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APPLICATION OF STOCHASTIC NUMERICAL METHODS IN MULTI-PERIOD OPTIMAL PORTFOLIO STRATEGIES:CASE STUDIES

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ABSTRACT. The authors propose a new algorithm combining a stochastic flow technique and the Ninomiya-Victoir method [14], to solve an optimal portfolio and consumption problem for a single-agent facing a Markovian security market setting. In that class, optimal feedback portfolio strategies are expressed by transition semigroups of the system of stochastic differential equations, which are induced by applying the differential rule of a composite function to stochastic flows. Some numerical examples are given.

1. INTRODUCTION

We consider a single-agent optimal portfolio and consumption problem in a continuous-time. Optimal portfolio and consumption choice in multi-period or in continuous-time settings were first investigated by Samuelson [15] and Merton [11] [12]. Many articles about optimal portfolio strategies are published since then. We have the general formula for optimal solutions in the case of complete market settings. See Cox-Huang [1] and Cvitanic-Karatzas [2].

However, for many financial problems which practitioners tackle in daily business, it is difficult to obtain tractable analytical optimal solutions. The difficulty requires us to apply the numerical methods especially when economy's state variables are stochastic such as in stochastic interest rate models, stochastic volatility models, bond portfolio strategies, bond-equity mix problems and so on. Recently some advanced stochastic methods using Malliavin calculus are extensively applied to obtain optimal portfolio strategies numerically. Detemple-Garcia-Rindisbacher [3] applied Malliavin calculus and the generalized Clark formula and obtain numerical results. Kunitomo-Takahashi [8] and Takahashi-Yoshida [16] used the combination of Malliavin Calculus and the asymptotic expansion approach.

In this paper, to solve optimal portfolio problems numerically, the Ninomiya-Victoir method [14](NV method for short), a version of Kusuoka approximation [9], is combined with a stochastic numerical algorithm using stochastic flows [5]. For a class of security market models specified later, solutions are represented in feedback form on some stochastic processes, by using transition semigroups and forward stochastic flows. Therefore, our proposal is relatively easy to implement compared to other numerical methods using Malliavin Calculus. Also we expand Karatzas-Shreve's setting of the single-agent optimal portfolio and consumption framework. In

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our setting some state-dependent utility functions are considered. Using Theorem 2.5, the optimal portfolio strategy $\hat{\varphi}(t)$ is given as a rational expression of expected values of Markovian-type diffusion processes. These diffusion processes are solutions of stochastic differential equations which are induced by applying the differential rule of a composite function to stochastic flows and multiplicative functionals. Theorem 2.5 gives a fundamental framework for numerical calculations of $\hat{\varphi}(t)$. We can directly apply the NV method [14] to, which is reported to be more time-effective in calculation of derivative prices.

The remainder of this paper is structured as follows. We review main results in [5] in Section 2. In Section 3, we apply the version of Kusuoka approximation to a Stock-Bond-Cash allocation problem. Concluding remark is in Section 4.

2. Application of stochastic flows

Throughout this paper we assume the following setting: Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $\{B(t) = (B^1(t), \cdots, B^d(t)); t \in [0,T]\}$ be a d-dimensional standard Brownian motion. The time interval is $[0,T]$, where $T > 0$. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the augmented Brownian filtration with usual conditions. We have the investment horizon T_0, where $0 < T_0 < T$.

Definition 2.1. We say that a function $f : [0,T] \times \mathbb{R}^m \to \mathbb{R}$, where $m \in \mathbb{N}$, is a member of a class $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^m)$, if the following conditions are satisfied:

1. $f(t,x)$ is continuous in t, x, and smooth in x for all t.
2. There exists a constant $C > 0$, such that
$$|f(t,x)| \leqslant C(1+|x|), \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^m.$$
3. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_m)$, there exists a constant depending on α, $C_\alpha > 0$, such that
$$|D^\alpha_x f(t,x)| \leqslant C_\alpha, \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^m.$$

An economy's state variables $X(t)$ is given by $\mathbb{R}^n$-valued continuous stochastic process $X(t) = (X^1(t), \cdots, X^n(t))$. We assume the following:

(S1): Coefficient functions $\mu^X_i(t,x)$, $\sigma^X_{i,j}(t,x), i = 1, \cdots, n$, $j = 1, \cdots, d$ of $X(t)$ are in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$.

We assume that $X(t)$ is a unique solution to the following stochastic differential equation in the sense of Itô and a stochastic process with spacial parameters (see, e.g., Kunita [7]).

$$X(t;s,x) = x + \int_s^t \mu^X(v, X(v;s,x))\mathrm{d}v + \int_s^t \sigma^X(v, X(v;s,x))\mathrm{d}B(v), \tag{1}$$

where $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$. Let μ^X be an $\mathbb{R}^n$-valued function $\mu^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, and σ^X be an $\mathbb{R}^n \otimes \mathbb{R}^d$-valued function $\sigma^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d$ by the following:

$$\mu^X(t,x) = \begin{pmatrix} \mu^X_1(t,x) \\ \vdots \\ \mu^X_n(t,x) \end{pmatrix}, \text{ and } \sigma^X(t,x) = \begin{pmatrix} \sigma^X_{1,1}(t,x) & \cdots & \sigma^X_{1,d}(t,x) \\ \vdots & & \vdots \\ \sigma^X_{n,1}(t,x) & \cdots & \sigma^X_{n,d}(t,x) \end{pmatrix}.$$

We may assume that $X(t;x)$ is a forward stochastic flow of C^∞-diffeomorphisms (see Kunita [7] Theorem 4.6.5). We denote this stochastic flow by $X(t;s,x)$ for $0 \leqslant s \leqslant t \leqslant T$ and for $x \in \mathbb{R}^n$.

At time $t=0$, choose a starting point $x_0 \in \mathbb{R}^n$, and fix it. Let $X(t) = X(t;0,x_0)$. Then for $0 \leqslant s \leqslant t \leqslant T$, we have $X(t) = X(t;s,X(s))$.

Let r be a function $r : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

(S2): $r(t,x)$ is in $C^{0,\infty}_{\text{ub}}(\mathbb{R}^n)$.

We define $r_t = r(t,X(t))$ and consider r_t as the risk free rate at time t. Let $S^0(t)$ be the money account:

$$S^0(t) = \exp\left\{\int_0^t r(v,X(v))\mathrm{d}v\right\}.$$

Let μ_i be a function $\mu_i : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ and $\sigma_{i,j}$ be a function $\sigma_{i,j} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfying the following:

(S3): $\mu_i(t,x)$, $\sigma_{i,j}(t,x)$,$i,j=1,\cdots,d$ are in $C^{0,\infty}_{\text{ub}}(\mathbb{R}^n)$.

