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<tr>
<td>Author(s)</td>
<td>BILER, Piotr</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1464: 107-121</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/48002">http://hdl.handle.net/2433/48002</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Radially symmetric solutions of a chemotaxis model in the critical case

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1 The formulation of the problem

This is a report on a joint work with Grzegorz Karch (Wrocław), Philippe Laurengot (Toulouse) and Tadeusz Nadzieja (Zielona Góra), cf. a part of published results in [5].

We investigate properties and large time asymptotics of radially symmetric solutions of a parabolic-elliptic model of chemotaxis (the simplified Keller-Segel system) either in a disc of $\mathbb{R}^2$ or in the whole plane $\mathbb{R}^2$, in the subcritical and critical cases.

Denoting by $u = u(x,t) \geq 0$ the density of microorganisms (e.g. amoeba), and by $\varphi = \varphi(x,t)$ the concentration of a chemoattractant secreted by themselves, the simplified Keller-Segel system we study herein reads

$$
\begin{align*}
  u_t &= \nabla \cdot (\nabla u + u \nabla \varphi), \\
  \varphi &= E_2 * u,
\end{align*}
$$

(1.1) (1.2)

with the space variable $x$ ranging either in $B(0,R) \equiv \{x \in \mathbb{R}^2, |x| < R\}$, $R > 0$, or $\mathbb{R}^2$, and the time variable $t \in (0,\infty)$. Here $E_2(z) = \frac{1}{2\pi} \log |z|$
denotes the fundamental solution of the Laplacian in $\mathbb{R}^2$, so that (1.2) leads to the Poisson equation $\Delta \varphi = u$. The system is supplemented with either the no flux boundary condition
\[
\frac{\partial u}{\partial \nu} + u \frac{\partial \varphi}{\partial \nu} = 0,
\]
where $\nu$ denotes the unit normal vector field to the boundary of $B(0, R)$, or a suitable decay condition $u(x, t) \to 0$ as $|x| \to \infty$ implying the integrability condition $\int_{\mathbb{R}^2} u(x, t) \, dx < \infty$. Moreover, an initial condition
\[
u(x, 0) = u_0(x) \geq 0
\]
is added. After a suitable reduction, see [5, (1.5)–(1.7)] (or [4]), the problem may be posed as a nonlinear nonuniformly parabolic equation for the cumulated mass variable $M(s, t) = \int_{B(0, \sqrt{s})} u(x, t) \, dx$
\[
M_t = 4s M_{ss} + \frac{1}{\pi} M M_s
\]
with a nondecreasing continuous initial condition
\[
M(s, 0) = M_0(s)
\]
on either the interval $(0, 1)$ or the half-line $(0, \infty)$, together with the boundary conditions:
\[
M(0, t) = 0, \quad M(1, t) = M,
\]
or
\[
M(0, t) = 0, \quad M(\infty, t) = M
\]
respectively. We study the problem (1.5)–(1.6) and either (1.7) or (1.8) when the total mass parameter $\overline{M}$ belongs to the interval $[0, 8\pi]$.

As it is well known, in the supercritical case $\overline{M} > 8\pi$ there occurs a lost of the boundary condition at $s = 0$: $\lim_{s \to 0} M(s, t) > 0$ for $t \geq T$ with some
$T > 0$, cf. e.g. [2], [11]. This is interpreted as a blow up of solutions of the original chemotaxis system (at $x = 0$ for radially symmetric solutions)

$$\lim_{t \nearrow T} \|u(t)\|_{H^1} = \lim_{t \nearrow T} |u(t)|_{L^p} = \lim_{t \nearrow T} \int_{\Omega} u(x, t) \log u(x, t) \, dx = \infty$$

