<table>
<thead>
<tr>
<th>Title</th>
<th>On the probabilities associated with unitary matrices (The 7th Workshop on Stochastic Numerics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Takahashi, Y.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録, 2006, 1462: 217-223</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47983">http://hdl.handle.net/2433/47983</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On the probabilities associated with unitary matrices

Y. Takahashi
RIMS, Kyoto University

1 Background

The series of joint works with T. Shirai on fermion point processes, boson point processes and others strongly suggested the following.

**Theorem 1.** For a given unitary matrix $U = (u_{ij})_{1 \leq j,k \leq n}$ there exists a probability $p$ on the symmetric group $S_n$ such that

$$| \det U_{AB} |^2 = \sum_{\sigma \in S_n, \sigma(A) = B} p(\sigma) \quad (A, B \subset \{1, 2, \ldots, n\}).$$

where $U_{AB} = (u_{jk})_{j \in A, k \in B}$ and we set $\det U_{AB} = 0$ unless $|A| \neq |B|$.

This result sharpens the following well-known theorem which shows the existence of an i.i.d sequence of permutations that drives a given symmetric Markov chain.

**Theorem 2.** A doubly stochastic matrix $P = (p_{jk}), \sum_k p_{jk} = \sum_k p_{kj} = 1$ is a convex combination of representation matrices of permutations, $E_\sigma = (\delta(k = \sigma(j)))_{1 \leq j,k \leq n}$.

The proof of Theorem 1 will be published elsewhere. Here we discuss the uniqueness problem for $| \det U_{AB} |^2$ appearing in the L.H.S. of the assertion.

**Theorem 3.** Let $X, Y$ be matrices of the same type and assume

$$\det(I + X^*SXT) = \det(I + Y^*SYT)$$

for any diagonal matrices $S$ and $T$. 
Then there exist unitary diagonal matrices $D_1$ and $D_2$ such that
\[ Y = D_1^*XD_2 \quad \text{or} \quad Y = D_1^*\overline{X}D_2 \]
where $\overline{X}$ stands for the component-wise complex conjugate of $X$.

It is obvious that the converse of Theorem 3 holds. The determinant $\det(I + X^*SXT)$ is a generating function in components of $S$ and $T$ with coefficients $|\det X_{AB}|^2$. Consequently, it solves the uniqueness problem stated above.

By the way, such a kind of uniqueness problem is not so simple in general. For instance, we have the following

**Theorem 4.** Let $X$ and $Y$ be hermitian matrices and assume
\[ \det(I + XT) = \det(I + YT) \]
for any diagonal matrix $T$.

Then, "generically", there exists a unitary diagonal matrix $D$ such that $Y = D^*XD$ or $Y = D^*\overline{X}D$ but there exist counter-examples if the size $n \geq 4$.

In deed, the "canonical form" of counter-examples for $n = 4$ is as follows. Consider
\[
\begin{bmatrix}
c_{11} & c_{12}e^{i\alpha} & c_{13} & c_{14} \\
c_{21}e^{-i\alpha} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34}e^{i\beta} \\
c_{41} & c_{42} & c_{43}e^{-i\beta} & c_{44}
\end{bmatrix}
\]
where $c_{jk} \geq 0$, $\alpha, \beta > 0$. If we choose distinct pairs of $\delta, \varepsilon \in \{\pm 1\}$ for $X$ and $Y$, we can find a counter-example.

## 2 The proof of Theorem 3

We employ the following notations for matrices $X = (x_{jk})_{l \leq j \leq m, l \leq k \leq n}$ and $Y = (y_{jk})_{l \leq j \leq m, l \leq k \leq n}$ with $x_{jk} \in \mathbb{C}$ and $y_{jk} \in \mathbb{C}$:

(a) $\overline{X} = (\overline{x}_{jk})_{l \leq j \leq m, l \leq k \leq n}$.