Let us introduce d individual securities, $S^i(t)$, $i=1,\cdots,d$, where each $S^i(t)$ is an $\mathbb{R}$-valued stochastic process and a unique solution of the following stochastic differential equation:

$$(2)\qquad S^i(t) = S^i_0 + \int_0^t \mu_i(v,X(v))S^i(v)\mathrm{d}v + \sum_{j=1}^d \int_0^t \sigma_{i,j}(v,X(v))S^i(v)\mathrm{d}B^j(v),\quad i=1,\cdots,d.$$

Let $S(t) = (S^1(t),\cdots,S^d(t))$, and

$$\mu(t,x) = \begin{pmatrix}\mu_1(t,x)\\ \vdots \\ \mu_d(t,x)\end{pmatrix},\quad \text{and}\quad \sigma(t,x) = \begin{pmatrix}\sigma_{1,1}(t,x) & \cdots & \sigma_{1,d}(t,x)\\ \vdots & & \vdots\\ \sigma_{d,1}(t,x) & \cdots & \sigma_{d,d}(t,x)\end{pmatrix}.$$

We assume the following condition.

(S4): The volatility matrix $\sigma(t,x)$ is invertible for all $t \in [0,T]$ and for all $x \in \mathbb{R}^n$.

Then we can define an $\mathbb{R}^d$-valued function $\lambda : [0,T]\times\mathbb{R}^n \to \mathbb{R}^d$ as follows:

$$\lambda(t,x) = \sigma(t,x)^{-1}\left(\mu(t,x) - r(t,x)\vec{1}\right),$$

where $\vec{1} = (1,\cdots,1) \in \mathbb{R}^d$. We denote the j-th element of $\lambda(t,x)$ by $\lambda_j(t,x)$. We assume the following:

(S5): $\lambda_j(t,x), j=1,\cdots,d$, are in $C^{0,\infty}_{\text{ub}}(\mathbb{R}^n)$.

We define a stochastic process with spacial parameters $\Pi(t;s,x)$ as follows:

$$(3)\qquad \begin{aligned}\Pi(t;s,x) = \exp\Bigg\{&-\int_s^t r(v,X(v;s,x))\mathrm{d}v - \sum_{j=1}^d\int_s^t \lambda_j(v,X(v;s,x))\mathrm{d}B^j(v)\\ &-\frac{1}{2}\int_s^t\sum_{j=1}^d \lambda_j(v,X(v;s,x))^2\mathrm{d}v\Bigg\},\quad 0\leqslant s\leqslant t\leqslant T,\quad x\in\mathbb{R}^n.\end{aligned}$$

Let $\Pi(t) = \Pi(t;0,x_0)$. Then we see that for any $0\leqslant s\leqslant t\leqslant T$

$$\Pi(t) = \Pi(s)\Pi(t;s,X(s)).$$

We see that $\Pi(t;s,x)$ is $\{\mathcal{F}_{s,t}\}_{0\leqslant s\leqslant t\leqslant T}$-measurable, where

$$\mathcal{F}_{s,t} = \sigma(X(s)) \vee \sigma(B_j(r) - B_j(s) : j=1,\cdots,d, s\leqslant r\leqslant t).$$

$\Pi(t)$ is the state price density process (see Duffie [4] and Karatzas-Shreve [6]). For each $j = 1, \cdots, n$, we define the following stochastic processes:

$$
\begin{aligned}
\pi_j(t;s,x) = &- \int_s^t \sum_{k=1}^n \partial_k r(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}v \\
&- \sum_{i=1}^d \int_s^t \sum_{k=1}^n \partial_k \lambda_i(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}B^i(v) \\
&- \int_s^t \sum_{i=1}^d \sum_{k=1}^n \lambda_i(v, X(v;s,x)) \partial_k \lambda_i(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}v, \quad j = 1, \cdots, n,
\end{aligned} \tag{4}
$$

where ∂_k means $\partial/\partial x^k$. We define the following local martingale $\Xi(t)$:

$$
\Xi(t) = \exp\left\{ -\sum_{j=1}^d \int_0^t \lambda_j(v, X(v))\mathrm{d}B^j(v) - \frac{1}{2}\int_0^t \sum_{j=1}^d \lambda_j(v, X(v))^2 \mathrm{d}v \right\}. \tag{5}
$$

We assume the following condition:

(S6): The local martingale $\Xi(t)$ is a martingale.

Let $g_0(t,x), g_1(t,x), \cdots, g_d(t,x)$ and $h_0(t,x), h_1(t,x), \cdots, h_d(t,x)$ be functions from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}$ satisfying the following:

(S7): $g_i(t,x)$, $i = 0, 1, \cdots, d$, and $h_i(t,x)$, $i = 0, 1, \cdots, d$ are in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$.

We introduce the following stochastic processes with spacial parameters; for $0 \leqslant s \leqslant t \leqslant T_0$, and for all $x \in \mathbb{R}^n$,

$$
\Delta(t;s,x) = \exp\left\{ \int_s^t g_0(v, X(v;s,x))\mathrm{d}v + \sum_{j=1}^d \int_s^t g_j(v, X(v;s,x))\mathrm{d}B^j(v) \right\}, \tag{6}
$$

$$
\begin{aligned}
\delta_j(t;s,x) = &\int_s^t \sum_{k=1}^n \partial_k g_0(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}v \\
&+ \sum_{i=1}^d \int_s^t \sum_{k=1}^n \partial_k g_i(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}B^i(v), \quad \text{for } j = 1, \cdots, d,
\end{aligned} \tag{7}
$$

$$
E(t;s,x) = \exp\left\{ \int_s^t h_0(v, X(v;s,x))\mathrm{d}v + \sum_{j=1}^d \int_s^t h_j(v, X(v;s,x))\mathrm{d}B^j(v) \right\}, \tag{8}
$$

$$
\begin{aligned}
\eta_j(t;s,x) = &\int_s^t \sum_{k=1}^n \partial_k h_0(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}v \\
&+ \sum_{i=1}^d \int_s^t \sum_{k=1}^n \partial_k h_i(v, X(v;s,x)) \frac{\partial X^k}{\partial x^j}(v;s,x)\mathrm{d}B^i(v), \quad \text{for } j = 1, \cdots, d.
\end{aligned} \tag{9}
$$

The following equations hold as for $\Pi(t;s,x)$: Let us define $\Delta(t)$ and $E(t)$ by

$$
\Delta(t) = \Delta(t;0,x_0), \quad E(t) = E(t;0,x_0). \tag{10}
$$

Then

$$\Delta(t) = \Delta(s)\Delta(t; s, X(s)), \quad E(t) = E(s)E(t; s, X(s)),$$

for all $0 \leqslant s \leqslant t \leqslant T_0$. We see that $\Delta(t; s, x)$ and $E(t; s, x)$ are $\mathcal{F}_{s,t}$-measurable.

