数理解析研究所講究録 1462 巻 2006 年 145-155

A review of recent results on weak approximation of solutions of stochastic differential equations*

Arturo Kohatsu-Higa
Osaka University
Graduate School of Engineering Sciences
Machikaneyama cho 1-3, Osaka 560-8531, Japan

December 15, 2005

Abstract

In this article, I give a brief review of some recent results concerning numerical schemes to approximate solutions of stochastic differential equations. We concentrate on results about weak approximation.

1 Introduction

The Euler-Maruyama scheme is a naive approximation method for the solution of various types of stochastic differential equations. It helps not only to simulate the solutions of stochastic equations but it also serves theoretical purposes (see e.g. the articles of E. Gobet on the LAMN property in statistics).

To introduce this notion consider the stochastic differential equation

$$X(t) = x + \int_0^t b(X(s))ds + \sum_{j=1}^{r} \int_0^t \sigma_j(X(s))dZ^j(s), \tag{1}$$

where $b, \sigma_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, ..., r$, are Lipschitz coefficients. Z is a Lévy process. That is, a stochastically continuous process with independent increments with characteristic function given by

$$E\left[\exp\left(i\left\langle \theta, Z(t)\right\rangle\right)\right] = \exp\left(-\frac{1}{2}\left\|\theta\right\|^2 t - \left\langle b, \theta\right\rangle t - \int_{\mathbb{R}^r} \left(\exp(i\left\langle \theta, x\right\rangle) - 1 - i\theta x 1\left\{x \leq 1\right\}\right)\nu(dx)\right)$$

where $\theta \in \mathbb{R}^r$ and ν is a measure satisfying $\int_{\mathbb{R}^r} \left(1 \wedge |x|^2\right)\nu(dx) < \infty$. When $b = \nu = 0$ then Z is a standard r-dimensional Wiener process. b denotes the drift of the process and ν is the Lévy measure associated to the process Z. We note that in comparison with the Wiener process case not all moments of Z are finite. In fact the moment of order k of Z is bounded if $\int_{\mathbb{R}^r} |x|^k 1(x \geq 1)\nu(dx) < \infty$.

The existence and uniqueness of the above equation (1) is assured by standard theorems that can be found in e.g. Protter under Lipschitz assumptions on the coefficients b and σ. Nevertheless it is not clear under which conditions the moments of the solution are finite if Z is a Lévy process, except for the case of bounded coefficients.

*Keywords: duality, Euler-Maruyama scheme, stochastic differential equations.
I would like to express my deep thanks to Prof. S. Ogawa for inviting me to this conference and his continuous support.

In particular, we do not know how the finite moment property transfers from Z into X when the coefficients are Lipschitz. These properties are important in order to determine the convergence properties of the Euler scheme. The situation in the case that σ is constant is already difficult enough. Nevertheless, this is an interesting problem.

We quote here some results of the article Kohatsu-Yamazato that study this problem in the particular case that σ is constant.

For example, consider for simplicity the one dimensional case $r = d = 1$ and ν is a measure concentrated on $(0, \infty)$. Then consider $E(X(t)^{\beta})$ for $\beta > 0$.

$b(y) = y^{\alpha}$	β	Criteria
$0 \leq \alpha \leq 1$	$\beta > 0$	$\int_1^{+\infty} y^{\beta} \nu(dy)$
$\alpha > 1$	$\beta < \alpha - 1$	always finite
$\alpha > 1$	$\beta = \alpha - 1$	$\int_1^{+\infty} \log(y)\, \nu(dy)$
$\alpha > 1$	$\beta > \alpha - 1$	$\int_1^{+\infty} y^{\beta - \alpha + 1} \nu(dy)$

The last column in the table above determines if the corresponding moment is finite or not. In the same lines of the above table, but in another set up, Grigoriu-Samorodnistsky studied the tail behavior of $X(t)$. In either case the conclusions are similar.

The rule seems to be that if the drift coefficient is sublinear then the drift does not influence the finite moment property of Z and it transfers directly to X. If the drift is superlinear then the situation is different. That is, the finite moment property depends on the difference of power between the drift and the moment to be evaluated. Therefore, it can be conjectured that this is the situation in the Lipschitz cases.

Currently, as far as my knowledge goes it is not known if X has finite moments even if the exponential moments of Z are bounded unless one imposes a series of stringent conditions. In most papers found in the literature besides this assumption one also has to make the assumption that the moments of X are bounded which is an unaccomplished feature of this problem.

