数理解析研究所講究録 1464 巻 2006 年 145-163

Generation and propagation of interface to a competition-diffusion system with a very large interaction rate

東京海洋大学・海洋科学部 中島主恵 (Kimie Nakashima)
Tokyo University of Marine Science and Technology

1 Introduction

Habitat segregation phenomena in mathematical ecology supply us with various problems which are interesting from the aspect of interfacial dynamics. We mathematically discuss regional partition by competitive two species and their competition for their own habitats. When the competition between two species is bitter, they cannot coexist at the same point. In such cases we can expect that the two species with a suitable initial state segregate their habitats and compete on the interface between both the habitats. Then it is significant to understand the dynamics of the segregation patterns.

In this article we treat a competition-diffusion system for two species in competition of the Lotka-Volterra type:

$$\begin{cases} u_t = d_1\Delta u + (a_1 - b_1 u - c_1 v)u, & \text{in } \Omega \times (0,\infty), \\ v_t = d_2\Delta v + (a_2 - b_2 v - c_2 u)v, & \text{in } \Omega \times (0,\infty), \end{cases}$$

with Neumann zero boundary condition on $\partial\Omega$. Here a_k, b_k, c_k and d_k $(k = 1,2)$ are positive constants; $u = u(t,x)$ and $v = v(t,x)$ are the population densities of competitive two species. Our concern is the situation where the interspecific competition is exceedingly bitter: in particular, the situation close to the singular limit as $c_1, c_2 \to \infty$ with c_1/c_2 fixed. Thus we simply rewrite the above system as

$$\begin{cases} u_t = \Delta u + (a-u)u - bMuv, & \text{in } \Omega \times (0,\infty), \\ v_t = D\Delta v + (d-v)v - cMuv, & \text{in } \Omega \times (0,\infty), \end{cases} \tag{1}$$

where a, b, c, d, D are fixed positive constants and M is a huge parameter. As seen in the following section, the spatial supports of u and v satisfying (3) become separated from each other by an interface in a short time-period. Then after that the *segregated* (u, v) behaves like a solution of a two phase free boundary problem for the Fisher equation. We will establish a rigorous mathematical theory both for the formation of interfaces at the initial stage and for the motion of those interfaces in the later stage. More precisely, we will show that, given virtually arbitrary smooth initial data, the solution develops interfaces within the time scale of $O(\epsilon^2)$. We will then prove that the motion of the interfaces converges to the following free boundary problem as $\epsilon \to 0$.

$$\begin{cases} u^*_t = \Delta u^* + (a - u^*)u^*, \quad v^* \equiv 0 \quad \text{in } R(t), \\ v^*_t = D\Delta v^* + (d - v^*)v^*, \quad u^* \equiv 0 \quad \text{in } \Omega\backslash R(t), \\ c\dfrac{\partial u^*}{\partial \nu} + bD\dfrac{\partial v^*}{\partial \nu} = 0 \quad \text{on } \Gamma(t), \end{cases} \tag{2}$$

where

$$\Gamma(t) = \partial R(t),$$

and ν an inner normal to $\Gamma(t)$.

There are several related works on singular limits of some reaction-diffusion systems as the effect of interaction tends to infinity: [1], [3], [4], [5] and [11] investigate the *fast reaction limit* of chemical reaction systems (see also the references therein). As for competition-diffusion systems, [2] investigates singular limits of the stationary problems as the interspecific competition rate tends to infinity. The most related work is [6], which we will mention after giving the formal derivation of the singular limit.

2 Formal derivation of the singular limit

We rewrite (1) as

$$\begin{cases} u_t = \Delta u + (a - u)u - \dfrac{b}{\epsilon^3}uv, \qquad \text{in } \Omega \times (0, \infty), \\ v_t = D\Delta v + (d - v)v - \dfrac{c}{\epsilon^3}uv, \qquad \text{in } \Omega \times (0, \infty), \end{cases} \tag{3}$$

with the boundary condition and the initial condition such that

$$\begin{aligned} &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad \text{on } \partial\Omega \times (0, \infty), \\ &u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \qquad \text{in } \Omega. \end{aligned}$$

Here $u_0(x) > 0$, $v_0(x) > 0$, in $\bar{\Omega}$, n is an outer normal to $\partial\Omega$, and ϵ is a small parameter, especially. In this section we present a formal derivation of the singular limit of (3).

Set

$$R(0) = \{\, x \in \Omega \mid cu_0(x) > bv_0(x) \,\}, \qquad \Omega\backslash\overline{R(0)} = \{\, x \in \Omega \mid cu_0(x) < bv_0(x) \,\},$$

and assume that both of $R(0)$ and $\Omega\backslash\overline{R(0)}$ possess interior points.

Let us consider the first stage of a short time period from $t = 0$ until $t = \epsilon^2$. Since the initial data is smooth, it is heuristically seen that the behavior of the solution of (3) is formally approximated by that of $(\tilde{u}, \tilde{v})$ below during the very early stage, where the diffusion terms, $u(a-u)$ and $v(d-v)$ are relatively small compared with the competition terms.

$$\begin{cases} \tilde{u}_t = -\dfrac{b\tilde{u}\tilde{v}}{\epsilon^3} \\ \tilde{v}_t = -\dfrac{c\tilde{u}\tilde{v}}{\epsilon^3} \\ \tilde{u}(x,0) = u_0(x), \quad \tilde{v}(x,0) = v_0(x). \end{cases} \tag{4}$$

The solution of (4) is given by

$$\tilde{u}(x,t) = \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big), \quad \tilde{v}(x,t) = \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big), \tag{5}$$

where $(\phi(\tau;\xi,\eta), \psi(\tau;\xi,\eta))$ is a solution of

$$\begin{cases} \dot{\phi} = -b\phi\psi, & \phi(0) = \xi > 0, \\ \dot{\psi} = -c\phi\psi, & \psi(0) = \eta > 0. \end{cases} \tag{6}$$