Let $U_0 : (w_0, \infty) \to \mathbb{R}$ and $u_0 : (c_0, \infty) \times [0, T_0] \to \mathbb{R}$, where $w_0 \geqslant 0$ and $c_0 \geqslant 0$ are functions satisfying the following conditions:

(U1): $U_0 : (w_0, \infty) \to \mathbb{R}$ is a C^3-function such that

1. $U_0'(w) > 0$ for all $w \in (w_0, \infty)$, and

$$\lim_{w\to\infty} U_0'(w) = 0, \qquad \lim_{w\to w_0} U_0'(w) = +\infty,$$

2. $U_0''(w) < 0$ for all $w \in (w_0, \infty)$,
3. $U_0'''(w) > 0$ for all $w \in (w_0, \infty)$.

(U2): $u_0 : (c_0, \infty) \times [0, T_0] \to \mathbb{R}$ is a continuous function in $w \in (c_0, \infty)$ and $t \in [0, T_0]$, and for all $t \in [0, T_0]$, $u_0(w, t)$ is a C^3-function in w such that for all $t \in [0, T_0]$,

1. $\frac{\partial u_0}{\partial w}(w, t) > 0$ for all $w \in (c_0, \infty)$, and

$$\lim_{w\to\infty} \frac{\partial u_0}{\partial w}(w, t) = 0, \qquad \lim_{w\to c_0} \frac{\partial u_0}{\partial w}(w, t) = +\infty,$$

2. $\frac{\partial^2}{\partial w^2} u_0(w, t) < 0$ for all $w \in (c_0, \infty)$,
3. $\frac{\partial^3}{\partial w^3} u_0(w, t) > 0$ for all $w \in (c_0, \infty)$.

Let us define $U : (w_0, \infty) \times \Omega \to \mathbb{R}$ and $u : (c_0, \infty) \times [0, T_0] \times \Omega \to \mathbb{R}$ by

$$U(w, \omega) = \frac{U_0(w)}{\Delta(T_0)}, \quad u(w, t, \omega) = \frac{u_0(w, t)}{E(t)}.$$

Let us define $V : D \to \mathbb{R}$ by

$$V(C, Z) = \mathbb{E}\left[\int_0^{T_0} u(C(v), v, \omega)\mathrm{d}v + U(Z, \omega)\right],$$

where D is given in Definition 2.3. Since U_0' and $\partial_w u_0$ are continuous, convex, positive, and strictly decreasing functions, there exist $I_1 : (0, \infty) \times [0, T_0] \to (c_0, \infty)$ and $I_2 : (0, \infty) \to (w_0, \infty)$ such that

$$\frac{\partial}{\partial w} u_0 \left(I_1(u, t), t\right) = u, \quad u \in (0, \infty), \quad U_0' \left(I_2(u)\right) = u, \quad u \in (0, \infty).$$

Then $I_1(u, t)$ and $I_2(u)$ are C^1-functions in u.

Definition 2.2. We say that $(\varphi_0(t), \varphi(t))$ is a portfolio process if $\varphi_0(t)$ is an $(\mathcal{F}_t)$-progressively measurable, $\mathbb{R}$-valued process and $\varphi(t) = (\varphi_1(t), \cdots, \varphi_d(t))$ is an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process and the followings are satisfied:

1. $\varphi_0(t) + \varphi_1(t) + \cdots + \varphi_d(t) = 1, \qquad$ for all t.
2. $\int_0^{T_0} \sum_{j=1}^d |\varphi_j(v)|^2 \mathrm{d}v < \infty, \qquad P$-a.s.

From 1. of Definition 2.2, $\varphi_0(t)$ is determined by $\varphi(t)$.

Definition 2.3. We say a triplet (C, Z, φ) is an admissible strategy at $x \geqslant 0$, if $C(t)$ is an $(\mathcal{F}_t)$-progressively measurable, non-negative stochastic process and Z is an $\mathcal{F}_{T_0}$-measurable, non-negative random variable, and $(\varphi_0(t), \varphi(t))$ is a portfolio process and the following conditions are satisfied:

1. $\int_0^{T_0} C(v)\mathrm{d}v < \infty, \quad P$-a.s.
2. Let $W^{x,C,\varphi}(t)$ be a stochastic process of a solution of the following stochastic differential equation:

$$W^{x,C,\varphi}(t) = x + \sum_{j=0}^{d} \int_0^t \frac{\varphi_j(v)W^{x,C,\varphi}(v)}{S^j(v)} \mathrm{d}S_j(v) - \int_0^t C(v)\mathrm{d}v.$$

We assume that

$$W^{x,C,\varphi}(t) \geqslant 0, \qquad \text{for all } t \in [0, T_0], \quad P\text{-a.s.}$$

3. $Z = \sum_{j=0}^{d} \varphi_j(T_0)W^{x,C,\varphi}(T_0)$.
4. $\mathbb{E}\left[\int_0^{T_0} u(C(v), v, \omega)^- \mathrm{d}v + U(Z, \omega)^-\right] < \infty$.

$\mathcal{A}(x)$ and D denote the set of admissible strategies at x and the space of pair (C, Z) respectively.

Let us define a function $\mathcal{Y} : (0, \infty) \to \mathbb{R}$ by

$$\mathcal{Y}(x) = \mathbb{E}\left[\int_0^{T_0} \Pi(v)I_1\left(x\Pi(v)E(v), v\right) \mathrm{d}v + \Pi(T_0)I_2\left(x\Pi(T_0)\Delta(T_0)\right)\right].$$

It is easy to show that $\mathcal{Y}(x)$ is a decreasing function of x. We assume the following condition.