for each $p > 1$, cf. [4, 3, 6]. A fine description of blowing up solutions is fairly complicated, see [12], but for radially symmetric solutions the situation is much simpler. The degeneracy of the elliptic operator $4 s M_{ss}$ at $s = 0$ does not allow the diffusion to compensate the growth induced by the convection term $\frac{1}{\pi} M M_s$ and $M(0, t) \neq 0$ for $t > T$ holds. On the one hand, we will show that, in the critical case $\bar{M} = 8\pi$, the blow up in the disc does not take place in a finite time but occurs in infinite time, i.e. the whole mass concentrates at $s = 0$ as $t \to \infty$. We also obtain some temporal decay estimates on $|M(t) - 8\pi|_{L^1}$ for large times. On the other hand, if $\bar{M} \in [0, 8\pi)$, we show the exponential convergence of $M(t)$ towards the unique stationary solution to (1.5)–(1.7) in the disc. The situation is completely different in the case of the whole plane.

2 (Sub)critical case in the disc

The problem (1.5)–(1.7) on $(0, 1)$ is well posed whenever $\bar{M} \in [0, 8\pi]$.

**Theorem 2.1** Consider $\bar{M} \in [0, 8\pi]$ and a continuous nondecreasing function $M_0$ satisfying

$$M_0(0) = 0 \quad \text{and} \quad M_0(1) = \bar{M}. \tag{2.1}$$

There exists a unique function $M \in C([0, \infty); L^2(0, 1)) \cap C^2_{s,t}((0,1) \times (0,\infty))$ such that

$$0 \leq M(s, t) \leq \bar{M}, \quad M_s(s, t) \geq 0 \quad \text{for} \quad (s, t) \in (0,1) \times (0,\infty), \tag{2.2}$$

$$M^*(t) \equiv \inf_{s \in (0,1)} M(s, t) = 0 \quad \text{a.e. in} \quad (0,\infty), \tag{2.3}$$
and

\begin{align}
M_t &= 4sM_{ss} + \frac{1}{\pi}MM_s, \quad (s, t) \in (0, 1) \times (0, \infty), \quad (2.4) \\
M(1, t) &= \overline{M}, \quad t \in (0, \infty), \quad (2.5) \\
M(s, 0) &= M_0(s), \quad s \in (0, 1). \quad (2.6)
\end{align}

Moreover, if there is \( \delta \in (0, 1) \) such that \( M_0(s) \leq (8\pi s)/\delta \) for \( s \in (0, 1) \), then \( \mathcal{M}^*(t) = 0 \) for each \( t \geq 0 \). Observe that if the derivative of \( M_0 \) is finite: \( M_0, s(0) < \infty \), then the above condition on \( M_0 \) is satisfied with a suitable \( \delta > 0 \).

The idea of the proof of Theorem 2.1 is to consider a uniformly parabolic regularized problem

\begin{align}
M_{\epsilon,t} &= 4(s + \epsilon)M_{\epsilon,ss} + \frac{1}{\pi}M_{\epsilon}M_{\epsilon,s}, \quad (s, t) \in (0, 1) \times (0, \infty), \quad (2.7) \\
M_{\epsilon}(0, t) &= \overline{M} - M_{\epsilon}(1, t) = 0, \quad t \in (0, \infty), \quad (2.8) \\
M_{\epsilon}(s, 0) &= M_{0\epsilon}(s), \quad s \in (0, 1). \quad (2.9)
\end{align}

This problem has a unique solution

\[ M_\epsilon \in C([0, 1] \times [0, \infty])) \cap C_{s,t}^{2,1}((0,1) \times (0, \infty)), \]

and we infer from (2.1), (2.7)–(2.8), and the comparison principle that

\[ 0 \leq M_\epsilon(s, t) \leq \overline{M} \quad \text{and} \quad M_\epsilon, s(s, t) \geq 0 \quad \text{for} \quad (s, t) \in [0, 1] \times (0, \infty). \quad (2.10) \]

Moreover, classical parabolic regularity results imply that

\[ \|M_\epsilon\|_{C_{s,t}^{2+\alpha,1+\alpha/2}([\delta,1] \times [\tau,T])} \leq C(\alpha, \delta, \tau, T) \quad (2.11) \]

for each \( T > 0 \), \( \tau \in (0, T) \) and \( \alpha \in (0, 1) \), where \( 0 < C(\alpha, \delta, \tau, T) < \infty \) is a constant depending on \( \alpha, \delta, \tau \) and \( T \) but independent of \( \epsilon \in (0, 1) \).