For a partition of the interval $[0, T]$ denoted as $\pi : 0 = t_0 < \ldots < t_n = T$, we define the norm of the partition as $\|\pi\| = \max\{t_{i+1} - t_i; i = 0, \ldots, n-1\}$ and $\eta_1(s) = \sup\{t_i; t_i \leq t\}$ and $\eta_2(t) = \inf\{t_i; t_i \geq t\}$. Then the Euler-Maruyama scheme is defined as

$$X^n(t_{i+1}) = X^n(t_i) + b(X^n(t_i))(t_{i+1} - t_i) + \sum_{j=1}^{r} \sigma_j(X^n(t_i))(Z^j(t_{i+1}) - Z^j(t_i)).$$

The simplicity of this scheme and the generality of the possibility of applications are the main attractions of this scheme. First we mention the strong convergence rate result.

Theorem 1 *Suppose that Z has exponential moments and that X has finite moments. Then*

$$E\left[\sup_{t \leq T} \|X(t) - X^n(t)\|^{2p}\right] \leq C \|\pi\|^p$$

where the constant C depends on T, x and the Lipschitz coefficients.

The proof of this result is standard and goes through the same methodology to prove existence of solutions. This result can also be generalized to various equations without changing the essential ideas.

One remarkable different case is the situation of reflecting stochastic differential equations. In general if the domain is closed and convex then the results can be usually obtain as generalizations of the non-reflecting case. The main difference lies in how the inequalities are obtained. In fact, instead of using strong type inequalities directly on the error process $X(t) - X^n(t)$, one has to use Ito's formula and the fact that when the reflecting process is acting then $\langle X(t) - X^n(t), d(K_t - K_t^n)\rangle$ where K and K^n are the reflecting processes (or local times) of X and X^n respectively. If the domain is more general then the results are no longer valid. In fact, as proven by Pettersson (later refined by Slominski) the rates can decay slightly depending on the properties of the domain.

This is true but we do not discuss here and we refer the reader to a recent thesis by Menozzi.

2 From the weak error a la Jacod-Kurtz-Protter to the weak error a la Talay-Platen etc.

If one is trying to approach the problem of weak convergence of the error process then the first natural approach is to study the weak convergence of the process

$$\sqrt{n}\,(X(t)-X^n(t)).$$

This is done in a series of articles by Jacod, Kurtz and Protter. To simplify the ideas suppose that we are dealing with the Wiener case in one dimension, $b\equiv 0$ and that the partition is uniform. Then we can write a continuous extension of the process X^n as

$$X^n(t)=x+\int_0^t \sigma(X^n(\eta(s)))dW(s).$$

Then we have that

$$X(t)-X^n(t)=\int_0^t \sigma_1^n(s)\,(X(s)-X^n(s))\,dW(s)+\int_0^t \sigma_2^n(s)\,(W(s)-W(\eta(s)))\,dW(s) \qquad (2)$$

where

$$\begin{aligned}\sigma_1^n(s)&=\int_0^1 \sigma'(\alpha X(s)+(1-\alpha)X^n(s))d\alpha\\ \sigma_2^n(s)&=\int_0^1 \sigma'(\alpha X^n(s)+(1-\alpha)X^n(\eta(s)))d\alpha\sigma(X^n(\eta(s))).\end{aligned}$$

Given the strong convergence result and assuming smoothness of σ one has that σ_1^n and σ_2^n converge in the $L^p(C[0,T],\mathbb{R})$-norm to

$$\begin{aligned}\sigma_1(s)&=\sigma'(X(s))\\ \sigma_2(s)&=\sigma'\sigma(X(s)).\end{aligned}$$

Therefore if we solve (2), we obtain that

$$\begin{aligned}X(t)-X^n(t)=&\,\mathcal{E}^n(t)^{-1}\int_0^t \mathcal{E}^n(s)\sigma_2^n(s)\,(W(s)-W(\eta(s)))\,dW(s)\\ &-\mathcal{E}^n(t)^{-1}\int_0^t \mathcal{E}^n(s)\sigma_1^n\sigma_2^n(s)\,(W(s)-W(\eta(s)))\,ds,\end{aligned}$$

where

$$\mathcal{E}^n(t)=\exp\left(-\int_0^t \sigma_1^n(s)dW(s)-\frac{1}{2}\int_0^t (\sigma_1^n(s))^2\,ds\right)$$