ASSUMPTION 2.4. For given $W > 0$,

$$\lim_{x\to 0} \mathcal{Y}(x) > W, \quad \text{and} \quad \lim_{x\to+\infty} \mathcal{Y}(x) < W.$$

Therefore, there exists $\hat{\lambda} > 0$ satisfying the following equation:

$$\mathcal{Y}(\hat{\lambda}) = W. \tag{11}$$

Let $\Theta = (0, \infty) \times \mathbb{R}^n \times (0, \infty) \times (0, \infty) \times [0, T_0]$. We define functions $H : \Theta \to \mathbb{R}$, $G : \Theta \to \mathbb{R}$, and $\mathcal{X}_i : \Theta \to \mathbb{R}$, $i = 1, \cdots, d$ as follows: For $(\xi, x, \zeta, \nu, t) \in \Theta$,

$$H(\xi, x, \zeta, \nu, t) = \mathbb{E}\left[\int_t^{T_0} \Pi(v; t, x)^2 E(v; t, x) \frac{\partial I_1}{\partial u}\left(\hat{\lambda}\xi\nu\Pi(v; t, x)E(v; t, x), v\right) \mathrm{d}v\right].$$

$$G(\xi, x, \zeta, \nu, t) = \mathbb{E}\left[\Pi(T_0; t, x)^2 \Delta(T_0; t, x) \frac{\mathrm{d}I_2}{\mathrm{d}u}\left(\hat{\lambda}\xi\zeta\Pi(T_0; t, x)\Delta(T_0; t, x)\right)\right].$$

For $i = 1, \cdots, n$,

$$\begin{aligned}
&\mathcal{X}_i(\xi, x, \zeta, \nu, t) \\
&= \xi\mathbb{E}\left[\int_t^{T_0} \frac{\partial\Pi}{\partial x^i}(v; t, x)I_1\left(\hat{\lambda}\xi\nu\Pi(v; t, x)E(v; t, x), v\right) \mathrm{d}v + \frac{\partial\Pi}{\partial x^i}(T_0; t, x)I_2\left(\hat{\lambda}\xi\zeta\Pi(T_0; t, x)\Delta(T_0; t, x)\right)\right] \\
&+\hat{\lambda}\xi^2\nu\mathbb{E}\left[\int_t^{T_0} \Pi(v; t, x)\left(\frac{\partial\Pi}{\partial x^i}(v; t, x)E(v; t, x) + \Pi(v; t, x)\frac{\partial E}{\partial x^i}(v; t, x)\right)\frac{\partial I_1}{\partial u}(\hat{\lambda}\xi\nu\Pi(v; t, x)E(v; t, x), v)\mathrm{d}v\right] \\
&+\hat{\lambda}\xi^2\zeta\mathbb{E}\left[\Pi(T_0; t, x)\left(\frac{\partial\Pi}{\partial x^i}(T_0; t, x)\Delta(T_0; t, x) + \Pi(T_0; t, x)\frac{\partial\Delta}{\partial x^i}(T_0; t, x)\right)\frac{\mathrm{d}I_2}{\mathrm{d}u}(\hat{\lambda}\xi\zeta\Pi(T_0; t, x)\Delta(T_0; t, x))\right].
\end{aligned}$$

$\frac{\partial \Pi}{\partial x^i}(s;t,x)$, $\frac{\partial \Delta}{\partial x^i}(s;t,x)$, and $\frac{\partial E}{\partial x^i}(s;t,x)$, for $i = 1, \cdots, n$, in the above equations are given by applying the differential rule of a composite function to $\Pi(s;t,s)$, $\Delta(s;t,x)$ and $E(s;t,x)$.

Also we define functions, $F : \Theta \to \mathbb{R}$, $F_\xi : \Theta \to \mathbb{R}$, $F_\nu : \Theta \to \mathbb{R}$, and $F_\zeta : \Theta \to \mathbb{R}$ as follows:

$$F(\xi,x,\zeta,\nu,t) = \xi\mathbb{E}\Bigg[\int_t^{T_0} \Pi(v;t,x)I_1\left(\hat{\lambda}\xi\nu\Pi(v;t,x)E(v;t,x),v\right)\mathrm{d}v + \Pi(T_0;t,x)I_2\left(\hat{\lambda}\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x)\right)\Bigg],$$

$$F_\xi(\xi,x,\zeta,\nu,t) = \frac{1}{\xi}F(\xi,x,\zeta,\nu,t) + \hat{\lambda}\xi\nu H(\xi,x,\zeta,\nu,t) + \hat{\lambda}\xi\zeta G(\xi,x,\zeta,\nu,t),$$
$$F_\nu(\xi,x,\zeta,\nu,t) = \hat{\lambda}\xi^2 H(\xi,x,\zeta,\nu,t),$$
$$F_\zeta(\xi,x,\zeta,\nu,t) = \hat{\lambda}\xi^2 G(\xi,x,\zeta,\nu,t).$$

We consider the following conditions:

(A1): For any compact set $K \subset \mathbb{R}^n$,

$$\sup_{x\in K}\mathbb{E}\left[\int_0^{T_0}\Pi(v;0,x)\mathrm{d}v + \Pi(T_0;0,x)\right] < \infty.$$

(A2): For any compact set $K \subset \mathbb{R}^n$, $y \in \mathbb{R}$, and $t \in [0,T_0]$, the following equations hold:

$$\sup_{x\in K}\mathbb{E}\left[\int_0^{T_0}\left(1+\sum_{j=1}^n(|\pi_j(v;0,x)|+|\eta_j(v;0,x)|)\right)\Pi(v;0,x)I_1(y\Pi(v;0,x)E(v;0,x),v)\mathrm{d}v\right] < \infty,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]}\mathbb{E}\left[\left(1+\sum_{j=1}^n(|\pi_j(T_0;t,x)|+|\delta_j(T_0;t,x)|)\right)\Pi(T_0;t,x)I_2(y\Pi(T_0;t,x)\Delta(T_0;t,x))\right] < \infty,$$

$$\sup_{x\in K}\mathbb{E}\left[\int_0^{T_0}\left(1+\sum_{j=1}^n(|\pi_j(v;0,x)|+|\eta_j(v;0,x)|)\right)\Pi(v;0,x)^2E(v;0,x) \times \left|\frac{\partial I_1}{\partial u}(y\Pi(v;0,x)E(v;0,x),v)\right|\mathrm{d}v\right] < \infty,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]}\mathbb{E}\left[\left(1+\sum_{j=1}^n(|\pi_j(T_0;t,x)|+|\delta_j(T_0;t,x)|)\right)\Pi(T_0;t,x)^2 \times \Delta(T_0;t,x)\left|\frac{\mathrm{d}I_2}{\mathrm{d}u}(y\Pi(T_0;t,x)\Delta(T_0;t,x))\right|\right] < \infty,$$

$$\sup_{x\in K}\mathbb{E}\left[\int_0^{T_0}\left(1+\sum_{j=1}^n(|\pi_j(v;0,x)|+|\eta_j(v;0,x)|)\right)\Pi(v;0,x)^2E(v;0,x) \times I_1(y\Pi(v;0,x)E(v;0,x),v)\mathrm{d}v\right] < \infty,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]}\mathbb{E}\left[\left(1+\sum_{j=1}^{n}(|\pi_j(T_0;t,x)|+|\delta_j(T_0;t,x)|)\right)\Pi(T_0;t,x)^2 \times\Delta(T_0;t,x)I_2(y\Pi(T_0;t,x)\Delta(T_0;t,x))\right]<\infty.$$

Then we have the following theorem.