Set $A(\xi,\eta) = c\xi - b\eta$, then we can easily observe that $A(\phi(\tau), \psi(\tau))$ is preserved for any $\tau > 0$; so that

$$\dot{\phi} = (A(\xi,\eta) - c\phi)\phi, \qquad \phi(0) = \xi. \tag{7}$$

Solving (7) explicitly, we have

$$\phi(\tau;\xi,\eta) = \frac{\xi A e^{A\tau}}{A + c\xi(e^{A\tau} - 1)}, \quad \psi(\tau;\xi,\eta) = \frac{\eta A e^{-A\tau}}{A + b\eta(1 - e^{-A\tau})}, \tag{8}$$

therefore,

$$\lim_{\tau\to+\infty} \phi(\tau;\xi,\eta) = \max\Big\{\frac{A(\xi,\eta)}{c}, 0\Big\}, \quad \lim_{\tau\to+\infty} \psi(\tau;\xi,\eta) = \max\Big\{0, -\frac{A(\xi,\eta)}{b}\Big\}. \tag{9}$$

Then it follows that the solution becomes close to the continuous function

$$(u_1(x), v_1(x)) = \begin{cases} (\omega(x)/c,\ 0) & \text{in } R(0), \\ (0,\ -\omega(x)/b) & \text{in } \Omega\backslash\overline{R(0)}, \end{cases} \tag{10}$$

after a short period of time scale t. The non-degeneracy of $\nabla\omega$ on $\partial R(0) = \{\, x |\ \omega(x) = 0 \,\}$ causes the gap of $(\nabla u_1, \nabla v_1)$ across the surface $\partial R(0)$. Thus sharp transition of $(\nabla u,\ \nabla v)$ appears near $\partial R(0)$. Namely the *corner layer* of $(u(t,\cdot),\ v(t,\cdot))$ is generated along the surface $\partial R(0)$ in a short time-period.

The second stage of the dynamics of (3) describes the propagation of the corner layer. The stretching (u, v) with a suitable scale makes the analysis of the corner layer easier. To rescale the system in the best possible way, we need to estimate the length scale $\epsilon = \epsilon(M)$ of the width of the corner layer. We note that u_1, v_1 are continuous functions with bounded gradients and that the mean curvature of the surface $\partial R(0)$ is bounded. It is natural to assume in the second stage that $u = O(\epsilon)$, $v = O(\epsilon)$, $u_t = O(1)$ and $\Delta u = O(\epsilon^{-1})$ on the corner layer for huge M and that the effects of Δu and Muv in (3) are well-balanced. Then we have $\epsilon = O(M^{-1/3})$.

Taking account of (10) and the argument for the first stage, we can expect that $u(t, x; \epsilon)$ almost vanishes in some region in Ω, namely $\mathbb{R}^N \backslash R^\epsilon(t)$, on the other hand $v(t, x; \epsilon)$ vanishes in $R^\epsilon(t)$. Further the corner layer of $(u(t, \cdot; \epsilon), v(t, \cdot; \epsilon))$ remains along the interface $\partial R^\epsilon(t)$. Around each point $y \in \partial R^\epsilon(t)$ we introduce a local orthogonal coordinate system (ξ, σ) such that $\sigma = (\sigma_1, \dots, \sigma_{N-1})$ is a local coordinate along $\partial R^\epsilon(t)$ whereas $\xi = \xi(x, \partial R^\epsilon(t))$ is the signed distance from x to $\partial R^\epsilon(t)$ locally defined near y so that $\xi > 0$ in $R^\epsilon(t)$. Around the corner layer we stretch the solution and see it using a moving coordinate system (t, ρ, σ), where $\rho = \xi/\epsilon$ is a rescaled coordinate in the normal direction to $\partial R^\epsilon(t)$. Suppose that $(u(t, x; \epsilon), v(t, x; \epsilon))$ is asymptotically written as

$$(u, v) = \begin{cases} (u^*, v^*) + O(\epsilon) & \text{away from the layer (outer expansion)}, \\ \epsilon(U_1, V_1) + \epsilon^2(U_2, V_2) + O(\epsilon^3) & \text{around the layer (inner expansion)}, \end{cases}$$

where (u^*, v^*) is a bounded continuous function of the fixed coordinate (t, x) and (U_1, V_1) and (U_2, V_2) are smooth functions of the moving coordinate (t, ρ, σ) with a bounded gradient; all of them are independent of ϵ. By a formal argument based on the *matched asymptotic expansion* method, we can formally conclude that (u^*, v^*) satisfy (2) and (U_1, V_1) satisfy

$$\begin{cases} U_{1\rho\rho} = cU_1V_1, & -\infty < \rho < +\infty, \\ dV_{1\rho\rho} = bU_1V_1, & -\infty < \rho < +\infty, \\ (U_1(t, \rho, \sigma),\ V_1(t, \rho, \sigma)) = \left(0,\ -\rho\dfrac{\partial v^*}{\partial \nu^o}(t, y)\right) & \text{as } \rho \to -\infty, \\ (U_1(t, \rho, \sigma),\ V_1(t, \rho, \sigma)) = \left(\rho\dfrac{\partial u^*}{\partial \nu^i}(t, y),\ 0\right) & \text{as } \rho \to +\infty, \end{cases} \tag{11}$$

and (U_2, V_2) satisfies (30) which is given later.

Here $R(t)$ is the formal limit of $R^\epsilon(t)$ as $\epsilon \to +0$, $\nu^i(\nu^o)$ inner (outer) normal to $\partial R(t)$, and y a point on $\partial R(t)$ corresponding to the coordinate $(0, \sigma)$. In (11) the boundary conditions at $\rho = \pm\infty$ reflect the request that (u^*, v^*) and $\epsilon(U_1, V_1)$ should be matched. The boundary condition on $\partial R(t)$ in (2) is requested for (u^*, v^*) in order that the elliptic boundary value problem (11) possesses a solution. Consequently, in the second stage the supports of $u(t, \cdot; \epsilon)$ and $v(t, \cdot; \epsilon)$ are almost separated by the corner layer which remains in a narrow range of $O(\epsilon)$ along the propagating interface $\partial R(t)$. The dynamics of the segregation pattern is essentially determined by the free boundary problem (2). We see from the elliptic equations in (11) that the population on the interface supplied by the diffusion from both the habitats instantly disappears by the strong competition between two species.