The key estimate which allows us to control the behavior of solutions for small \( s > 0 \) is

\[ 0 \leq \int_0^T \int_0^1 \frac{M_\epsilon(s, t) (8\pi - M_\epsilon(s, t))}{s + \epsilon} \, ds \, dt \leq C_1(T) \quad (2.12) \]
for every $\epsilon \in (0, 1)$ and a constant $0 < C_1(T) < \infty$ independent of $\epsilon$. This is obtained by multiplying (2.7) by $-\log(s + \epsilon)$ and integrating over $(0, 1)$. Here we use crucially the relation $0 \leq M_\epsilon \leq \overline{M} \leq 8\pi$.

The behaviour of $M_\epsilon$ for small times can be inferred from the estimate

$$\int_0^T \int_0^1 (s + \epsilon) |M_{\epsilon,s}(s, t)|^2 \, ds \, dt + \int_0^T \|M_{\epsilon,t}(t)\|_{H^{-1}}^2 \, dt \leq C_2(T)$$

(2.13)

for every $\epsilon \in (0, 1)$ and a constant $0 < C_2(T) < \infty$ independent of $\epsilon$.

The above estimates permit us to pass to the limit $\epsilon \to 0$ with the approximate solutions $M_\epsilon$ and obtain a solution $M$.

In fact, for each continuous increasing initial data $M^*(t) = 0$ holds for every $t \in (0, \infty)$, not merely for a.e. $t$. Moreover there is a regularizing parabolic effect for (1.5) on the derivatives of solutions. Namely, the estimate $M_s(s, t) \leq C/t$ holds for each $s > 0$ and $t > 0$. These properties are shown by a local comparison with self-similar solutions discussed in Section 3.

Remark. Using the methods above, similar existence and regularity results can be obtained for the “star problem” considered in [6, Theorem 1(i)] and describing a cloud of self-attracting particles in the gravitational field of a fixed point mass (“star”). Namely, the equation (1.5) with the boundary conditions $M(0, t) = m^* \in (0, 4\pi)$, $M(1, t) = \overline{M} \leq 8\pi - m^*$, and suitable initial conditions, has global solutions satisfying properties similar to those in Theorem 2.1.

Since (1.5) is a convection-diffusion equation, we anticipate that it may enjoy some contraction property with respect to some $L^1$-norm. We actually show the following $L^1$-stability property for solutions.

**Theorem 2.2** If $M$, $\overline{M}$ are two solutions to (1.5)-(1.7) (as in Theorem 2.1) with initial data $M_0$ and $\overline{M}_0$ satisfying (2.1) with the same $\overline{M}$, $\overline{M} \in [0, 8\pi]$, then $t \mapsto |\rho(M(t) - \overline{M}(t))|_{L^1}$ is a nonincreasing function of time for each
nonnegative, nonincreasing and concave weight $\rho \in W^{2,\infty}(0,1)$. Furthermore, if $\overline{M} \in [0,8\pi)$,

$$ |M(t) - \overline{M}(t)|_{L^1} \leq 2 |M_0 - \overline{M}_0|_{L^1} e^{-(4-(\overline{M}/2\pi))t}. $$

(2.14)

To prove Theorem 2.2 we consider the difference $N = M - \overline{M}$ which satisfies the equation

$$ N_t = \frac{\partial}{\partial s} \left( 4s N_s + \frac{1}{2\pi} N(M + \overline{M} - 8\pi) \right) $$