THEOREM 2.5. *We assume Conditions **(U1)**, **(U2)**, **(S1)**, **(S2)**, **(S3)**, **(S4)**, **(S5)**, **(S6)**, **(S7)**, **(A1)**, **(A2)**, and Assumption 2.4. Then there exists an optimal portfolio strategy $\hat{\varphi}(t)$ of the following equation:*

$$J(W,x_0,\eta_0,\delta_0)=\sup_{(C,Z,\varphi)\in\mathcal{A}(W)}V(C,W^{W,C,\varphi}(T_0)). \tag{12}$$

$\hat{\varphi}(t)$ is given by the following feedback form:

$$\begin{aligned}\hat{\varphi}(t)=&\left(1-\frac{1}{\Pi(t)}\right)(\sigma(t,X(t))^*)^{-1}\lambda(t,X(t))\\&-\hat{\lambda}\frac{\Pi(t)E(t)}{W(t)}H(\Pi(t),X(t),\Delta(t),E(t),t)(\sigma(t,X(t))^*)^{-1}(\lambda(t,X(t))-h(t,X(t)))\\&-\hat{\lambda}\frac{\Pi(t)\Delta(t)}{W(t)}G(\Pi(t),X(t),\Delta(t),E(t),t)(\sigma(t,X(t))^*)^{-1}(\lambda(t,X(t))-g(t,X(t)))\\&+\frac{1}{W(t)}\frac{1}{\Pi(t)}(\sigma(t,X(t))^*)^{-1}(\sigma^X(t,X(t))^*)\begin{pmatrix}\mathcal{X}_1(\Pi(t),X(t),\Delta(t),E(t),t)\\ \vdots\\ \mathcal{X}_n(\Pi(t),X(t),\Delta(t),E(t),t)\end{pmatrix},\end{aligned}\tag{13}$$

where $W(t)=W^{W,\hat{C},\hat{\varphi}}(t)$ and $\hat{C}(t)$ is an optimal consumption strategy.

The proof is given in Theorem 2.5. of [5].

REMARK 2.6. Confirming **(S3)** may not be feasible when bonds or other derivative securities are included in tradable securities. In that case, using $\mu^X(t,x)$ and $\sigma^X(t,x)$, we calculate the following:

$$\hat{\mu}(t,x)=\begin{pmatrix}\mu_1(t,x)\\ \vdots\\ \mu_{d-n}(t,x)\\ \mu_1^X(t,x)\\ \vdots\\ \mu_n^X(t,x)\end{pmatrix},\ \hat{\sigma}(t,x)=\begin{pmatrix}\sigma_{1,1}(t,x)&\cdots&\sigma_{1,d}(t,x)\\ \vdots&&\vdots\\ \sigma_{d-n,1}(t,x)&\cdots&\sigma_{d-n,d}(t,x)\\ \sigma_{1,1}^X(t,x)&\cdots&\sigma_{1,d}^X(t,x)\\ \vdots&&\vdots\\ \sigma_{n,1}^X(t,x)&\cdots&\sigma_{n,d}^X(t,x)\end{pmatrix},$$

$$\text{and}\quad \lambda(t,x)=\hat{\sigma}(t,x)^{-1}\left(\hat{\mu}(t,x)-\begin{pmatrix}r(t,x)\\ \vdots\\ r(t,x)\\ \bar{\mu}_1^X(t,x)\\ \vdots\\ \bar{\mu}_n^X(t,x)\end{pmatrix}\right),$$

where $\bar{\mu}_j^X(t,x)$, $j = 1, \cdots, n$ are drift terms of $X(t)$ with respect to the equivalent martingale measure ([4] and [6]). Then **(S3)** and **(S5)** will be satisfied with these processes, and Theorem 2.5 also holds.

REMARK 2.7. In the case of some HARA utility functions, Condition **(A1)** and **(A2)** are replaced by more straightforward conditions. We can also show that optimal portfolio strategies are continuous processes. See Corollary 5.2. and Corollary 5.3. of [5].

3. NUMERICAL EXAMPLES

This section gives examples of optimal portfolio strategies, Stock-Bond-Cash allocation problems, using a combination of stochastic flow technique and a new version of Kusuoka approximation. We re-examine the same example in [5], where the Euler-Maruyama scheme is applied. An investor has an initial endowment W at time 0. Her utility function is of power type $(\gamma, \beta, 0, 0)$. Also her utilities of consumptions are discounted by a proportion of interest rate. Utilities of terminal wealths are discounted by a linear combination of interest rates and stock returns. In this setting, her terminal wealth may be hedged partially against stock returns.

3.1. **Settings and optimal portfolio strategies.** The market is modeled as follows. Let $d = 2$ and $n = 1$. Let $X(t)$ be

$$X(t) = x_0 - a\int_0^t X(v)\mathrm{d}v + b\int_0^t \mathrm{d}B^1(v) = x_0 e^{-at} + be^{-at}\int_0^t e^{av}\mathrm{d}B^1(v),$$

where $a > 0, b \neq 0$. The short rate is modeled by

$$r_t = r(X(t)) = c\left(\log\left(1 + e^{X(t)}\right)\right)^{\alpha},$$

for some $\alpha \in (0,1)$ and $c > 0$. Money account $S^0(t)$ is given by $S^0(t) = \exp\{\int_0^t r(X(v))\mathrm{d}v\}$. A stock, $S^1(t)$, is traded in the market.

$$S^1(t) = S\exp\{(\mu - \frac{1}{2}(\rho^2 + \sigma^2))t + \rho B^1(t) + \sigma B^2(t)\},$$

where $\sigma > 0$ and $\rho \neq 0$.