is the Doleans-Dade exponential. Now consider the process

$$\sqrt{n}\int_0^t (W(s)-W(\eta(s)))\,dW(s)=\frac{\sqrt{n}}{2}\left(\sum_{i=0}^{j(t)}(W(t_{i+1})-W(t_i))^2+(W(t)-W(\eta(t)))^2-t\right),$$

where $t_{j(t)}=\eta(t)$. Then using Donsker's theorem (see e.g. Billingsley p. 68) we have that

$$\sqrt{n}\int_0^{\cdot} (W(s)-W(\eta(s)))\,dW(s)\Longrightarrow W'$$

where W' is a Wiener process independent of W. This follows because

$$\left\langle \sqrt{n} \int_0^{\cdot} (W(s) - W(\eta(s)))\, dW(s), W \right\rangle = \sqrt{n} \int_0^{\cdot} (W(s) - W(\eta(s)))\, ds$$

and

$$\sqrt{n} \int_0^{\cdot} H_s \,(W(s) - W(\eta(s)))\, dW(s) \to 0$$

in probability in the space $C[0,T]$. In fact, first suppose that H is a simple bounded process, that is

$$H_t = \sum_{j=1}^{m} H_i 1(s_j < t \leq s_{j+1}).$$

Then

$$\begin{aligned}\sqrt{n} \int_0^t H_s \,(W(s) - W(\eta(s)))\, ds &= \sum_{i=1}^{m} \sqrt{n} H_i \sum_{j=j(i)}^{j(i+1)} \int_{t_j}^{t_{j+1}} (W(s) - W(t_j))\, ds \\ &= \sum_{i=1}^{m} \sqrt{n} H_i \sum_{j=j(i)}^{j(i+1)} \int_{t_j}^{t_{j+1}} (t_{j+1} - s) dW(s).\end{aligned}$$

Therefore

$$\begin{aligned} E\left|\sum_{i=1}^{m} \sqrt{n} H_i \sum_{j=j(i)}^{j(i+1)} \int_{t_j}^{t_{j+1}} (t_{j+1} - s) dW(s)\right|^2 &\leq C \sum_{i=1}^{m} n \sum_{j=j(i)}^{j(i+1)} \left(\int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 ds \right) \\ &\leq C n^{-1}. \end{aligned}$$

Now suppose that H is a bounded process and let H_m be a sequence of bounded simple process that converges to H in the following sense

$$\lim_{m \to \infty} E\left[\int_0^T (H_m(s) - H(s))^2 ds\right] = 0.$$

Then

$$\begin{aligned} & E\left[\left|\int_0^T (H_m(s) - H(s))\sqrt{n}(W(s) - W(\eta(s)))ds\right|^2\right] \\ & \leq C \left(E\left[\int_0^T (H_m(s) - H(s))^2 ds\right]\right)^{1/2} \left(E\left[n \int_0^T (W(s) - W(\eta(s)))^2 ds\right]\right)^{1/2} \\ & \leq C \left(E\left[\int_0^T (H_m(s) - H(s))^2 ds\right]\right)^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} & E\left[\left(\int_0^T H_s \sqrt{n}(W(s) - W(\eta(s)))ds\right)^2\right] \\ & \leq C \left(E\left[\int_0^T (H_m(s) - H(s))^2 ds\right]\right)^{1/2} + E\left[\left(\sqrt{n} \int_0^T H_m(s)\,(W(s) - W(\eta(s)))\, ds\right)^2\right]. \end{aligned}$$

Therefore the argument follows by taking limits with respect to n and then with respect to m. To prove that the sequence is tight is not difficult as

$$E\int_u^t \left|H_s\sqrt{n}(W(s)-W(\eta(s)))\right| ds \leq C\left(E\left[\int_u^t |H_s|^2\, ds\right]\right)^{1/2}.$$

Therefore we have that

$$\left(W, \sqrt{n}\int_0^{\cdot} (W(s)-W(\eta(s)))\,dW(s)\right) \Longrightarrow (W,W')$$

where W and W' are two independent Wiener processes. Putting together all the above calculations, we have that

$$\sqrt{n}\,(X(t)-X^n(t)) \Longrightarrow \mathcal{E}(t)^{-1}\int_0^t \mathcal{E}(s)\sigma_2(s)dW(s),$$

where

$$\mathcal{E}(t)=\exp\left(-\int_0^t \sigma_1(s)dW(s)-\frac{1}{2}\int_0^t (\sigma_1(s))^2\, ds\right)$$

and (W,W') is a 2 dimensional Wiener process.