3 Main result

The formal derivation of the free boundary problem (2) from (3) as $\epsilon \to +0$ is justified by [6] on a bounded domain in $\mathbb{R}^N$ under the no-flux boundary condition in the framework of weak topology of H^1. It also gives a result on the uniqueness and existence of a Hölder-continuous weak solution to (2). However we need to justify the derivation of (2) at least in the framework of C^0-topology in order to investigate the dynamics of the segregating interface. To accomplish this end we impose the existence of a classical solution to (2) as follows.

Before stating the results, we will make some assumptions.

Assumption 1 *(nondegeneracy condition) Suppose that*

$$\inf_{\Gamma(0)} |c\nabla u_0 - b\nabla v_0| > 0.$$

Here $\Gamma(0) = \partial R(0)$

Remark 1 Assumption 1 assures that Γ_0 is an $N-1$ dimensional hypersurface with bounded mean curvature.

Let $(u^*(x,t), v^*(x,t), \Gamma(t))$ be a solution to the free boundary problem (2) with an initial data

$$u^*(x,0) = \frac{\omega(x)}{c}, \quad \text{in } R(0), \qquad v^*(x,0) = -\frac{\omega(x)}{b} \quad \text{in } \Omega\backslash\overline{R(0)}. \tag{12}$$

Assumption 2 *$(u^*(x,t), v^*(x,t), \Gamma(t))$ satisfies (2) with the initial data (12) in a classical sense for $(x,t) \in \Omega \times [0,T]$. $\Gamma(t)$ is a closed hypersurface in Ω and is in C^2 for each t and in C^1 with respect to t.*

Assumption 3 *u^* and v^* be nonnegative continuous functions, $|u^*|, |\nabla u^*|, |\Delta u^*|$ are bounded in $R(t)$ uniformly with respect to t, and $|v^*|, |\nabla v^*|, |\Delta v^*|$ are bounded in $\Omega\backslash\overline{R(t)}$ uniformly with respect to t;*

Assumption 4 $\displaystyle\inf_{y\in\Gamma(t)} \lim_{\substack{x\to y \\ x\in R(t)}} |\nabla u^*(x)| > 0, \quad \inf_{y\in\Gamma(t)} \lim_{\substack{x\to y \\ x\in \Omega\backslash\overline{R(t)}}} |\nabla v^*(x)| > 0.$

Remark 2 If the free boundary condition in (2) is replaced by

$$\mu\frac{d}{dt}\Gamma(t) = c\frac{\partial u^*}{\partial\nu^i} - bD\frac{\partial v^*}{\partial\nu^o} \quad \text{on } \Gamma(t),$$

where μ is a positive constant and $\frac{d}{dt}\Gamma(t)$ denotes the propagation speed of $\Gamma(t)$ in the outer normal direction, then the regularity of $\Gamma(t)$ will be assured by the parabolicity as treated in [8] and [10]. However, in our case which corresponds to the case $\mu = 0$, it is not easy to deduce the regularity of $\Gamma(t)$ in (2), because the parabolicity is partially broken on $\Gamma(t)$. Nevertheless, a recent result in [11] suggests that the partial regularity of $\Gamma(t)$ in the classical sense can hold also for (2). Thus we believe the above assumptions natural.

Now we will give our main theorem.

Theorem 1 *There exist a positive constant $C > 0$ such that for sufficiently small $\epsilon > 0$, the following hold:*

$$|u_\epsilon(x,t) - u^*(t,x)| < C\epsilon|\log\epsilon|,$$

$$|v_\epsilon(x,t) - v^*(t,x)| < C\epsilon|\log\epsilon| \quad \textit{for } (t,x) \in [\epsilon^2, T] \times \Omega,$$

where $(u_\epsilon(x,t),\ v_\epsilon(x,t))$ is a nonnegative solution of (3). More precisely, there exists $C', C'', C''' > 0$ such that for sufficiently small ϵ, the following holds:

$$|u_\epsilon(x,t)| < C'\exp\left(-\frac{C'''|\tilde{d}(x,t)|}{\epsilon}\right), \qquad \textit{for}\{x \in \Omega\backslash\overline{R(t)}\ ;\ |\tilde{d}(x,t)| > C''\epsilon|\log\epsilon|\},$$

$$|v_\epsilon(x,t)| < C'\exp\left(-\frac{C'''|\tilde{d}(x,t)|}{\epsilon}\right), \qquad \textit{for}\{x \in R(t)\ ;\ |\tilde{d}(x,t)| > C''\epsilon|\log\epsilon|\}.$$

Theorem 1 shows that , for virtually arbitrary smooth initial data, the solution develops interfaces in time $t = \epsilon^2$ and the motion of the interface is approximated by the free boundary problem (2) for $t \in [\epsilon^2, T]$.

Our main tool for deriving the above results is the method of upper and lower solutions. We will use two different pairs of upper and lower solutions, namely $(U^\pm, V^\pm)$ and $(u^\pm, v^\pm)$. The first one $(U^\pm, V^\pm)$ is used to analyze the generation of the interface that takes place in a very fast time scale. The second one $(u^\pm, v^\pm)$ is used to study the motion of the interface in a relatively slow time scale. The transition from the initial stage to the second stage occurs within a time scale of ϵ^2. Since the behaviors of solutions are so different between the two stages, it is important to construct suitable upper and lower solutions for each stage and to know the right timing to switch from $(U^\pm, V^\pm)$ to $(u^\pm, v^\pm)$.

In the following Section 4, we deal with the generation of the interface, and in Section 6, the motion of the interface. Section 4 is depend on [9], and Section 6 is on [7].

4 Generation of interface

In this section we study the generation of interface that takes place in the initial stage. We will construct an upper and lower solution for this stage.

