for MA(G, d, c). □

Lemma 6. *Let $x \in \mathbb{R}_+^E$ satisfy* (c) *and* (d) *for* MA(G,d,c) *and* $\beta_2 = \lfloor \min_{e \in E, b(e) \neq 0} b(e)/2 \rfloor$. *Then vector*

$$\left(1 - \frac{1}{2\lfloor 3\beta_2/2 \rfloor + 1}\right) x$$

satisfies (e) *for* MA(G,d,c).

Proof: Let U be a nonempty subset of V, and F be a subset of $\delta(U)$ which can be empty. It suffices to show that

$$x(E[U]) + x(F) \leq \left\lfloor \frac{d(U) + c(F)}{2} \right\rfloor.$$

We can assume that U contains no vertices v such that $d(v) = 0$ (i.e., $x(\delta(v)) = 0$) because the above inequality for such U and any F is obtained from the one for $U - \{v\}$ and $F - \delta(v)$.

Since x satisfies (d) for MA(G, d, c), it holds

$$2x(E[U]) + x(\delta(U)) = \sum_{v \in U} x(\delta(v)) \leq \sum_{v \in U} d(v) = d(U),$$

from which we have

$$x(E[U]) \leq \frac{d(U) - x(\delta(U))}{2}. \tag{9}$$

From (c), $x(F) = \sum_{e \in F} x(e) \leq \sum_{e \in F} c(e) = c(F)$ holds. From this inequality and (9), we have

$$x(E[U]) + x(F) \leq \frac{d(U) + c(F) - (x(\delta(U)) - x(F))}{2}.$$

Since $x(\delta(U)) - x(F) \geq 0$ holds by $F \subseteq \delta(U)$, it holds

$$x(E[U]) + x(F) \leq \frac{d(U) + c(F)}{2}. \tag{10}$$

The gap between $(d(U)+c(F))/2$ and $\lfloor (d(U) + c(F))/2 \rfloor$ depends on the parity of $d(U)+c(F)$. Therefore we only have to consider the case where $d(U) + c(F)$ takes a minimum odd value. We consider the following three sub-cases.

Case 1: $|U| = 1$. Let $U = \{v\}$. Then $x(E[U]) = 0$. Therefore the left hand side of (e) equals to $x(F)$. Since $d(U) + c(F) = d(v) + c(F)$ is assumed to be odd, it holds $d(v) \neq c(F)$, which implies $d(u) + c(F) \geq 2\min\{d(u), c(F)\} + 1$. From (c), $x(F) \leq c(F)$ holds. Moreover, $x(F) \leq x(\delta(v)) \leq d(v)$ holds since $F \subseteq \delta(v)$. Therefore we have

$$x(F) \leq \min\{c(F), d(v)\} \leq \frac{d(U) + c(F) - 1}{2} = \left\lfloor \frac{d(U) + c(F)}{2} \right\rfloor.$$

Case 2: $|U| = 2$. Let $U = \{v_1, v_2\}$, $F_1 = \delta(v_1) \cap F$, and $F_2 = \delta(v_2) \cap F$. Then $d(U) + c(F) = d(v_1) + d(v_2) + c(F_1) + c(F_2)$. From the facts that $\delta(v_1) \cup F_2 \supseteq E[U] \cup F$ and that $\delta(v_2) \cup F_1 \supseteq E[U] \cup F$, we have

$$x(E[U]) + x(F) \leq \min\{x(\delta(v_1)) + x(F_2), x(\delta(v_2)) + x(F_1)\}. \tag{11}$$

It holds that $x(\delta(v_1)) \leq d(v_1)$ and $x(\delta(v_2)) \leq d(v_2)$ from (d). Moreover, we have $x(F_1) \leq c(F_1)$ and $x(F_2) \leq c(F_2)$ from (c). These relations and inequality (11) lead to

$$x(E[U]) + x(F) \leq \min\{d(\delta(v_1)) + c(F_2), d(\delta(v_2)) + c(F_1)\}. \tag{12}$$

On the other hand, since $d(U) + c(F)$ is assumed to be odd, it holds $d(\delta(v_1)) + c(F_2) \neq d(\delta(v_2)) + c(F_1)$, which implies that

$$\min\{d(\delta(v_1)) + c(F_2), d(\delta(v_2)) + c(F_1)\} \leq \left\lfloor \frac{d(U) + c(F)}{2} \right\rfloor. \tag{13}$$

From (12) and (13), we have (e) for $\mathrm{MA}(G, d, c)$.