(b) $X \approx Y$ if for any $p = 1, 2, \ldots, \min\{m, n\}$ and for any $j_1 < \cdots < j_p$ and $k_1 < \cdots < k_p$
\[ |\det(y_{j_r k_s})_{l \leq r, s \leq p}| = |\det(x_{j_r k_s})_{l \leq r, s \leq p}|. \]
(c) $X \sim Y$ if there exist $\theta_1, \ldots, \theta_m, \varphi_1, \ldots, \varphi_m \in \mathbb{R}$ (precisely, $\mathbb{R}/2\pi\mathbb{Z}$) such that

\[ y_{jk} = e^{i(\theta_j - \varphi_k)} x_{\dot{g},k} \]

for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$.

Moreover we write

\[ X \overset{\prec}{\sim} Y \quad \text{if} \quad X \sim Y \quad \text{and} \quad X \overset{\succ}{\sim} Y \quad \text{if} \quad X \sim Y. \]

Under above notations the statement of Theorem can be restated as follows:

if $X \approx Y$, then $X \overset{\prec}{\sim} Y$ or $X \overset{\succ}{\sim} Y$.

2.1 Preliminary

Lemma 1. Let $a, b, \theta, \varphi \in \mathbb{R}$ and assume

\[ |e^{i\theta}a - b| = |e^{i\varphi}a - b|. \]

Then one of the following holds:

(a) $a = 0$  \hspace{1cm} (b) $b = 0$  \hspace{1cm} (c) $\theta = \varphi \pmod{2\pi}$  \hspace{1cm} (d) $\theta = -\varphi \pmod{2\pi}$.

Conversely, if one of (a)-(d) holds then $|e^{i\theta}a - b| = |e^{i\varphi}a - b|$.

Proof. The cases (a) and (b) are trivial. Assume $a \neq 0$ and $b \neq 0$. Then $|z - b| = r$ and $|z| = |a|$ are two distinct circles on the complex plane which are symmetric with respect to the real axis. Hence they interested at most two points which are complex conjugate. \[ \square \]

Lemma 2. Let $U_{jk} \in \mathbb{C}$, $j, k = 1, 2$. Then the identity

\[ U_{11} + U_{22} = U_{12} + U_{21} \]

holds if and only if there exist $v_1, v_2, w_1, w_2 \in \mathbb{C}$ such that

\[ u_{jk} = v_j - w_k. \]
Proof. The "if" part is obvious. To prove the "only if" part set

\[ U_{11} + U_{22} = U_{12} + U_{21} = s, \]
\[ U_{21} - U_{11} = U_{22} - U_{12} = a, \]
\[ U_{12} - U_{11} = U_{22} - U_{21} = b \]

and

\[ v_1 = \frac{s - a}{2}, \]
\[ v_2 = \frac{s + a}{2}, \]
\[ w_1 = \frac{b}{2}, \]
\[ w_2 = -\frac{b}{2} \]

Then

\[ u_{jk} = v_j - w_k \text{ for } j, k = 1, 2. \]

Lemma 3. Let \( X \) and \( Y \) be matrices of type \((m, n)\) and set

\[ X' = (x_{jk})_{l \leq j \leq m, l \leq k \leq n-1}, \quad X'' = (x_{jk})_{l \leq j \leq m, 2 \leq k \leq n}, \]
\[ Y' = (y_{jk})_{l \leq j \leq m, l \leq k \leq n-1}, \quad Y'' = (y_{jk})_{l \leq j \leq m, 2 \leq k \leq n}. \]

Assume that

\[ X' \sim Y' \text{ and } X'' \sim Y''. \]

In addition, assume that \( x_{jk} \neq 0 \) for some \( j \) and \( k \) with \( 1 \leq j \leq m \) and \( 2 \leq k \leq n - 1 \). Then

\[ X \sim Y. \]

Proof. By the assumption there exist \( \theta'_1, \ldots, \theta'_m, \varphi'_1, \ldots, \varphi'_{n-1} \) and \( \theta''_1, \ldots, \theta''_m, \varphi''_2, \ldots, \varphi''_n \) such that

\[ y_{jk} = e^{i(\theta'_j - \varphi'_k)} x_{jk} \text{ for } l \leq j \leq m \text{ and } 1 \leq k \leq n - 1, \]
\[ y_{jk} = e^{i(\theta''_j - \varphi''_k)} x_{jk} \text{ for } l \leq j \leq m \text{ and } 2 \leq k \leq n. \]