As we have mentioned in Section 2, we can expect that the solution $(u(x,t), v(x,t))$ would be approximated by

$$(\phi(\frac{t}{\epsilon^3}; u_0(x), v_0(x)),\ \psi(\frac{t}{\epsilon^3}; u_0(x), v_0(x))) \tag{13}$$

by a formal argument. Let $d^* > 0$ be sufficiently small constant such that $dist(x, \Gamma(t))$ be signed distance function defined in $\{x \in \Omega \mid dist(x, \Gamma(t)) \leq 3d^* \}$ We will introduce cut-off functions. We define $\tilde{d}$ as a modification of $dist(x, \Gamma(t))$ such that $\tilde{d}\, d \geq 0$.

$$\tilde{d}(x,t) = \begin{cases} dist(x, \Gamma(t)) & \text{if} \quad |dist(x,\Gamma(t))| \leq d^*, \\ \\ d^* \leq |dist(x,\Gamma(t))| \leq 2d^* & \text{if} \quad d^* \leq |dist(x,\Gamma(t))| \leq 2d^*, \\ \\ |dist(x,\Gamma(t))| = 2d^* & \text{for} \quad \Omega \setminus \{x \in \Omega \mid dist(x,\Gamma(t)) \leq 2d^* \}. \end{cases} \tag{14}$$

Set

$$\Gamma_0 = \Gamma(0).$$

It is easily seen that there exists $0 < C_0 < C_1$ such that

$$C_0 |\tilde{d}(x,0)| < |\omega(x)| < C_1 |\tilde{d}(x,0)|.$$

Therefore, we obtain the following theorem:

Theorem 2 (*Nakashima-Wakasa [9]*) *Then there exist C_1, $C_2 > 0$ such that for sufficiently small $\epsilon > 0$, the solution (u_ϵ, v_ϵ) of (3) satisfies the following estimate:*

$$\left| u_\epsilon(x,t) - \phi\left(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\right) \right| < C_1 \epsilon, \quad (x,t) \in \Omega \times (0, \epsilon^2),$$

$$\left| v_\epsilon(x,t) - \psi\left(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\right) \right| < C_1 \epsilon \quad (x,t) \in \Omega \times (0, \epsilon^2),$$

and

$$\left| u_\epsilon(x, \epsilon^2) - \max\left\{ \frac{\omega(x)}{c}, 0 \right\} \right| < C_2 \epsilon, \quad x \in \Omega$$

$$\left| v_\epsilon(x, \epsilon^2) - \max\left\{ 0, -\frac{\omega(x)}{b} \right\} \right| < C_2 \epsilon \quad x \in \Omega.$$

Moreover, under Assumption 1, there exist $C_3, C_4, C_5 > 0$ such that for sufficiently small $\epsilon > 0$,

$$\begin{aligned} &|u_\epsilon(x,\epsilon^2)| < C_5 \exp(-\frac{C_3 |\ \tilde{d}(x,0)|}{\epsilon}), \quad \textit{in } \{x \in \Omega \backslash \overline{R(0)}\ ;\ |\tilde{d}(x,0)| > C_4 \epsilon |\log \epsilon|\}, \\ &|v_\epsilon(x,\epsilon^2)| < C_5 \exp(-\frac{C_3 |\ \tilde{d}(x,0)|}{\epsilon}), \quad \textit{in } \{x \in R(0)\ ;\ |\tilde{d}(x,0)| > C_4 \epsilon |\log \epsilon|\}. \end{aligned} \tag{15}$$

Theorem 2 shows that, for virtually arbitrary initial data, the solution forms interfaces in time $t = \epsilon^2$. More precisely, at time $t = \epsilon^2$, $(u^{\pm}, v^{\pm})$ stays between another pair of upper and a lower solution which are given in the next section, Motion of interface. This makes it possible to combine two different pairs of upper and lower solutions.

5 Proof of Theorem 2

In this section, we will prove Theorem 2 The proof of this theorem is due to constructing upper and lower solutions, which are modifications of the approximate solutions $(\tilde{u}, \tilde{v})$ in (4).

5.1 Some estimates for solutions to O.D.E. system

Let us consider (ϕ, ψ), solutions to (6). We will give some estimates for several quantities of ϕ, ψ, and their derivatives with respect to ξ and η. From (8) and (9), we can see that sign of $A(\xi, \eta)$ plays an essential role to determine asymptotic behavior of (ϕ, ψ) as $\tau \to +\infty$. In order to show Theorem 2, we need two kind of different estimates. One of these are given uniformly in A, that is, these estimates are independent of $A(\xi, \eta)$. We also need estimates for ϕ (resp. ψ), if $(\xi, \eta) \in \{A(\xi, \eta) < 0\}$ (resp. if $(\xi, \eta) \in \{A(\xi, \eta) > 0\}$).

Lemma 1 (i) *For all $\tau, \xi, \eta > 0$,*

$$0 < \phi(\tau; \xi, \eta) - \max\left\{\frac{A(\xi, \eta)}{c}, 0\right\} \leq \frac{\xi}{(1 + c\xi\tau)},$$

and

$$0 < \psi(\tau; \xi, \eta) - \max\left\{0, -\frac{A(\xi, \eta)}{b}\right\} \leq \frac{\eta}{(1 + b\eta\tau)}.$$

(ii) *If $\xi, \eta > 0$ satisfies $A(\xi, \eta) < 0$ (resp. $A(\xi, \eta) > 0$), then*

$$0 < \phi(\tau; \xi, \eta) \leq \xi e^{A(\xi,\eta)\tau} \quad (\textit{resp. } 0 < \psi(\tau; \xi, \eta) \leq \eta e^{-A(\xi,\eta)\tau})$$

for any $\tau > 0$.

Lemma 2 *For all $\tau > 0, \xi > 0, \eta > 0$,*

$$0 < \phi_\xi(\tau; \xi, \eta) < 1, \quad -\frac{b}{c} < \phi_\eta(\tau; \xi, \eta) < 0,$$

and

$$-\frac{b}{c} < \psi_\xi(\tau; \xi, \eta) < 0, \quad 0 < \psi_\eta(\tau; \xi, \eta) < 1.$$

Lemma 3 *If $\xi, \eta > 0$ satisfies $A(\xi,\eta) < 0$ (resp. $A(\xi,\eta) > 0$), then there exists $M_0 > 0$, which is independent of ξ and η, such that*

$$|\phi_\xi(\tau;\xi,\eta)| \le M_0(1+\xi\tau)e^{A(\xi,\eta)\tau}, \quad |\phi_\eta(\tau;\xi,\eta)| \le M_0\xi\tau e^{A(\xi,\eta)\tau}$$

$$(resp.\ |\psi_\xi(\tau;\xi,\eta)| \le M_0\eta\tau e^{-A(\xi,\eta)\tau}, \quad |\psi_\eta(\tau;\xi,\eta)| \le M_0(1+\eta\tau)e^{-A(\xi,\eta)\tau})$$

for any $\tau > 0$.