This result in a variety of forms and generalizations have been extensively proved by Jacod, Kurtz and Protter.

In particular from this result one obtains that for any continuous bounded functional F in $C[0,T]$ one has that $E\left[F\left(\sqrt{n}\,(X-X^n)\right)\right]$. Nevertheless, this does not give full information about the rate of convergence of various other functionals that may be interesting from an application point of view. For example, $E(X(t)^2)-E(X^n(t)^2)$. For this reason other efforts have been directed into extending the type of convergence into stronger topologies than than one given by weak convergence of processes. In [13], the authors prove that for any bounded continuous real valued function f and any bounded real variable Y we have that

$$E(Yf(\sqrt{n}\,(X-X^n))) \to E(Yf(\mathcal{E}(\cdot)^{-1}\int_0^{\cdot} \mathcal{E}(s)\sigma_2(s)dW(s)U)).$$

This type of convergence is called stable convergence in law. This type of results are promising but still it does not allow for the analysis of the convergence of quantities like $E(X(t)^2)$.

In order to analyze this problem, there is another "parallel" theory called weak approximation that deals particularly with the error

$$E\left[f(X)-f(X^n)\right].$$

This theory started by D. Talay which is based on the Feyman-Kac formula and the partial differential equation satisfied by the fundamental solution (or density) of the solution of (1) is the central point. The state of the art using this technique is more advanced than the one given previously by the theory of Jacod-Kurtz-Protter. In fact one is able to deal with non bounded, non continuous and even Schwartz distribution functions f. On the other hand one is not able to give precise information on the distribution of the limit error. Nevertheless in the above calculation there are a variety of ideas that can be used in the previous proposed problem. In fact, just to explain in the light of the Jacod-Kurtz-Protter approach the ideas behind an approach for weak approximation, let's explain in simple terms a complex result due to V. Bally and D. Talay.

To clarify the methodology, we consider a real diffusion process (that is $Z=W$ a Wiener process)

$$X_t = x+\int_0^t \sigma(X_s)dW_s,\, t\in[0,T]$$

and its Euler approximation

$$X_t^n = x+\int_0^t \sigma(X^n_{\eta(s)})dW_s,\, t\in[0,T]$$

where $\eta(s) = kT/n$ for $kT/n \le s < (k+1)T/n$. The error process $Y = X - X^n$ solves

$$Y_t = \int_0^t (\sigma(X_s) - \sigma(X^n_{\eta(s)}))dW_s = \int_0^t \int_0^1 \sigma'(aX_s + (1-a)X^n_{\eta(s)})da(X_s - X^n_{\eta(s)})dW_s,$$

this can be written

$$Y_t = \int_0^t \sigma_1^n(s) Y_s dW_s + G_t, \quad 0 \le t \le T,$$

with

$$\sigma_1^n(s) = \int_0^1 \sigma'(aX_s + (1-a)\overline{X}_{\eta(s)})da$$

$$G_t = \int_0^t \sigma_1^n(s)(X_s^n - \overline{X}_{\eta(s)})dW_s = \int_0^t \sigma_1^n(s)\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})dW_s.$$

In this simple case we have an explicit expression for Y_t,

$$Y_t = \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1}(dG_s - \sigma_1^n(s)d < G, W >_s)$$

where $\mathcal{E}$ is the unique solution of

$$\mathcal{E}_t = 1 + \int_0^t \sigma_1^n(s)\mathcal{E}_s dW_s.$$

Finally we obtain

$$Y_t = \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1}\sigma_1(s)\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})dW_s - \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1}\sigma_1^n(s)^2\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})ds.$$

Now let f be a smooth function with possibly polynomial growth at infinity. We are interested in obtaining the rate of convergence of$Ef(X_T)$ to $Ef(X_T^n)$. We first write the difference

$$Ef(X_T) - Ef(X_T^n) = E\int_0^1 f'(aX_T + (1-a)X_T^n)daY_T.$$

Replacing Y_T by its expression, we obtain with the additional notation $F^n = \displaystyle\int_0^1 f'(aX_T + (1 - a)X_T^n)da$,

$$Ef(X_T) - Ef(X_T^n) = E\left[F^n\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}\sigma_1^n(s)\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})dW_s\right] -$$
$$E\left[F^n\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}\sigma_1^n(s)^2\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})ds\right]. \tag{3}$$