Let us introduce a zero bond $S^2(t)$ whose maturity is T.

$$S^2(t) = \mathbb{E}^Q\left[\exp\{-\int_t^T r(X(v))\mathrm{d}v\} \,\middle|\, \mathcal{F}_t\right],$$

where Q is the equivalent martingale measure, which is supposed to be defined by the following market price of risk process $(\lambda_1(t), \lambda_2(t))$:

$$\lambda_1(t, X(t)) = \lambda = \text{constant}, \quad \lambda_2(t, X(t)) = c_1 - c_2 r(X(t)),$$

where $c_1 = (\mu - \rho\lambda)/\sigma$ and $c_2 = 1/\sigma$. Then, the state price deflater $\Pi(t;x) = \Pi(t;0,x)$ is given by

$$\Pi(t;x) = \exp\left\{-\int_0^t r(X(v;x))\mathrm{d}v - \int_0^t \lambda\mathrm{d}B^1(v)\right.$$
$$\left. - \int_0^t (c_1 - c_2 r(X(v;x)))\mathrm{d}B^2(v) - \frac{1}{2}\int_0^t (\lambda^2 + (c_1 - c_2 r(X(v;x)))^2)\mathrm{d}v\right\}.$$

Using $\partial X(t;x)/\partial x = e^{-at}$, we have

$$\pi(t;x) = -\int_0^t \{c_1c_2 - 1 - {c_2}^2 r(X(v;x))\} r'(X(v;x)) e^{-av} \mathrm{d}v + c_2 \int_0^t r'(X(v;x)) e^{-av} \mathrm{d}B^2(v).$$

The volatility matrix of $S^1(t)$ and $S^2(t)$ at time 0 is given by

$$\sigma(0,x) = \begin{pmatrix} \rho & \sigma \\ \sigma_2 & 0 \end{pmatrix}, \text{ where } \sigma_2 = \frac{b}{S^2(0;x)} \frac{\partial S^2}{\partial x}(0;x),$$

$$S^2(0;x) = \mathbb{E}\Big[\Pi(T;x)\Big], \text{ and } \quad \frac{\partial S^2}{\partial x}(0;x) = \mathbb{E}\Big[\pi(T;x)\,\Pi(T;x)\Big].$$

In this example, we assume that an investor has a utility function of power type $(\gamma,\beta,0,0)$, $\gamma \in (0,1), \beta > 0$:

$$u(w,t,\omega) = \beta \frac{w^{1-\gamma}}{1-\gamma} \frac{1}{E(t)}, \quad U(w,\omega) = \frac{w^{1-\gamma}}{1-\gamma} \frac{1}{\Delta(T_0)},$$

where for some $0 < \beta_1, \beta_2, \beta_3$,

$$E(t;x) = \exp\left\{ \beta_1 \int_0^t r(X(v;x)) \mathrm{d}v \right\},$$

$$\Delta(t;x) = \exp\left\{ \beta_2 \int_0^t r(X(v;x)) \mathrm{d}v \right\} \exp\left\{ \beta_3 \left((\mu - \frac{1}{2}(\rho^2 + \sigma^2))t + \rho B^1(t) + \sigma B^2(t) \right) \right\}.$$

Let $h(t,x) = (0,0), \quad g(t,x) = (\beta_3\rho, \beta_3\sigma)$. In this case, we have

$$\eta(t;x) = \beta_1 \int_0^t r'(X(v;x)) e^{-av} \mathrm{d}v, \text{ and } \quad \delta(t;x) = \beta_2 \int_0^t r'(X(v;x)) e^{-av} \mathrm{d}v.$$

From Theorem 2.5, we have the following formula.

$$\hat{\varphi}(0) = \begin{pmatrix} \varphi_s \\ \varphi_b \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} \frac{1}{\sigma}(c_1 - c_2 r(x_0)) - \frac{\beta_3 A_2(0)}{\beta^{1/\gamma} A_1(0) + A_2(0)} \\ \frac{\lambda}{\sigma_2} - \frac{\rho(c_1 - c_2 r(x_0))}{\sigma\sigma_2} - \frac{b}{\sigma_2} \frac{D(0)}{\beta^{1/\gamma} A_1(0) + A_2(0)} \end{pmatrix}, \tag{14}$$

where

$$A_1(0) = \mathbb{E}\left[\int_0^{T_0} \Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}} \mathrm{d}v \right], \quad A_2(0) = \mathbb{E}\left[\Pi(T_0)^{1-\frac{1}{\gamma}} \Delta(T_0)^{-\frac{1}{\gamma}} \right],$$

$$D(0) = (1-\gamma)\mathbb{E}\left[\beta^{1/\gamma} \int_0^{T_0} \pi(v)\Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}} \mathrm{d}v + \pi(T_0)\Pi(T_0)^{1-\frac{1}{\gamma}} \Delta(T_0)^{-\frac{1}{\gamma}} \right]$$
$$+ \mathbb{E}\left[\beta^{1/\gamma} \int_0^{T_0} \eta(v)\Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}} \mathrm{d}v + \delta(T_0)\Pi(T_0)^{1-\frac{1}{\gamma}} \Delta(T_0)^{-\frac{1}{\gamma}} \right].$$

3.2. **The NV method.** To calculate Equation (14), we apply the NV method. First of all we set up a system of stochastic differential equations. Let $Y(t;y)$ be a solution of the system. Then we calculate $\mathbb{E}[\Phi(Y(\tau;y))]$, where $\Phi(\cdot)$ is a function and $\tau < T$. Lastly rational functions of $\mathbb{E}[\Phi_j(Y(\tau;y))]$ are calculated based on Equation (14).

The corresponding system of stochastic differential equations in the sense of Stratonovich:

$$\mathrm{d}Y(t;y) = \sum_{j=0}^{2} V_j(Y(t;y)) \circ \mathrm{d}B^j(t),$$

where vector fields V_0, V_1, and V_2 are given as

$$V_0(y) = \sum_{k=1}^{11} V_0{}^k(y)\frac{\partial}{\partial y_k}, \quad V_1(y) = \sum_{k=1}^{11} V_1{}^k(y)\frac{\partial}{\partial y_k}, \quad \text{and } V_2(y) = \sum_{k=1}^{11} V_2{}^k(y)\frac{\partial}{\partial y_k},$$

and given as follows. Let $B^0(t) = t, t \in [0,T]$. Regarding $V_0(y)$, we have

$$V_0{}^1(y) = -a\,y_1, \quad V_0{}^2(y) = \left(-r(y_1) - \frac{1}{2}\lambda^2 - \frac{1}{2}(c_1 - c_2\,r(y_1))^2\right) y_2,$$

$$V_0{}^3(y) = -(c_1c_2 - 1 - c_2{}^2 r(y_1))r'(y_1)y_{11}, \quad V_0{}^4(y) = \left(\beta_2 r(y_1) + \beta_3\mu - \frac{1}{2}\beta_3{}^2\rho^2 - \frac{1}{2}\beta_3{}^2\sigma^2\right) y_4,$$

$$V_0{}^5(y) = \beta_2 r'(y_1)y_{11}, \quad V_0{}^6(y) = \beta_1 r(y_1)y_6, \quad V_0{}^7(y) = \beta_1 r'(y_1)y_{11},$$

$$V_0{}^8(y) = (y_2)^{1-\frac{1}{\gamma}}(y_6)^{-\frac{1}{\gamma}}, \quad V_0{}^9(y) = \beta^{1/\gamma}y_3(y_2)^{1-\frac{1}{\gamma}}(y_6)^{-\frac{1}{\gamma}},$$

$$V_0{}^{10}(y) = \beta^{1/\gamma}y_7(y_2)^{1-\frac{1}{\gamma}}(y_6)^{-\frac{1}{\gamma}}, \quad V_0{}^{11}(y) = -a\,y_{11}.$$