(2.15)

with $N(0,t) = N(1,t) = 0$ for a.e. $t \in (0,\infty)$. We prove the $L^1((0,1);\rho(s) \, ds)$ contraction property of solutions. For $\delta \in (0,1)$ and $r \in \mathbb{R}$, we use a convex approximation of $r \mapsto |r|$, e.g.,

$$ \Phi_\delta(r) \equiv \left\{ \begin{array}{ll}
\frac{1}{\delta} \left( |r| - \frac{\delta}{2} \right)^2 & \text{if } |r| \in [0,\delta], \\
|r| - \frac{3}{4}\delta & \text{if } |r| \in (\delta,\infty),
\end{array} \right. $$

We multiply (2.15) by $\rho \, \Phi_\delta'(N)$ and integrate over $(0,1)$ to obtain

$$ \frac{d}{dt} \int_0^1 \rho(s) \Phi_\delta(N) \, ds $$

$$ = \left. 4s\rho(s) N_s \Phi_\delta'(N) \right|_0^1 + \left. \frac{1}{2\pi} \rho(s) \Phi_\delta'(N) N(M + \overline{M} - 8\pi) \right|_0^1 $$

$$ - \int_0^1 4s\rho(s) \Phi_\delta''(N) N_s^2 \, ds - \int_0^1 4s\rho'(s) \Phi_\delta'(N) N_s \, ds $$

$$ - \frac{1}{2\pi} \int_0^1 \rho(s) \Phi_\delta''(N) N_s N(M + \overline{M} - 8\pi) \, ds $$

$$ - \frac{1}{2\pi} \int_0^1 \rho'(s) \Phi_\delta'(N) N(M + \overline{M} - 8\pi) \, ds $$

$$ \leq - \frac{1}{2\pi} \int_0^1 \rho(s) \Phi_\delta''(N) N N_s (M + \overline{M} - 8\pi) \, ds $$

$$ - \frac{1}{2\pi} \int_0^1 \rho'(s) \Phi_\delta'(N) N (M + \overline{M} - 16\pi) \, ds $$

$$ + 4 \int_0^1 s\rho''(s) \Phi_\delta(N) \, ds + 4 \int_0^1 \rho'(s) (\Phi_\delta(N) - N\Phi_\delta'(N)) \, ds. $$
Observe that $N_s$ belongs to $L^\infty((0, \infty); L^1(0, 1))$, $M$, $\overline{M}$ and $N$ are bounded, and $r \mapsto r \Phi'_\delta(r)$ is bounded and converges a.e. towards zero as $\delta \to 0$. Thus, the Lebesgue dominated convergence theorem ensures that the first term of the right-hand side of the above inequality converges to zero as $\delta \to 0$. On the other hand, both $r \mapsto \Phi_\delta(r)$ and $r \mapsto r \Phi'_\delta(r)$ converge uniformly towards $r \mapsto |r|$ on $\mathbb{R}$. Thanks to the boundedness of $M$, $\overline{M}$ and $N$, we can pass to the limit as $\delta \to 0$ in the other terms of the above inequality, and end up with

$$\frac{d}{dt} \int_0^1 \varrho(s)|N| \, ds \leq -\frac{1}{2\pi} \int_0^1 \varrho'(s)|N|(M + \overline{M} - 16\pi) \, ds + 4 \int_0^1 s \varrho''(s)|N| \, ds,$$

(2.16)

Since $M + \overline{M} \leq 2\overline{M} \leq 16\pi$ and $\varrho'$ and $\varrho''$ are both nonpositive, the right-hand side of (2.16) is nonpositive, from which the first assertion of Theorem 2.2 follows.