Lemma 4 *(i) For all $\tau > 0,\ \xi > 0,\ \eta > 0$, it holds that*

$$\left|\frac{\phi(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| < 2\xi, \qquad \left|\frac{\psi(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| < 2\eta.$$

(ii) There exist $M_1, M_2 > 0$, which are independent of ξ and η, such that

$$\left|\frac{\phi_{\xi\xi}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| + \left|\frac{\phi_{\xi\eta}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| + \left|\frac{\phi_{\eta\eta}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| \le \frac{M_1}{\xi} + M_2\tau,$$

$$\left|\frac{\psi_{\xi\xi}(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| + \left|\frac{\psi_{\xi\eta}(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| + \left|\frac{\psi_{\eta\eta}(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| \le \frac{M_1}{\eta} + M_2\tau,$$

for all $\tau > 0,\ \xi > 0,\ \eta > 0$.

Proofs of above lemmas are ommited.

5.2 Definition of upper and lower solutions

Let $(u(x,t), v(x,t))$ be a smooth function defined on $\bar{\Omega} \times [t_0, t_1]$. We say (u,v) is an *upper solution* for equation (3) (in the time interval $t_0 \le t \le t_1$) if it satisfies

$$\begin{cases} u_t - \Delta u - (a-u)u + \dfrac{buv}{\epsilon^3} \ge 0, \\[2ex] v_t - d\Delta v - (d-v)v + \dfrac{cuv}{\epsilon^3} \le 0 \end{cases} \tag{16}$$

for $x \in \Omega, t_0 \le t \le t_1$ along with the boundary condition

$$\frac{\partial u}{\partial n} \ge 0, \frac{\partial v}{\partial n} \le 0,\ (x \in \partial\Omega,\ t_0 \le t \le t_1).$$

We say (u,v) is a *lower solution* for equation (3) if it satisfies

$$\begin{cases} u_t - \Delta u - u(a-u) + \dfrac{buv}{\epsilon^3} \le 0, \\[2ex] v_t - d\Delta v - v(d-v) + \dfrac{cuv}{\epsilon^3} \ge 0 \end{cases} \tag{17}$$

for $x \in \Omega, t_0 \le t \le t_1$ along with the boundary condition

$$\frac{\partial u}{\partial n} \le 0, \frac{\partial v}{\partial n} \ge 0,\ (x \in \partial\Omega,\ t_0 \le t \le t_1).$$

The following is a consequence of the maximum principle.

Proposition 1 *Let (u^+, v^+) be an upper solution and (u^-, v^-) be a lower solution of* (3) *for $t_0 \le t \le t_1$. Suppose that a solution (u, v) of* (3) *satisfies $u^-(x,t_0) \le u(x,t_0) \le u^+(x,t_0)$, $v^-(x,t_0) \ge v(x,t_0) \ge v^+(x,t_0)$ for $x \in \bar{\Omega}$. Then the solution (u,v) satisfies $u^-(x,t) \le u(x,t) \le u^+(x,t)$ and $v^-(x,t) \ge v(x,t) \ge v^+(x,t)$ for $t \in [t_0, t_1]$ and $x \in \bar{\Omega}$.*

The following is also an immediate consequence of the above proposition:

Cororrary 1 Comparison principle. *If (u,v) and $(\tilde{u}, \tilde{v})$ are two solutions of* (3) *and if $u \le \tilde{u}$ and $v \ge \tilde{v}$ for $t = t_0$, then $u \le \tilde{u}$ and $v \ge \tilde{v}$ for $t \ge t_0$.*

Remark. This comparison principle reduces to Proposition 1 in the case of the ODE system (6). More precisely

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \succeq \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} \quad \text{implies} \quad \begin{pmatrix} \phi(\tau;\xi,\eta) \\ \psi(\tau;\xi,\eta) \end{pmatrix} \succeq \begin{pmatrix} \phi(\tau;\tilde{\xi},\tilde{\eta}) \\ \psi(\tau;\tilde{\xi},\tilde{\eta}) \end{pmatrix} \text{ for } \tau \ge 0. \tag{18}$$

5.3 Construction of an upper and a lower solution

The upper and lower solutions for the early stage are constructed by modifying the solution of the following problem:

Define

$$\begin{aligned} U^+(x,t) &= \phi(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}),\ v_0(x) - \gamma_2\epsilon\exp(\frac{t}{\epsilon^2})), \\ V^+(x,t) &= \psi(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_2\epsilon\exp(\frac{t}{\epsilon^2})), \\ U^-(x,t) &= \phi(\frac{t}{\epsilon^3}; u_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}),\ v_0(x) + \gamma_2\epsilon\exp(\frac{t}{\epsilon^2})), \\ V^-(x,t) &= \psi(\frac{t}{\epsilon^3}; u_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}),\ v_0(x) + \gamma_2\epsilon\exp(\frac{t}{\epsilon^2})), \end{aligned} \tag{19}$$

where $\gamma_1, \gamma_2 > 0$ is constants determined later.

Lemma 5 *There exists $\gamma_1 > 0$, $\gamma_2 > 0$ such that for sufficiently small $\epsilon > 0$, the functions $U^\pm, V^\pm$ are pair of upper and lower solutions of* (3) *for $0 \le t \le \epsilon^2$.*

Proof. We first consider the case where

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0.$$

on $\partial\Omega$. Consequently, we have

$$\frac{\partial U^+}{\partial n} = \frac{\partial U^-}{\partial n} = \frac{\partial V^+}{\partial n} = \frac{\partial V^-}{\partial n} = 0$$

on $\partial\Omega$. The general case will be considered in Remark below. We will show that (U^+, V^+), (U^-, V^-) satisfy inequalities (16) and (17) respectively. We set

$$\begin{cases} \mathcal{L}_1(u,v) &= u_t - \Delta u - (a-u)u + \dfrac{buv}{\epsilon^3}, \\ \mathcal{L}_2(u,v) &= v_t - d\Delta v - (d-v)v + \dfrac{cuv}{\epsilon^3}. \end{cases} \tag{20}$$

Our goal is to show that

$$\mathcal{L}_1(U^+, V^+) \geq 0, \quad \mathcal{L}_1(U^-, V^-) \leq 0, \quad \mathcal{L}_2(U^+, V^+) \leq 0, \quad \mathcal{L}_2(U^-, V^-) \geq 0.$$