Applying the duality formula for stochastic integrals ($E[< DF, u >_{L^2[0,T]}] = E[F\delta(u)]$) where D stands for the stochastic derivative and δ stands for the adjoint of the stochastic derivative. This gives

$$\begin{aligned} Ef(X_T) - Ef(X_T^n) &= E\left[\int_0^T D_s(F^n\mathcal{E}_T)\mathcal{E}_s^{-1}\sigma_1^n(s)\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})ds\right] \\ &-E\left[F^n\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}\sigma_1^n(s)^2\sigma(X^n_{\eta(s)})(W_s - W_{\eta(s)})ds\right]. \end{aligned}$$

Consequently, the difference$Ef(X_T) - Ef(X_T^n)$ has the simple expression

$$Ef(X_T) - Ef(X_T^n) = E\left[\int_0^T U_s^n(W_s - W_{\eta(s)})ds\right],$$

with

$$U_s^n = (D_s(F^n\mathcal{E}_T) - F^n\mathcal{E}_T\sigma_1(s))(\mathcal{E}_s^{-1}\sigma_1^n(s)\sigma(X_{\eta(s)}^n)).$$

We finally obtain the rate of convergence by applying once more the duality for stochastic integrals

$$Ef(X_T) - Ef(X_T^n) = E\left[\int_0^T \int_{\eta(s)}^s D_u U_s^n du ds\right].$$

This last formula makes clear that $|Ef(X_T) - Ef(X_T^n)| \leq T/n$ and leads to an expansion of $Ef(X_T) - Ef(\overline{X}_T)$ with some additional work. Furthermore the above argument extends easily in the case that f is an irregular function through the use of the integration by parts formula of Malliavin Calculus.

The idea explained about appeared for the first time at this workshop proceedings (in a joint paper with R. Pettersson) and later was used by various authors between them Gobet and Munos and Gobet, Pages, Pham and Printems to prove weak approximations errors in other contexts. In some stochastic equations, one cannot explicitly solve the stochastic linear equation satisfied by Y, but in a recent joint article with E. Clement and D. Lamberton, we have developed a general framework that allows treating a great variety of equations. As example we have developed the case of delay equations.

In fact, considering these articles, what was considered before just another way of proving the classical results of weak approximation of Talay through the PDE method has taken a completely new methodology that can go beyond the classical method using the Feyman-Kac formula. To explain this with a concrete example, I will briefly describe the problem with delay equations which is solved in my joint paper with E. Clement and D. Lamberton. In few words the problem with the Euler approximation for delay equations is that if one tries to use the Talay method one gets into infinite dimensional problems quite rapidly and therefore the degree of generalization is quite limited. In fact, consider (see the article of Buckwar and Shardlow) the following one dimensional delay equation

$$dX(t) = \left(\int_{-\tau}^0 X(t+s)dm(s) + b(X(t))\right) dt + \sigma(X(t))dW(t)$$

with initial conditions $X(s) = x(s)$ for $s \in [-\tau, 0]$ and m is a deterministic finite measure on the interval $[-\tau, 0]$. The natural definition of the Euler scheme is obviously obtained by discretization of the integral in the drift term. That is,

$$X^n(t_{i+1}) = X^n(t_i) + \sum_{j=0}^m X(t_i + s_j)m(s_j, s_{j+1}]b(X^n(t_i))(t_{i+1} - t_i) + \sigma(X^n(t_i))(W(t_{i+1}) - W(t_i))$$

where s_j is a partition of the interval $[-\tau, 0]$ in such a way that $t_i + s_j = t_l$ for some $l \leq i$. In this situation, the natural way to extend the classical argument of Talay is to consider this system as an infinite dimensional stochastic differential equation so as to retain the Markov property. If one does so, one obtains that the solution can be written as

$$X(t) = S(t)x + \int_0^t S(t-s)b(X(s))ds + \int_0^t S(t-s)\sigma(X(s))dW(s)$$

where S is the semigroup associated with the linear term in the equation for X. Similarly, one finds that X^n is generated using instead of S the Yoshida approximations to this operator. Then the partial differential equation associated with this problem is

$$u_t(t,x) = \frac{1}{2}u_{xx}(t,x)\sigma(x_0)^2 + u_x(t,x)\left(Ax + b(x_0)\right)$$

where $x(0) = x_0$ and $Ax(t) = \int_{-\tau}^0 x(t+s)dm(s)$ for $x \in L^2[-\tau, 0]$. The (non-trivial) argument is then similar to the classical Talay argument. Nevertheless, it is also clear from the above set-up that this approach has its limitations. For example, one can not suppose that there is also a continuous delay in the diffusion coefficient or that the delay term is non-linear.