We now turn to the decay rate (2.14) and assume that $\overline{M} \in [0, 8\pi)$. We take $\varrho(s) = 2 - s$ in (2.16). Since $M + \overline{M} \leq 2\overline{M} < 16\pi$, we infer from (2.16) that

$$\frac{d}{dt} \int_0^1 (2 - s)|N| \, ds \leq \frac{1}{2\pi} \int_0^1 |N|(2\overline{M} - 16\pi) \, ds \leq \frac{\overline{M} - 8\pi}{2\pi} \int_0^1 (2 - s)|N| \, ds,$$

whence

$$\int_0^1 (2 - s)|N(t)| \, ds \leq \int_0^1 (2 - s)|N(0)| \, ds \, e^{-(4-(\overline{M}/2\pi))t},$$

from which (2.14) readily follows.

An immediate consequence of (2.14) with $\overline{M} = M_b$ — the (unique) steady state such that $M_b(1) = \overline{M}$, i.e.

$$M_b(s) = 8\pi \frac{s}{s + b}, \quad s \in (0, 1), \quad \text{with} \quad b = \frac{8\pi}{\overline{M}} - 1 > 0,$$

(2.17)
is the exponential decay 
\[ |M(t) - M_b|_{L^1} \leq 2 |M_0 - M_b|_{L^1} e^{-(4 - (\overline{M}/2\pi))t}. \]

The exponential decay rate does not hold true for the critical case \( \overline{M} = 8\pi \) but the following weaker assertion is available
\[ |M(t) - 8\pi|_{L^1} \leq \frac{8\pi}{t}. \] (2.18)

For the proof, we put \( N(s, t) = M - 8\pi, \phi(s) = 2 - s \). We notice that \( N \) solves
\[ N_t = \frac{\partial}{\partial s} \left( 4sN_s + \frac{1}{2\pi}NM \right) \] (2.19)
with \( N(0, t) = -8\pi \) and \( N(1, t) = 0 \) for a.e. \( t \in (0, \infty) \). Keeping the notations from the proof of Theorem 2.2, we multiply (2.19) by \( \phi'(\delta)(N) \) and integrate over \((0, 1)\) to obtain
\[
\frac{d}{dt} \int_0^1 \phi(s) \Phi_{\delta}(N) \, ds \\
\leq -\frac{1}{2\pi} \int_0^1 \phi(s) \Phi''_{\delta}(N) NN_s M \, ds - \frac{1}{2\pi} \int_0^1 \phi'(s) \Phi_{\delta}(N) NM \, ds \\
+ 4 \int_0^1 s\phi''(s) \Phi_{\delta}(N) \, ds + 4 \int_0^1 \phi'(s) \Phi_{\delta}(N) \, ds ,
\]
since \( \Phi''_{\delta} \) vanishes on a neighbourhood of 0 and \( M^*(t) = 0 \), so the boundary terms vanish. We then proceed as in the proof of (2.16) to pass to the limit as \( \delta \to 0 \) and end up with
\[
\frac{d}{dt} \int_0^1 \phi(s)|N| \, ds \leq \frac{1}{2\pi} \int_0^1 \phi'(s)(8\pi - M)|N| \, ds ,
\]
i.e.
\[
\frac{d}{dt} \int_0^1 (2 - s)|N| \, ds \leq -\frac{1}{2\pi} \int_0^1 |N|^2 \, ds .
\]
We infer from the Cauchy–Schwarz inequality that
\[
\frac{d}{dt} \int_0^1 (2 - s)|N| \, ds \leq -\frac{1}{2\pi} \left( \int_0^1 |N| \, ds \right)^2 \leq -\frac{1}{8\pi} \left( \int_0^1 (2 - s)|N| \, ds \right)^2 ,
\]
whence
\[ |M(t) - 8\pi|_{L^1} \leq \int_0^1 (2 - s)|N(t)|ds \leq \frac{8\pi}{t + 4\pi|8\pi - M_0|_{L^1}^{-1}}. \]

3 The problem in the whole plane

The equation (1.5) for \( s \in (0, \infty) \) is invariant under the space-time scaling
\[ s \mapsto Rs, \quad t \mapsto Rt, \quad R > 0. \] (3.1)

This property has important consequences for the analysis of the problem (1.5)--(1.6) on \((0, \infty) \times (0, \infty)\).