We will only prove $\mathcal{L}_1(U^+, V^+) \geq 0$ and $\mathcal{L}_2(U^+, V^+) \leq 0$, since the other inequalities can be proved similarly. $\mathcal{L}_1(U^+, V^+)$ and $\mathcal{L}_2(U^+, V^+)$ are given by

$$\begin{aligned} \mathcal{L}_1(U^+, V^+) =& \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2})(\gamma_1\phi_\xi - \gamma_2\phi_\eta) \\ &- \phi_{\xi\xi}|\nabla u_0(x)|^2 - 2\phi_{\xi\eta}\nabla u_0(x)\nabla v_0(x) - \phi_{\eta\eta}|\nabla v_0(x)|^2 \\ &- \phi_\xi \Delta u_0(x) - \phi_\eta \Delta v_0(x) - (a-\phi)\phi \\ =& \phi_\xi(\gamma_1 \cdot \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_1) + (-\phi_\eta)(\gamma_2 \cdot \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_2), \end{aligned} \tag{21}$$

and

$$\begin{aligned} \mathcal{L}_2(U^+, V^+) =& \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2})(\gamma_1\psi_\xi - \gamma_2\psi_\eta) \\ &- D\psi_{\xi\xi}|\nabla u_0(x)|^2 - 2D\psi_{\xi\eta}\nabla u_0(x)\nabla v_0(x) - D\psi_{\eta\eta}|\nabla v_0(x)|^2 \\ &- D\psi_\xi \Delta u_0(x) - D\psi_\eta \Delta v_0(x) - (d-\psi)\psi \\ =& D\psi_\xi(\frac{\gamma_1}{D} \cdot \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_3) + (-D\psi_\eta)(\frac{\gamma_2}{D} \cdot \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_4). \end{aligned} \tag{22}$$

Here R_i $(i = 1, \cdots, 4)$ are

$$\begin{aligned} &R_1 = \tfrac{\phi}{\phi_\xi}(a-\phi) + \Delta u_0 + \tfrac{\phi_{\xi\xi}}{\phi_\xi}|\nabla u_0|^2 + 2\tfrac{\phi_{\xi\eta}}{\phi_\xi}\nabla u_0 \nabla v_0 + \tfrac{\phi_{\eta\eta}}{\phi_\xi}|\nabla v_0|^2, \\ &R_2 = -\Delta v_0, \quad R_3 = \Delta u_0, \\ &R_4 = -\tfrac{\psi}{\psi_\eta}(d-\psi) - \Delta v_0 - \tfrac{\psi_{\xi\xi}}{\psi_\eta}|\nabla u_0|^2 - 2\tfrac{\psi_{\xi\eta}}{\psi_\eta}\nabla u_0 \nabla v_0 - \tfrac{\psi_{\eta\eta}}{\psi_\eta}|\nabla v_0|^2. \end{aligned}$$

In above expressions, we also use several notations as follows:

$$\begin{aligned} \phi &= \phi(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2})), \\ \phi_\xi &= \frac{\partial\phi}{\partial\xi}(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2})), \\ \phi_{\xi\xi} &= \frac{\partial^2\phi}{\partial\xi^2}(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2})), etc. \end{aligned}$$

By Lemma 2, we can see followings: ϕ_ξ, $-\phi_\eta$ are positive and ψ_ξ, $-\psi_\eta$ are negative. Additionally, from Lemma 4, we can observe that if ϵ is sufficiently small, then

$$\begin{aligned} \max\{|R_1|, |R_3|\} &\le M_1\Big(\frac{1}{\inf_{x\in\Omega} u_0(x)} + \frac{t}{\epsilon^3}\Big), \\ \max\{|R_2|, |R_4|\} &\le M_2\Big(\frac{1}{\inf_{x\in\Omega} v_0(x)} + \frac{t}{\epsilon^3}\Big), \end{aligned} \tag{23}$$

for $x \in \Omega$, and $t > 0$. Here M_1, M_2 are positive constants and, depend on bounds of $|u_0|$, $|\nabla u_0|$, $|\Delta u_0|$, $|v_0|$, $|\nabla v_0|$, $|\Delta v_0|$ in Ω. Therefore applying (23) to (21) and (22), we obtain $\mathcal{L}_1(U^+, V^+) \ge 0$ for sufficiently large γ_1, $\gamma_2 > 0$ independently of $\epsilon > 0$. The proof is complete. $\square$

Lemma 6 *There exist $C_1, C_2 > 0$ such that for sufficiently small $\epsilon > 0$, the solution (u_ϵ, v_ϵ) of* (3) *satisfies the following estimate:*

$$\Big|U^\pm(x,t) - \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\Big| < C_1\epsilon, \quad (x,t) \in \Omega\times(0,\epsilon^2),$$

$$\Big|V^\pm(x,t) - \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\Big| < C_1\epsilon \quad (x,t) \in \Omega\times(0,\epsilon^2),$$

and

$$\Big|U^\pm(x,\epsilon^2) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\}\Big| < C_2\epsilon, \quad x\in\Omega$$

$$\Big|V^\pm(x,\epsilon^2) - \max\Big\{0, -\frac{\omega(x)}{b}\Big\}\Big| < C_2\epsilon \quad x\in\Omega.$$

Moreover, for any $\beta > 0$, there exist $C_3 > 0$ such that for sufficiently small $\epsilon > 0$,

$$|U^\pm(x,\epsilon^2)| < C_3\epsilon^\beta, \quad \textit{in } \{x \in \Omega\backslash\overline{R(0)}\ ;\ \omega(x) < -\beta\epsilon|\log\epsilon|\},$$

$$|V^\pm(x,\epsilon^2)| < C_3\epsilon^\beta, \quad \textit{in } \{x \in R(0)\ ;\ \omega(x) > \beta\epsilon|\log\epsilon|\}.$$