Case 3: $|U| \geq 3$. Since $b(e) \geq \min_{e \in E, b(e) \neq 0} b(e)$ for all $e \in E$, it holds $d(v) \geq \beta_2$ for all $v \in V$. Hence $d(U) \geq 3\beta_2$. Considering that $d(U) + c(F)$ is odd, we have

$$d(U) + c(F) \geq 2 \left\lfloor \frac{3\beta_2}{2} \right\rfloor + 1.$$

From (10) and the above inequality,

$$\frac{\lfloor (d(U) + c(F))/2 \rfloor}{x(E[U]) + x(F)} \geq 1 - \frac{1}{d(U) + c(F)} \geq 1 - \frac{1}{2\lfloor 3\beta_2/2 \rfloor + 1}.$$

This completes the proof of the lemma. □

Theorem 10. *Let d be a vector constructed in Step 2 of algorithm* PACK. *Then $MA(G, d, c)$ is a polyhedron whose maximum cost extreme points are $\rho(\beta_1, \beta_2)$-approximate solutions of the (b, c)-edge packing problem for a graph G, where*

$$\rho(\beta_1, \beta_2) = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\beta_1}\right) \cdot \left(1 - \frac{1}{2\lfloor 3\beta_2/2 \rfloor + 1}\right) & \textit{if there is an edge } e \textit{ with odd } b(e), \\ \frac{1}{2}\left(1 - \frac{1}{2\lfloor 3\beta_2/2 \rfloor + 1}\right) & \textit{otherwise,} \end{cases}$$

$\beta_1 = \min_{e \in E, b(e) \text{ is odd}} b(e)$ *and* $\beta_2 = \lfloor \min_{e \in E, b(e) \neq 0} b(e)/2 \rfloor$.

Proof: We have already observed that an integer vector $x \in \mathbb{Z}_+^E$ in $\mathrm{MA}(G, d, c)$ is a (b, c)-edge packing. Since $\mathrm{MA}(G, d, c)$ is an integer polytope, every extreme point is a (b, c)-edge packing.

Denote by OPT the maximum cost of a solution to problem (7) with (G, b, c, w). In what follows, we show that $f(\beta_1, \beta_2)\mathrm{OPT} \leq w^T \bar{x}$ holds, where $\bar{x}$ is a maximum cost extreme point of $\mathrm{MA}(G, d, c)$. Let $x^* \in \mathrm{EP}(G, b, c)$ be a vector of the maximum cost $w^T x^*$ for the cost $w \in \mathbb{Q}_+^E$. Because $\mathrm{EP}(G, b, c)$ contains an optimal solution to problem (7), it holds

$$\mathrm{OPT} \leq w^T x^*.$$

By Lemmas 5 and 6, we can see that vector $\rho(\beta_1, \beta_2)x^*$ belongs to $\mathrm{MA}(G, d, c)$. By the maximality of $w^T\bar{x}$ over $\mathrm{MA}(G, d, c)$, it holds

$$\rho(\beta_1, \beta_2)w^T x^* \leq w^T \bar{x}.$$

From the above two inequalities, we have

$$\rho(\beta_1, \beta_2)\mathrm{OPT} \leq w^T \bar{x},$$

as required. □

The above theorem is equivalent to saying that the approximation factor of algorithm PACK is $\rho(\beta_1, \beta_2)$ because algorithm PACK outputs a maximum cost vector over the polyhedron $\mathrm{MA}(G, d, c)$.

Corollary 1. *Let* $\beta_1 = \min_{e \in E, b(e) \text{ is odd}} b(e)$ *and* $\beta_2 = \lfloor \min_{e \in E, b(e) \neq 0} b(e)/2 \rfloor$. *Then the approximation factor of algorithm* PACK *is* $\rho(\beta_1, \beta_2)$.

Note that $\rho(\beta_1, \beta_2) = 0$ if E contains an edge e such that $b(e) = 1$. We know that this problem cannot be approximated within a constant factor in the case where $b(e) = 1$ for all $e \in E$ [8]. Hence it would be appropriate to assume that $b(e) = 0$ or $b(e) \geq 2$ for all $e \in E$ (i.e., $\beta_1 \geq 3$ and $\beta_2 \geq 1$). Particularly in the worst case, where $\beta_1 = 3$ and $\beta_2 = 1$, $\rho(\beta) = 2/9$ holds.

Acknowledgment

This research was partially supported by the Scientific Grant-in-Aid from Ministry of Education, Culture, Sports, Science and Technology of Japan.

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