Regarding $V_1(y)$, we have

$$V_1{}^1(y) = b, \quad V_1{}^2(y) = -\lambda y_2, \quad V_1{}^3(y) = 0,$$

$$V_1{}^4(y) = \beta_3\rho y_4, \quad V_1{}^5(y) = \cdots = V_1{}^{11}(y) = 0.$$

Regarding $V_2(y)$, we have

$$V_2{}^1(y) = 0, \quad V_2{}^2(y) = -(c_1 - c_2 r(y_1))y_2, \quad V_2{}^3(y) = c_2 r'(y_1)y_{11},$$

$$V_2{}^4(y) = \beta_3\sigma y_4, \quad V_2{}^5(y) = \cdots = V_2{}^{11}(y) = 0.$$

Then $S^2(0;x_0)$, $\partial S^2(0;x_0)/\partial x$, $A_1(0)$, $A_2(0)$, and $D(0)$ are given by

$$S^2(0;x_0) = \mathbb{E}\left[Y^2(T)\right], \quad \frac{\partial S^2}{\partial x}(0;x_0) = \mathbb{E}\left[Y^3(T)\,Y^2(T)\right],$$

$$A_1(0) = \mathbb{E}\left[Y^8(T_0)\right], \quad A_2(0) = \mathbb{E}\left[Y^2(T_0)^{1-\frac{1}{\gamma}}Y^4(T_0)^{-\frac{1}{\gamma}}\right],$$

$$D(0) = (1-\gamma)\left\{\beta^{1/\gamma}\mathbb{E}[Y^9(T_0)] + \mathbb{E}[Y^3(T_0)Y^2(T_0)^{1-\frac{1}{\gamma}}Y^4(T_0)^{-\frac{1}{\gamma}}]\right\} + \beta^{1/\gamma}\mathbb{E}[Y^{10}(T_0)] + \mathbb{E}[Y^5(T_0)Y^2(T_0)^{1-\frac{1}{\gamma}}Y^4(T_0)^{-\frac{1}{\gamma}}].$$

Therefore we define a function $\Phi(y) = (\Phi_1(y), \cdots, \Phi_5(y))$ as follows:

$$\Phi_1(y) = y_2, \quad \Phi_2(y) = y_3\,y_2, \quad \Phi_3(y) = y_8, \quad \Phi_4(y) = (y_2)^{1-1/\gamma}(y_4)^{-1/\gamma},$$

$$\Phi_5(y) = (1-\gamma)\left\{\beta^{1/\gamma}y_9 + y_3\cdot(y_2)^{1-\frac{1}{\gamma}}\cdot(y_4)^{-\frac{1}{\gamma}}\right\} + \beta^{1/\gamma}y_{10} + y_5\cdot(y_2)^{1-\frac{1}{\gamma}}\cdot(y_4)^{-\frac{1}{\gamma}}.$$

$\exp(tV)$ denotes a deterministic flow generated by a smooth vector field V. By direct calculation, it is easy to show that

$$\exp(tV_1)(y) = \begin{pmatrix} 1 & & & & & & 0 \\ & e^{-\lambda t} & & & & & \\ & & 1 & & & & \\ & & & e^{\beta_3 \rho t} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} y + \begin{pmatrix} b\,t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{15}$$

$$\exp(tV_2)(y) = \begin{pmatrix} 1 & & & & & & 0 \\ & e^{-(c_1 - c_2 r(y_1))t} & & & & & \\ & & 1 & & & & \\ & & & e^{\beta_3 \sigma t} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} y + \begin{pmatrix} 0 \\ 0 \\ c_2 r'(y_1)\, y_{11}\, t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{16}$$

and

$$\exp(tV_1) \circ \exp(tV_2) = \exp(tV_2) \circ \exp(tV_1). \tag{17}$$

Therefore, from Theorem 2.1. of [14], an approximation equation using the NV method is given as follows.

PROPOSITION 3.1. *Let $(\xi_i)_{i \in \{1,\cdots,n\}}$ be n independent standard 2-dimensional normal random variables. Let $\{Y_{s\,k}^{K,n}\}_{k=1,2,\cdots,n}$ be a family of random variables defined as follows: Let $s = \tau/n$, and*

$$Y_0^{K,n} = y,$$

$$Y_{s(k+1)}^{K,n} = \exp\left(\frac{s}{2}V_0\right) \circ \exp\left(\sqrt{s}\xi_k{}^1 V_1\right) \circ \exp\left(\sqrt{s}\xi_k{}^2 V_2\right) \circ \exp\left(\frac{s}{2}V_0\right)(Y_{s\,k}^{K,n}).$$

Then, we have an order 2 approximation.

$$\mathbb{E}\left[\Phi(Y(\tau; y))\right] \sim \mathbb{E}\left[\Phi(Y_{s\,n}^{K,n})\right].$$

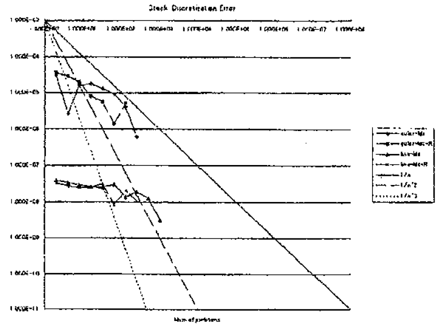
REMARK 3.2. Regarding $\exp(tV_0)$, we use the improved Euler method, a version of 2-stage 2-order Runge-Kutta method, which produces enough precision for our case. See Remark 3.1.2 of [14]. $\exp(tV_0)$ is approximated as follows:

$$\exp(tV_0)(y) \approx y + \frac{t}{2}(V_0(y) + V_0(y + tV_0(y))), \quad \text{for small } t.$$

TABLE 1. Base case parameters

x_0	a	b	α	c	μ	ρ	σ
1.0	1.0	2.0	0.9	0.01	0.08	-0.14	0.20
T	T_0	λ	γ	β	$\beta 1$	$\beta 2$	$\beta 3$
2	1	-0.165	0.90	2.0	0.1	0.05	0.05

FIGURE 1. Error coming from the discretization error: Stock



3.3. **Results of simulations.** In this subsection, we implement stochastic numerical simulations. First of all, we compare numerically the NV method to the Euler-Maruyama scheme with and without Romberg extrapolation.[1] The quasi-Monte Carlo method is applied for numerical integrations. Also, the Monte Carlo method is applied for comparison.[2] A base case of parameters is presented in Table 1.