All the limitations cited so far appeared because of the need of using infinite dimensional partial differential equations. Nevertheless using the method explained previously, we have obtained the following result: (for details, see Clement-Kohatsu-Lamberton)

Let (X_t) be the solution stochastic delay equation :

$$\begin{cases} dX_t &= \sigma\left(\int_{-r}^{0} X_{t+s}d\nu(s)\right)dW_t + b\left(\int_{-r}^{0} X_{t+s}d\nu(s)\right)dt \\ X_s &= \xi_s, s\in[-r,0], \end{cases}$$

where $r>0$, $\xi\in\mathcal{C}([-r,0],\mathbb{R})$ and ν is a finite measure.

We consider the Euler approximation of (X_t) with step $h=r/n$

$$\begin{cases} dX_t^n &= \sigma\left(\int_{-r}^{0} X^n_{\eta(t)+\eta(s)}d\nu(s)\right)dW_t + b\left(\int_{-r}^{0} X^n_{\eta(t)+\eta(s)}d\nu(s)\right)dt \\ X_s^n &= \xi_s, s\in[-r,0], \end{cases}$$

with $\eta(s)=\frac{[ns/r]}{n/r}$, where $[t]$ stands for the entire part of t. We assume that the functions f, σ and b are C_b^3. Then we obtain that

$$\mathbb{E}f(X_T)-\mathbb{E}f(X_T^n)=hC_f+I^h(f)+o(h)$$

where $C_f=C(U^0)$ and $I^h(f)=I^h(U^0)$ are defined in Clement-Kohatsu-Lamberton. In particular $|I^h(f)|\le Ch$ and

$$\begin{aligned} U_s^0 &= \sigma'\left(\int_{-r}^{0} X_{s+u}d\nu(u)\right)D_sf'(X_T)+b'\left(\int_{-r}^{0} X_{s+u}d\nu(u)\right)f'(X_T)+ \\ &\quad \sigma'\left(\int_{-r}^{0} X_{s+u}d\nu(u)\right)D_s\left(\int_0^T\theta_tdt\right)+b'\left(\int_{-r}^{0} X_{s+u}d\nu(u)\right)\int_s^T\theta_tdt \end{aligned}$$

and θ is the unique solution of

$$\theta_t=\alpha^*\left(J\left(f'(X_T)+\int_0^T\theta_sds\right)\right)(t)+\beta^*\left(E\left(f'(X_T)+\int_.^T\theta_sds|\mathcal{F}_.\right)\right)(t)$$

with

$$\begin{aligned} \alpha^*(X)(t) &= \mathbb{E}\left(\int_{\max(t-T,-r)}^0\sigma'\left(\int_{-r}^{0} X_{t-u+v}d\nu(v)\right)X_{t-u}d\nu(u)|\mathcal{F}_t\right) \\ \beta^*(X)(t) &= \mathbb{E}\left(\int_{\max(t-T,-r)}^0 b'\left(\int_{-r}^{0} X_{t-u+v}d\nu(v)\right)X_{t-u}d\nu(u)|\mathcal{F}_t\right) \end{aligned}$$

As a brief comment about the fact that we based our explanation on the one dimensional case we present in the next section an interesting recent result on exact simulation of one dimensional diffusions.

3 An exact simulation method for one dimensional uniformly elliptic diffusions

Recently in an article by Beskos et.al. an interesting exact method of simulation has been introduced. Therefore this result excludes the widespread use of the Euler-Maruyama scheme in one dimension.We describe it shortly here. Consider the one dimensional diffusion

$$X(t)=x+\int_0^t b(X(s))ds+\int_0^t\sigma(X(s))dW(s)$$

First, suppose that $\sigma(x) \geq c > 0$ for any $x \in \mathbb{R}$ with $\sigma \in C^1(\mathbb{R})$. Then perform the change of variables $Y_t = \eta(X_t)$ where $\eta(x) = \int_0^x \frac{1}{\sigma(u)} du$. Then using Ito's formula, Y satisfies the following sde:

$$Y(t) = \eta(x) + \int \alpha(Y(s))ds + W(t)$$

where $\alpha(x) = b\sigma^{-1}(x) + 2^{-1}\sigma'(x)$. Suppose that we want to compute $E(f(X_T))$. Then using Girsanov's Theorem we have that

$$Ef(X_T) = E\left[f(B_T)\exp\left(\int_0^T \alpha(B_s)dB_s - \frac{1}{2}\int_0^T \alpha(B_t)^2 dt\right)\right]$$

where B is another Wiener process starting at $\eta(x)$ and here we assume that α is bounded. This idea is usually found when one proves existence of weak solutions for stochastic differential equations.