Moreover, by the additional assumption \( x_{jk} \neq 0 \) and \( y_{jk} \neq 0 \) for some \( j \) and \( k \) with \( l \leq j \leq m \) and \( 2 \leq k \leq n - 1 \). Hence

\[ \theta'_j - \varphi'_k = \theta''_j - \varphi''_k \quad \text{or} \quad \theta'_j - \theta''_j = \varphi''_k - \varphi'_k = \alpha \]

for such \((j, k)\). Consequently,

\[ y_{jk} = e^{i(\theta_j - \varphi_k)} x_{jk} \text{ for } l \leq j \leq m \text{ and } l \leq k \leq n \]

with \( \theta_j = \theta'_j (l \leq j \leq m), \varphi_k = \varphi_k (l \leq k \leq n - 1) \) and \( \varphi_n = \varphi''_n + \alpha \).
2.2 Proof of Theorem 3

Step 1: \( m = n = 2 \).

Let \( X = (x_{jk})_{1 \leq j,k \leq 2} \) and \( Y = (y_{jk})_{1 \leq j,k \leq 2} \). Since \( X \approx Y \),

\[
|x_{jk}| = |y_{jk}| \quad (j, k = 1, 2) \quad \text{and} \quad |x_{11}x_{22} - x_{12}x_{21}| = |y_{11}y_{22} - y_{12}y_{21}|.
\]

Set \( x_{jk} = c_{jm}e^{i\xi_{jk}} \) and \( y_{jk} = c_{jk}e^{i\eta_{jk}} \) where \( c_{jk} = |x_{jk}| \). Then

\[
|e^{i(\xi_{11}+\xi_{22}-\xi_{12}-\xi_{21})}c_{11}c_{22} - c_{12}c_{21}| = |e^{i(\eta_{11}+\eta_{22}-\eta_{12}-\eta_{21})}c_{11}c_{22} - c_{12}c_{21}|.
\]

By Lemma 1, it follows either \( c_{11}c_{22}c_{12}c_{21} = 0 \) or

\[\eta_{11} + \eta_{22} - \eta_{12} - \eta_{21} = \pm(\xi_{11} + \xi_{22} - \xi_{12} - \xi_{21}).\]

In the latter case, by Lemma 2 there exist \( \theta_1, \theta_2, \varphi_1 \) and \( \varphi_2 \) such that

\[\eta_{jk} = \xi_{jk} + \theta_j - \varphi_k, \quad j, k = 1, 2.\]

Hence

\[Y \sim X \quad \text{or} \quad Y \sim \overline{X}\]

according to the sign \( \pm \).

If \( c_{11}c_{22}c_{12}c_{21} = 0 \), \( X \) is one of the following form

\[
\begin{cases}
(a) & \begin{pmatrix} 0 & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, x_{12}x_{21}x_{22} \neq 0
\quad (a') & \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & 0 \end{pmatrix}, x_{11}x_{12}x_{21} \neq 0
\end{cases}
\]

\[
\begin{cases}
(b) & \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix}, x_{11}x_{21}x_{22} \neq 0
\quad (b') & \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}, x_{11}x_{12}x_{22} \neq 0
\end{cases}
\]

\[
\begin{cases}
(c) & \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}
\quad (c') & \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}
\end{cases}
\]

\[
\begin{cases}
(c'') & \begin{pmatrix} 0 & x_{11} \\ 0 & x_{21} \end{pmatrix}
\quad (c''') & \begin{pmatrix} 0 & 0 \\ x_{21} & x_{22} \end{pmatrix}
\end{cases}
\]

In case (a), setting \( \varphi_1 = 0, \theta_1 = \eta_{11} - \xi_{11}, \theta_2 = \eta_{21} - \xi_{21} \) and \( \varphi_2 = \eta_{22} - \xi_{22} - \theta_2 \) one finds \( \eta_{jk} = \xi_{jk} + \theta_j - \varphi_k \).
In case (b), setting $\theta_2 = 0$, $\varphi_1 = \xi_{21}-\eta_{21}, \varphi_2 = \xi_{22}-\eta_{22}$ and $\theta_1 = \eta_{12}-\xi_{12}+\varphi_2$ one finds $\eta_{jk} = \xi_{jk} + \theta_j - \varphi_k$.