Second, sensitivity analyses are carried out by changing the initial value of X, x_0, and the risk aversion factor γ using the NV method.

3.3.1. *Discretization Error.* Figure 1 and 2 show relations between the number of partitions of the investment horizon and errors of the methods for optimal holding ratios of the stock and bond. Here, we consider that true values of those ratios are obtained by the NV method with extrapolation, quasi-Monte Carlo. The number of partitions are $1024 + 512$, and the number of samples is 10^7.

Note that some $V_j(\cdot)$ and $\Phi(\cdot)$ are not members of C_b^∞. Also, our optimal solutions are expressed as rational functions of the approximated expected values. Therefore our problem does not satisfies conditions of [9] nor [14]. However, similar to the Euler-Maruyama scheme with Romberg extrapolation, we observe that the NV method gives nice approximations of better order than the Euler-Maruyama scheme.

3.3.2. *Convergence Error.* Figure 3 and 4 show that the performance of the convergence of the Monte Carlo and quasi-Monte Carlo methods depends on the number of partitions and on the algorithms. 2 times standard deviations for the Monte Carlo method and absolute differences

[1]See Talay [17].

[2]Mersenne Twister(Matsumoto-Nishimura [10]) and a generalized Niederreiter sequence(Ninomiya-Tezuka [13]) are used in our numerical simulations.

FIGURE 2. Error coming from the discretization error: Bond

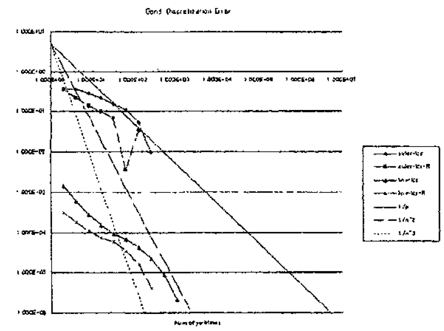


FIGURE 3. Convergence error from quasi-Monte Carlo and Monte Carlo: Stock

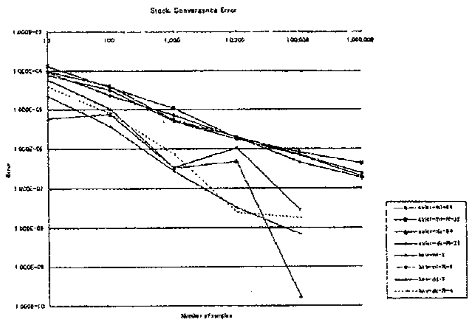
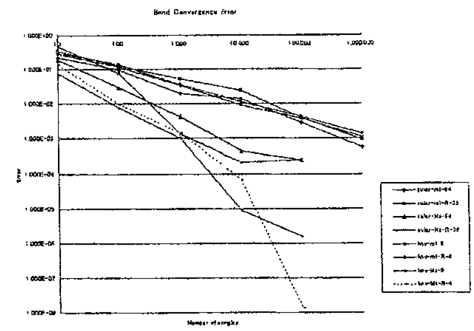


FIGURE 4. Convergence error from quasi-Monte Carlo and Monte Carlo: Bond



for the quasi-Monte Carlo are used as measures of convergence errors. Absolute differences are calculated with respect to values of maximal samples. Number of partitions are set to 32 and 64 for the Euler-Maruyama method with/without the extrapolation. For the NV method with/without the extrapolation, number of partitions are set to 4 and 8. The quasi-Monte Carlo method outperforms the Monte Carlo method when used with the Euler-Maruyama method and the NV method. The used methods does not result significant differences in convergence errors.

TABLE 2. Optimal portfolios for various γ

γ	β_1	β_2	β_3	J	φ_s	φ_b
0.50	0.50	0.25	0.25	4.55506	2.10740	4.37665
0.60	0.40	0.20	0.20	5.96274	1.75914	2.61065
0.70	0.30	0.15	0.15	8.37760	1.51795	1.80583
0.80	0.20	0.10	0.10	13.29601	1.34186	1.37021
0.90	0.10	0.05	0.05	28.21577	1.20801	1.10451

TABLE 3. Optimal portfolio for various x_0

x_0	$r(x_0)$	y	J	φ_s	φ_b
0.70	0.01092	0.01112	28.21789	1.25955	0.46512
0.80	0.01153	0.01136	28.21719	1.24279	0.67310
0.90	0.01215	0.01160	28.21648	1.22560	0.88633
1.00	0.01278	0.01184	28.21577	1.20801	1.10451
1.10	0.01343	0.01208	28.21505	1.19004	1.32734
1.20	0.01409	0.01233	28.21433	1.17171	1.55453
1.30	0.01476	0.01258	28.21360	1.15305	1.78577

3.3.3. *Sensitivity analyses.* The quasi-Monte Carlo NV method with Romberg extrapolation is used for sensitivity analyses. The number of samples is set to 10^6. The number of partitions is set to $128 + 256$.

Table 2 shows values of objective functions and the optimal portfolio for various γ, $\beta_1, \beta_2, \beta_3$. Regarding these constants, we set the following relations,

$$\beta_1 = 1 - \gamma = 2\beta_2 = 2\beta_3,$$

which means that the investor's consumptions are discounted by short rates. The terminal wealth is discounted by the average of short rates and returns of stocks to measure her utilities. As γ increases, holding ratios of stock and bond decrease. This is quite reasonable because γ represents a risk aversion tendency of this investor. Also, it is quite interesting that as γ increases, J, the value of objective function increases.

Table 3 shows values of objective functions and the optimal portfolio for various x_0. As x_0 increases, an initial short rate r_0 increases, and in our setting this means that an expected return of bond increases. Therefore, the holding ratio of bond increases. It is meaningful that the holding ratio of stock decreases as x_0 increases.

4. CONCLUDING REMARKS

This paper gives the validity of the combination of the stochastic flow technique and the NV method for the calculation of optimal strategies when the market is modeled by a Markovian setting. Using a nice quasi-Monte Carlo method with the above combination, we can expect better convergence speeds of approximations. By using the proposed method, time-consuming sensitivity analyses are performed effectively from the points of view of practitioners.

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