Next one defines the function $A(u) = \int_0^u \alpha(y)dy$. With this definition we have applying Ito's formula that

$$A(B_T) - A(x) = \int_0^T \alpha(B_s)dB_s + \frac{1}{2}\int_0^T \alpha'(B_s)ds.$$

Therefore

$$Ef(X_T) = E\left[f(B_T)\exp\left(A(B_T) - A(\eta(x)) - \frac{1}{2}\int_0^T \left(\alpha(B_t)^2 + \alpha'(B_t)\right)dt\right)\right].$$

If one where to simulate the above quantity one will need the whole path of the Wiener process B. In fact this is done in a series of papers by Detemple et. al. where the Doss-Sussman formula is used to improve the approximation scheme to obtain an scheme which is of strong order one. Instead, Beskos et.al. proposes to use a Poisson process to simulate the exponential in the above expression. In fact, define $\phi(x) = \frac{1}{2}\alpha(x)^2 + \alpha'(x)$ and let N be a point Poisson process in the interval $[0,T] \times [0,M]$, independent of B, where we suppose without loss of generality that $0 \leq \alpha(x) \leq M$. Then the we have the following result

$$P(\text{the Poisson point process } N \text{ does not hit any point below the graph of } \phi(B_s) \text{ in the interval } s \in [0,T]/B)$$

$$= \exp\left(-\int_0^T \phi(B_s)ds\right)$$

In other words, if we let $N_1(t)$ be the Poisson process that counts the number of times until time t that the Poisson point process has hit point under the curve of $\phi(B)$, then the above statement can be simply written as

$$P(N_1(T) = 0/B) = \exp\left(-\int_0^T \phi(B_s)ds\right)$$

and the simulation scheme follows from the following equality

$$Ef(X_T) = E\left[f(B_T)\exp\left(A(B_T) - A(\eta(x))\right)1(N_1(T) = 0)\right].$$

How is the simulation done? First one simulates independent exponential random variables with parameter $\lambda = 1$. Say $X_1, ..., X_n$ until $\sum_{i=1}^n X_i > T$. For each of these n occurrences one simulates the independent increments of the Wiener process B. That is, $B(X_1), ..., B(\sum_{i=1}^n X_i) - B(\sum_{i=1}^{n-1} X_i)$. Then for each $i = 1, ..., n-1$ one simulates a uniform random variable on the interval $[0, M]$. If its value is smaller that $\phi(B(\sum_{j=1}^i X_j))$ then we count it as one occurrence of N_1 or that the Poisson point process has hit the region below the graph of $\phi(B)$. Obviously there are various issues that have not been considered in this short introduction which rest as open problems or that had already been treated by the authors.

Also as it was also well known before the one dimensional case always permit various reductions that do not happen in higher dimensions. Nevertheless, the one dimensional case always remains as a testing ground for new methodology as it was proven by our recent development in Clement et.al.

In the multidimensional case one can use this idea similarly with the Doss-Sussman formula to produce a simulation scheme of order 2 under the Frobenius condition on σ.

References

[1] A. Beskos, Papaspiliopoulos and Robert G. (2004): Retrospective exact simulation of diffusion sample paths with Applications. Preprint.

[2] V. Bally, D. Talay. The law of the Euler scheme for stochastic differential equations (II) : convergence rate of the density. Monte Carlo Methods and Applications, 2:93-128, 1996.

[3] V. Bally, D. Talay. The law of the Euler scheme for stochastic differential equations (I) : convergence rate of the distribution function. Probability Theory and Related Fields, 104:43-60, 1995.

[4] E. Buckwar and T. Shardlow (2004). Weak approximation of stochastic differential delay equations.

[5] P. Billingsley. Convergence of Probability Measures. John Wiley and Sons 1968.

[6] E. Clement, A. Kohatsu-Higa and D. Lamberton: A duality approach for the weak approximations of stochastic differential equations. To appear in Annals of Applied Probability, 2006.

[7] Gobet E.: LAMN property for elliptic diffusion: a Malliavin Calculus approach. Bernoulli 7 (2001) 899-912.

[8] Gobet E. : LAN property for ergodic diffusions with discrete observations. Ann. Inst. H. Poincare Probab. Statist. 38 (2002) 711-737.