The global in time existence of solutions of that problem can be proved using the ideas of regularizations of the nonlinear term in [11]. An alternative way is to use our previous construction in Theorem 2.1 and the scaling property (3.1) of (1.5). More precisely, if \( 0 \leq M_0 \nearrow \overline{M} \leq 8\pi \) is a subcritical initial data, then we consider its restriction to the interval \((0, R)\). Rescaling \( M_0 \) to \( M_{0R} \) defined on \((0, 1)\), \( M_{0R}(s/R) = M_0(s) \leq \overline{M} \) for \( s \in (0, R) \), we construct the solution \( M_R \) of (1.5)--(1.7) with the initial condition \( M_R(s, 0) = M_{0R}(s) \).

For each \( s \in (0, 1) \) the functions \( M_{0R}(s) \leq \overline{M} \) increase with \( R \nearrow \infty \) so that, by the comparison principle, \( M_R(s, t) \leq \overline{M} \) are also increasing with respect to \( R \). The functions \( \overline{M}_R(s, t) = M_R(s/R, t/R) \) defined for \((s, t) \in (0, R) \times (0, \infty)\) solve the equation (1.5) with \( \overline{M}_R(s, 0) = M_0(s), \ s \in (0, R) \). To obtain a global in time solution with analogous regularity properties as in Theorem 2.1, we perform the passage with \( \overline{M}_R \) to the limit \( R \to \infty \).

Since (1.5) is invariant under the scaling (3.1) it is natural to consider self-similar solutions of (1.5), i.e. those satisfying \( M(Rs, Rt) \equiv M(s, t) \) for each \( R > 0 \). They have the form \( M(s, t) = m(s/t) \) for a function \( m \). The existence of self-similar solutions in the range \( \overline{M} \in [0, 8\pi) \) has been established in,
e.g., [2] and [10] (not necessarily radially symmetric case of the chemotaxis system).

Let us briefly recall the reasoning from [2, Prop. 3, i)]. For $M(s, t) = 2\pi \zeta(s/t)$ (1.5) reads

$$\zeta'' + \frac{1}{4} \zeta' + \frac{1}{2y} \zeta'' = 0 \quad \text{with} \quad y = \frac{s}{t}. \quad (3.2)$$

The change of variables $\tau = \frac{1}{2} \log y$, $v(\tau) = 2y \frac{d\zeta}{dy}(y)$, $w(\tau) = \zeta(y)$ transforms (3.2) into the nonautonomous problem for $(u, v)$ in the plane

$$\begin{align*}
u' &= (2 - w)v - \frac{e^{2\tau}}{2}v, \quad w' = v, \quad \tau' = \frac{d}{d\tau}, \\
v(-\infty) &= 0, \quad w(-\infty) = 0. \quad (3.3)
\end{align*}$$

Evidently, $\lim_{\tau \to \infty} w(\tau) < 4$ because the function $(w - 2)^2 + 2v$ is strictly decreasing along the phase trajectories of the above system.

We consider also an autonomous system

$$\begin{align*}
\underline{v}' &= (2 - \underline{w})\underline{v} - \epsilon \underline{v}, \quad \underline{w}' = \underline{v},
\end{align*}$$

where $\epsilon > 0$, $v = \underline{v}$, $w = \underline{w}$, with the same condition at $\tau = -\infty$. A comparison of these vector fields gives the relation $\underline{w}(\tau) \leq w(\tau)$ for all $\tau \leq \tau_\epsilon$ with $e^{2\tau_\epsilon} = 2\epsilon$. Since $\underline{w}(\tau) = 2(2 - \epsilon) A e^{(2 - \epsilon)\tau} (1 + A e^{(2 - \epsilon)\tau})^{-1}$ with an arbitrary $A > 0$ is a solution of the auxiliary system, so $\underline{w}(\tau_\epsilon) = \frac{2(2 - \epsilon) A e^{(2 - \epsilon)\tau}}{1 + A e^{(2 - \epsilon)\tau}} \geq \lim_{\epsilon \to 0} \underline{w}(\tau) = 4$.