Proof of Theorem 2. If we apply Proposition 1 to Lemma 5, then for $(x,t) \in \Omega\times[0,\epsilon^2]$, $U^-(x,t) \le u_\epsilon(x,t) \le U^+(x,t)$, $V^-(x,t) \ge v_\epsilon(x,t) \ge V^+(x,t)$ and especially,

$$\Big|u_\epsilon(x,t) - \phi\Big(\frac{t}{\epsilon^3}; u_0(x), v_0(x)\Big)\Big| \le \max_{U^+,U^-}\Big|U^\pm(x,t) - \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\Big|,$$

$$\Big|v_\epsilon(x,t) - \phi\Big(\frac{t}{\epsilon^3}; u_0(x), v_0(x)\Big)\Big| \le \max_{V^+,V^-}\Big|V^\pm(x,t) - \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\Big|.$$

and

$$\Big|u_\epsilon(x,\epsilon^2) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\}\Big| \le \max_{U^+,U^-}\Big|U^\pm(x,\epsilon^2) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\}\Big|,$$

$$\left|v_\epsilon(x,\epsilon^2)-\max\left\{0,-\frac{\omega(x)}{b}\right\}\right|\le\max_{V^+,V^-}\left|V^\pm(x,t)-\max\left\{0,-\frac{\omega(x)}{b}\right\}\right|.$$

Therefore from Lemma 6, we obtain the proof. □

Proof of Lemma 6. We only show the inequalities for U^+, since the other inequalities are shown in the same way. Using mean value theorem, we have

$$\begin{aligned}&\left|U^+(x,t)-\phi\Big(\frac{t}{\epsilon^3}\ ;\ u_0(x),v_0(x)\Big)\right|\\ \le&\epsilon\gamma_1\exp\frac{t}{\epsilon^2}\phi_\xi\Big(\frac{t}{\epsilon^3}\ ;\ u_0(x)+\epsilon\theta_1\gamma_1\exp\frac{t}{\epsilon^2},v_0(x)-\epsilon\theta_2\gamma_2\exp\frac{t}{\epsilon^2}\Big)\\ &-\epsilon\gamma_2\exp\frac{t}{\epsilon^2}\phi_\eta\Big(\frac{t}{\epsilon^3}\ ;\ u_0(x)+\epsilon\theta_3\gamma_1\exp\frac{t}{\epsilon^2},v_0(x)-\epsilon\theta_4\gamma_2\exp\frac{t}{\epsilon^2}\Big),\end{aligned}\tag{24}$$

for some $0\le\theta_i\le1$ $(i=1,2,3,4)$. It follows from Lemma 2 below and (24) that there exists $C_1'>0$ such that

$$\left|U^+(x,t)-\phi\Big(\frac{t}{\epsilon^3}\ ;\ u_0(x),v_0(x)\Big)\right|\le C_1'\epsilon,\quad\text{for }(x,t)\in\Omega\times[0,\epsilon^2].\tag{25}$$

Hereafter, C_i' $(i\in\mathbb{N})$ denotes a positive constant independent of $\epsilon>0$. Set $t=\epsilon^2$. It holds that

$$\begin{aligned}\left|U^+(x,\epsilon^2)-\max\left\{\frac{\omega(x)}{c},0\right\}\right|\ \le\ &\left|U^+(x,\epsilon^2)-\phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x),v_0(x)\Big)\right|\\ &+\left|\phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x),v_0(x)\Big)-\max\left\{\frac{\omega(x)}{c},0\right\}\right|\end{aligned}\tag{26}$$

By (i) of Lemma 1,

$$\left|\phi\Big(\frac{1}{\epsilon};u_0(x),v_0(x)\Big)-\max\left\{\frac{\omega(x)}{c},0\right\}\right|<c\epsilon$$

in Ω. Hence combining this inequality, (25), and (26) we obtain the second inequality for U^+ in Lemma 6. Now let us consider the third inequality for U^+ in Lemma 6. Fix $\beta>0$, and recall (24) and (26). Using of (ii) of Lemma 1 and Lemma 3 at $t=\epsilon^2$ $(\tau=\epsilon^{-1})$, if

$\omega(x) < -\beta\epsilon|\log\epsilon|$, then for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
|U^+(x,\epsilon^2)| \le & \left|U^+(x,\epsilon^2) - \phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big)\right| + \left|\phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big)\right| \\
\le & \epsilon\gamma_1 e\phi_\xi\Big(\frac{1}{\epsilon}\ ;\ u_0(x) + \epsilon\theta_1\gamma_1 e, v_0(x) - \epsilon\theta_2\gamma_2 e\Big) \\
& - \epsilon\gamma_2 e\phi_\eta\Big(\frac{1}{\epsilon}\ ;\ u_0(x) + \epsilon\theta_3\gamma_1 e, v_0(x) - \epsilon\theta_4\gamma_2 e\Big) \\
& + \left|\phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big)\right| \\
\le & \epsilon(\gamma_1+\gamma_2)e \cdot M_0'\frac{1}{\epsilon}\sup_{x\in\Omega} u_0(x)\exp\frac{\omega(x)+O(\epsilon)}{\epsilon} \\
& + \sup_{x\in\Omega} u_0(x)\exp\frac{\omega(x)}{\epsilon} \\
\le & C_3\exp(-\beta|\log\epsilon|) \le C_3\epsilon^\beta.
\end{aligned} \tag{27}$$

Therefore we obtain the inequality for U^+. The proof is complete. □

Remark We finally consider the case where $u_0(x)$ and $v_0(x)$ do not satisfy the Neumann zero boundary conditions. Since ϵ is small, $d(x,\partial\Omega)$ is smooth in $\{x \in \Omega \mid dist(x,\partial\Omega) \le 3\epsilon\}$. Set

$$\theta(x) = \begin{cases} 0 & \text{if} \quad dist(x,\partial\Omega) \ge 2\epsilon, \\ 1 & \text{if} \quad dist(x,\partial\Omega) \le \epsilon \end{cases} \tag{28}$$

Choose $\eta_1, \eta_2, \eta_3, \eta_4$ such that

$$-\eta_1 < \min_{x\in\partial\Omega}\left\{\frac{\partial u_0(x)}{\partial n}, 0\right\} \le 0, \quad \eta_2 > \max_{x\in\partial\Omega}\left\{\frac{\partial v_0(x)}{\partial n}, 0\right\} \ge 0,$$
$$\eta_3 > \max_{x\in\partial\Omega}\left\{\frac{\partial u_0(x)}{\partial n}, 0\right\} \ge 0, \quad -\eta_4 < \min_{x\in\partial\Omega}\left\{\frac{\partial v_0(x)}{\partial n}, 0\right\} \le 0,$$

we define

$$\begin{aligned}
u_0^+(x) &= u_0(x) + \eta_1 dist(x,\partial\Omega)\theta(x), \\
v_0^+(x) &= v_0(x) - \eta_2 dist(x,\partial\Omega)\theta(x), \\
u_0^-(x) &= u_0(x) - \eta_3 dist(x,\partial\Omega)\theta(x), \\
v_0^-(x) &= v_0(x) + \eta_4 dist(x,\partial\Omega)\theta(x),
\end{aligned}$$