[9] Gobet E., Munos R.: Sensitivity analysis using ItoMalliavin calculus and martingales. Application to stochastic optimal control. SIAM Journal on Control and Optimization, Vol.43(5), pp.1676-1713 - 2005.

[10] Gobet E., G. Pages, H. Pham and J. Printemps: Discretization and simulation for a class of SPDEs with applications to Zakai and McKean-Vlasov equations. Rapport interne # 565 du CMAP - 42 pages - 2005.

[11] Grigoriu M., Samorodnitsky G. Tails of solutions of certain nonlinear stochastic differential equations driven by heavy tailed Levy processes. Stochastic Proc. Appl. 105 (2003) 69-97.

[12] Ikeda N. and Watanabe S.: Stochastic Differential Equations and Diffusion Processes. Amsterdam Oxford New York: North-Holland/Kodansha 1989.

[13] Jacod, Jean; Protter, Philip Asymptotic error distributions for the Euler method for stochastic differential equations. Ann. Probab. 26 (1998), no. 1, 267–307.

[14] Kloeden, P. E. and Platen, E.: Numerical Solution of Stochastic Differential Equations. Berlin Heidelberg New York: Springer 1992.

[15] A.Kohatsu-Higa, R. Pettersson. On the simulation of some functionals of diffusion processes. Proceedings of the "Fourth Workshop on Stochastic Numerics". Research Institute for the Mathematical Sciences, Kyoto 2000.

[16] A. Kohatsu-Higa and M. Yamazato. On moments and tail behaviors of storage processes. Journal of Applied Probability, 20, 1069-1086, 2003.

[17] Kurtz, Thomas G.; Protter, Philip E. Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), 197–285, Lecture Notes in Math., 1627, Springer, Berlin, 1996.

[18] Kurtz, Thomas G.; Protter, Philip E. Weak convergence of stochastic integrals and differential equations. Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), 1–41, Lecture Notes in Math., 1627, Springer, Berlin, 1996.

[19] Jacod, Jean; Kurtz, Thomas G.; Meleard, Sylvie; Protter, Philip The approximate Euler method for Levy driven stochastic differential equations. Ann. Inst. H. PoincareProbab. Statist. 41 (2005), no. 3, 523–558.

[20] Kurtz, Thomas G.; Protter, Philip Characterizing the weak convergence of stochastic integrals. Stochastic analysis (Durham, 1990), 255–259, London Math. Soc. Lecture Note Ser., 167, Cambridge Univ. Press, Cambridge, 1991.

[21] Kurtz, Thomas G.; Protter, Philip Wong-Zakai corrections, random evolutions, and simulation schemes for SDEs. Stochastic analysis, 331–346, Academic Press, Boston, MA, 1991.

[22] Kurtz, Thomas G.; Protter, Philip Weak limit theorems for stochastic integrals and stochastic differential equations. Ann. Probab. 19 (1991), no. 3, 1035–1070.

[23] Mikulevicius, Remigijus; Platen, Eckhard Time discrete Taylor approximations for Ito processes with jump component. Math. Nachr. 138 (1988), 93–104.

[24] Nualart, D.: The Malliavin Calculus and Related Topics. Berlin Heidelberg New York: Springer 1995.

[25] Nualart, D: Analysis on Wiener space and anticipating stochastic calculus. Lecture Notes in Math., 1690 (1998) 123-227.

[26] Pettersson R. Approximations for stochastic differential equations with reflecting convex boundaries. Stochastic Proc. Appl. 59 (1995), 295-308.

[27] Platen, E. An approximation method for a class of Itoprocesses with jump component. Litovsk. Mat. Sb. 22 (1982), no. 2, 124–136.

[28] P. Protter, D. Talay. Convergence rate of the Euler scheme for stochastic differential equations driven by Le'vy processes. Annals of Probability , 25(1):393-423, 1997.

[29] Slominski L. On approximation of solutions of multidimensional SDEs with reflecting boundary condition. Stochastic Proc. Appl. 50 (1994), 197-219.

[30] Slominski L. Euler approximation of solutions of SDEs with reflecting boundary. Stochastic Proc. Appl. 94 (2001) 317-337.

[31] D. Talay, L. Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. Stochastic Analysis and Applications, 8(4):94–120, 1990.

[32] Talay D. and Zheng Z.: Quantiles of the Euler scheme for diffusion processes and financial applications. Mat. Finance 13 (2003), 187–199.