We prove that the asymptotics of general solutions of (1.5)-(1.6), (1.8) for $0 < \overline{M} < 8\pi$ is described by that of self-similar solutions, i.e.

$$0 \leq \frac{m(s/t) - M(s, t)}{m(s/t)} \to 0 \quad \text{as} \quad t \to \infty.$$

Here $m$ denotes the self-similar solution with $m(\infty) = \overline{M}$. The proof involves analysis of the family of suitable scalings of the solution $M$, and the
uniqueness property of self-similar solutions with a given mass $\overline{M} \in [0,8\pi)$. A related result for the original chemotaxis system has been recently announced in [8].

Looking at the problem on a finite interval $(0,1)$, one might suspect that $M(s,t) \rightarrow 8\pi$ as $t \rightarrow \infty$ but for $s \in (0,\infty)$ the picture is much more complicated. First of all, nontrivial solutions of the steady state problem (1.5)–(1.6), (1.8) on $(0,\infty)$ exist for $\overline{M} = 8\pi$ (only!) and are parametrized by $b > 0$:

$$M_b(s) = 8\pi \frac{s}{s+b}, \quad b > 0. \quad (3.4)$$

Second, if $M_0$ satisfies the condition $\int_0^\infty (8\pi - M(s,t)) \, ds < \infty$, the solution $M(.,t)$ converge pointwise to $8\pi$ as $t \rightarrow \infty$, but does not converge to $8\pi$ in the $L^1$ sense. Indeed, for those solutions (they correspond to solutions $u$ of the original chemotaxis system (1.1)–(1.2) possessing the second moment, i.e. $\int_{\mathbb{R}^2} |x|^2 u(x,t) \, dx < \infty$) we have

$$\frac{d}{dt} \int_0^\infty (8\pi - M(s,t)) \, ds = 32\pi - \frac{(8\pi)^2}{2\pi} = 0.$$ 

since $4sM_s(s,t) \rightarrow 0$ as $s \rightarrow 0$ and as $s \rightarrow \infty$. To prove the above, we begin with $M_0$ such that $(8\pi - M_0)$ has compact support in $[0,\infty)$. From the construction of $M$ as the limit of $\overline{M}_R$'s, it is easy to conclude using comparison principle that $M(s,t) \rightarrow 8\pi$ for each $s > 0$ when $t \rightarrow \infty$. The remaining part follows from the $L^1$ contraction property $|M(t) - M(t)|_{L^1} \leq |M_0 - \overline{M}_0|_{L^1}$ proved as in Theorem 2.2 with $\rho(s) \equiv 1$. Indeed, $M_0$ such that $(8\pi - M_0) \in L^1(0,\infty)$ can be approximated by initial data with $(8\pi - M_0)$ of compact support. Combining monotonicity properties of $M$'s and the $L^1$ contraction property, the desired pointwise convergence follows.