In these cases, it is easy to find $\theta_1, \theta_2, \varphi_1$ and $\varphi_2$ such that

$$y_{jk} = e^{i(\theta_j - \varphi_k)} x_{jk} \quad \text{for } j, k = 1, 2.$$  

For instance,

\begin{itemize}
  \item case(a): $\varphi_1 = 0, \theta_1 = \eta_{11}-\xi_{11}, \theta_2 = \eta_{21}-\xi_{21}$ and $\varphi_2 = \eta_{22}-\xi_{22}-\theta_2$.
  \item case(b): $\theta_2 = 0, \varphi_1 = \xi_{21}-\eta_{21}, \varphi_2 = \xi_{22}$ - $\eta_{22}$ and $\theta_1 = \eta_{12}-\xi_{12}+\varphi_2$.
\end{itemize}

Consequently, in these degenerated cases we obtain

$$Y \sim X.$$

\[ \square \]

Step 2: $m = 2, n = 3$.

Let $X = (x_{jk})_{l\leq j\leq 2, l\leq k\leq 3}$ and $Y = (y_{jk})_{l\leq j\leq 2, l\leq k\leq 3}$ and define

$$X' = (x_{jk})_{l\leq j\leq 2, l\leq k\leq 2}, \quad X'' = (x_{jk})_{l\leq j\leq 2, 2\leq k\leq 3},$$

and $Y', Y'', Y'''$ in a similar manner. Since $X \approx Y$ implies $X' \approx Y', X'' \approx Y'', X''' \approx Y'''$ it follows from Step 1 that

$$X' \sim Y', X'' \sim Y'', X''' \sim Y'''$$

for some $\varepsilon', \varepsilon'', \varepsilon''' \in \{\pm 1\}$. Then at least two of $\varepsilon', \varepsilon''$ and $\varepsilon'''$ coincide. For simplicity, assume $\varepsilon' = \varepsilon'' = +$. Then

$$X' \approx Y' \quad \text{and} \quad X'' \approx Y''.$$  

By Lemma 3 one can conclude $X \sim Y$ if $x_{12} \neq 0$ or $x_{22} \neq 0$. If $x_{12} = x_{22} = 0$, then relation $X''' \sim Y'''$ is equivalent to the relation $X \approx Y$.

Step 3: $m = 2, n \geq 4$.

We appeal to the induction on $n$. In Step 2 we proved the assertion for $n = 3$. Let us assume we have proved for $n - 1$ and show the case for $n$.

If $X$ and $Y$ are matrices of type $(2, n)$ and $X \approx Y$, then we have $n$ submatrices $X_1, \ldots, X_n$ of $X$ and $Y_1, \ldots, Y_n$ of $Y$ of type $(2, n - 1)$. 

By induction assumption, we have $X_i \sim Y_i$ for each $i$ with $\epsilon_i = \pm$. Since $n \geq 4$, we can find at least two $i$'s for which $\epsilon_i$'s coincide with each other. Thus, a similar argument to Step 2 shows that $X \sim Y$ or $X \sim Y$.

**Step 4**: $m \geq 3, n \geq 3$.

We appeal to the induction on $m$ fixing $n$.

Let $X$ and $Y$ be matrices of type $(m, n)$ and $X \approx Y$. Then we can find at least two par submatrices $X', X'', Y', Y''$ of type $(m - 1, n)$ and $\epsilon \in \{\pm\}$ such that

\[ X' \sim^\epsilon Y' \quad \text{and} \quad X'' \sim^\epsilon Y''. \]

By Lemma 3 if $X'$ and $X''$ have a common nonzero entry, we have $X \sim^\epsilon Y$. If they have no common nonzero entries, then $X$ and $Y$ are essentially of type $(2, n)$. Hence by Step 3 we obtain $X \sim Y$ or $X \sim Y$. □

**References**