To prove the stability of steady states (3.4), we will interprete (1.5) as a nonlinear Fokker–Planck type equation considered in [1], and we will employ a family of Lyapunov functionals for the dynamical system associated with (1.5)–(1.6), (1.8) in the $L^1(0,\infty)$-metric.
Theorem 3.1 The function \( W_b(M) = \int_0^\infty w_b(M(s, t)) \, ds \), where the entropy density \( w_b \) is defined as
\[
  w_b(M) = M \log \frac{M}{M_b} + (8\pi - M) \log \frac{8\pi - M}{8\pi - M_b},
\]
is finite for each \( M \) such that \((M - M_b) \in L^1(0, \infty), M_{b_1} \leq M \leq M_{b_2} \) for some \( b_1 > b > b_2 > 0 \). Moreover, this is nonincreasing along the trajectories \( M(t) = M(., t) \) of the dynamical system (1.5)–(1.6). (L8)
\[
  \frac{dW_b}{dt} \leq -\frac{1}{2\pi} \int_0^\infty s M(8\pi - M) \left| \frac{\partial}{\partial s} \left( \log \frac{M}{8\pi - M} \frac{8\pi - M_b}{M_b} \right) \right|^2 \, ds \leq 0. \tag{3.6}
\]
This implies that if \( M_0 \) is such that \( W_b(M_0) < \infty \) and \((M_0 - M_b) \in L^1(0, \infty) \) for some \( b > 0 \), then \( \lim_{t \to \infty} W_b(t) = 0 \), and therefore (by a Csiszar–Kullback type lemma)
\[
  \lim_{t \to \infty} |M(t) - M_b|_{L^1} = 0.
\]

Local attracting property of the stationary solutions \( M_b \) is a rather weak property. In particular, this does not give any information on the asymptotic behavior of solutions starting from data like, e.g., \( M_0(s) = 8\pi \frac{s}{s + 2 + \cos s} \) which satisfy the relation \( M_3 \leq M_0 \leq M_1 \), but \( M_0 - M_b \notin L^1(0, \infty) \) for any \( b > 0 \). All this shows that the long time behavior of solutions in the critical case may be extremely complicated and even chaotic.

Remark. The problem of the chemotaxis (1.1)–(1.4) in the whole plane in the subcritical case \( \overline{M} < 8\pi \), without radial symmetry assumptions, has been recently studied in [9]. In particular, the authors proved the global in time existence of solutions using logarithmic Sobolev inequalities.

Using the approach via radially symmetric decreasing rearrangements in [7] we might use the results here to give an alternative construction of global in time solutions for \( \overline{M} \leq 8\pi \), and to give a flavor of the diversity of locally attracting solutions for the problem without radial symmetry. Indeed, results from [7] imply that, roughly speaking, the existence of solutions of (1.1)–(1.4)
is controlled by the existence of solutions to the radially symmetric problem given by (1.5)-(1.6), (1.8) with the initial condition \(M_0\) obtained from the radially symmetric decreasing rearrangement of \(u_0\).

4 Supercritical case in \(\mathbb{R}^2\)

Let us recall some results from the preprint [11] (Theorems 2.7, 3.5, 4.4) related to the supercritical case of equation (1.5) on \((0, \infty)\), i.e. for \(\overline{M} > 8\pi\).

First, the classical solution of (1.5) (that possesses the second moment — which was not explicitly stated in [11], cf. [3], [4]) blows up in a finite time: there is \(0 < T < \infty\) such that \(\lim_{t \to T} M(s, t) \geq 8\pi\) for each \(s > 0\). This means that the boundary condition at \(s = 0\) is lost, \(M^*(t)\) jumps to \(8\pi\) instantaneously at \(t = T\).

Moreover, there exists a continuation of \(M, M \in C^\infty((0, \infty) \times (0, \infty))\), past the blow up time \(T\), satisfying (1.5), (1.6) for all \(t > 0\), and the quantity \(M^*(t)\) strictly increases for \(t > T\). Such a global in time smooth solution — a continuation of the classical solution for \(t < T\) — is unique in \(C^\infty((0, \infty) \times (0, \infty))\), and satisfies \(\lim_{t \to \infty} M(s, t) = \overline{M}\) for each \(s \geq 0\). Moreover, \(\lim_{t \to \infty} M^*(t) = \overline{M}\): the whole mass concentrates at the origin in the infinite time, unlike the critical \(\overline{M} = 8\pi\) (nontrivial steady states exist) and subcritical cases \(M^* < 8\pi\) (mass spreads to infinity).

Acknowledgements. The preparation of this paper was partially supported by the KBN (MNI) grant 2/P03A/002/24, and by the EU network HYKE under the contract HPRN-CT-2002-00282.

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