If we replace $u_0(x)$ and $v_0(x)$ in (U^+, V^+) in (19) by u_0^+ and v_0^+, respectively, we obtain the upper solutions. On the other hand, if we replace $u_0(x)$ and $v_0(x)$ in (U^-, V^-) in (19) by u_0^- and v_0^-, respectively, we obtain the lower solutions. The proof is completed by repeating the proof of Lemmas 5 and 6. Almost all the arguments are the same as the case of Neumann zero except for two differences. One is that $\Delta u_0^{\pm}, \Delta v_0^{\pm}$ are $O(1/\epsilon)$ in Lemma 5. Therefore, we obtain

$$\begin{aligned} \max\{|R_1|,|R_3|\} &\leq M_3\Big(\frac{1}{\epsilon}+\frac{1}{\inf_{x\in\Omega}u_0(x)}+\frac{t}{\epsilon^3}\Big), \\ \max\{|R_2|,|R_4|\} &\leq M_4\Big(\frac{1}{\epsilon}+\frac{1}{\inf_{x\in\Omega}v_0(x)}+\frac{t}{\epsilon^3}\Big), \end{aligned} \tag{29}$$

insted of (23). The other difference is that we need to check the Neumann zero boundary condition of u_0^+, v_0^+, u_0^-, and v_0^-. To show this we remark that

$$\theta(x)=1,\ \ \frac{\partial\theta}{\partial n}\geq 0,\ \ \text{on}\ \ \partial\Omega.$$

Then it follows that

$$\frac{\partial}{\partial n}(u_0^+(x))=\frac{\partial u_0(x)}{\partial n}+\eta_1\theta(x)+\eta_1 dist(x,\partial\Omega)\frac{\partial\theta}{\partial n}=0\ \ \text{on}\ \ \partial\Omega.$$

The conditions for v_0^+, u_0^-, v_0^- can be obtained in the same way.

6 Motion of interface

In this section we construct another pair of upper and lower solutions for the second stage, motion of interface. This upper and lower solutions $(u^{\pm}, v^{\pm})$ has interface near $\Gamma(t)$, the solution of the free boundary problem (2).

We first construct upper and lower solutions $(U_{in}^{\pm}, V_{in}^{\pm})$ in a tubular neighborhood of $\Gamma(t)$ by modifying the first two terms of the inner expansion. After that we construct an upper and a lower solution $(U_{out}^{\pm}, V_{out}^{\pm})$ outside the tubular neighborhood using the first term of outer expansion. Then we match $(U_{in}^{\pm}, V_{in}^{\pm})$ and $(U_{out}^{\pm}, V_{out}^{\pm})$, then obtain $(u^{\pm}, v^{\pm})$. Once $(u^{\pm}, v^{\pm})$ are obtained, they will later be combined with another set of upper and lower solutions $(U^{\pm}, V^{\pm})$ that take care of the generation of interface at the initial stage.

6.1 An upper and a lower solution near the interface

We define an upper and a lower solutions in the following form:

$$\begin{aligned} U_{in}^+(x,t) &= \epsilon U_1\left(\frac{\tilde d(x,t)}{\epsilon}-\eta(t),\sigma\right)+\epsilon^2 U_2\left(\frac{\tilde d(x,t)}{\epsilon}-\eta(t),\sigma,t\right)+\epsilon^3 q(t), \\ V_{in}^+(x,t) &= \epsilon V_1\left(\frac{\tilde d(x,t)}{\epsilon}-\eta(t),\sigma\right)+\epsilon^2 V_2\left(\frac{\tilde d(x,t)}{\epsilon}-\eta(t),\sigma,t\right)-\epsilon^3 \hat q(t), \\ U_{in}^-(x,t) &= \epsilon U_1\left(\frac{\tilde d(x,t)}{\epsilon}+\eta(t),\sigma\right)+\epsilon^2 U_2\left(\frac{\tilde d(x,t)}{\epsilon}+\eta(t),\sigma,t\right)-\epsilon^3 q(t), \\ V_{in}^-(x,t) &= \epsilon V_1\left(\frac{\tilde d(x,t)}{\epsilon}+\eta(t),\sigma\right)+\epsilon^2 V_2\left(\frac{\tilde d(x,t)}{\epsilon}+\eta(t),\sigma,t\right)+\epsilon^3 \hat q(t). \end{aligned}$$

Here $\tilde{d}(x,t)$ is defined in (14),

$$\eta(t) = \left(\log\frac{1}{\epsilon}\right)\gamma\exp(Mt), \quad q(t) = \sigma\exp(Mt), \quad \hat{q}(t) = \hat{\sigma}\exp(Mt),$$

where $\gamma, \sigma, \hat{\sigma}$ and M are positive constants to be determined appropriately, and (U_1, V_1) satisfies (11) and (U_2, V_2) satisfies

$$\begin{cases} -U_{2\xi\xi} + c(U_1V_2 + U_2V_1) = -U_{1\xi}(d_t - \Delta d) & -\infty < \rho < +\infty, \\ -DV_{2\xi\xi} + b(U_1V_2 + U_2V_1) = -V_{1\xi}(d_t - D\Delta d) & -\infty < \rho < +\infty, \\ (U_2(t,\rho,\sigma),\ V_2(t,\rho,\sigma)) = (0,\ 0) & \text{as } \rho \to -\infty, \\ (U_2(t,\rho,\sigma),\ V_2(t,\rho,\sigma)) = (0,\ 0) & \text{as } \rho \to +\infty. \end{cases} \tag{30}$$

(30) is obtained by the formal argument based on the *matched asymptotic expansion.* The following lemma assures the existence of the first and second term of upper and lower solutions, whose proofs are omitted.

Lemma 1 *(i) There exists a unique positive solution of* (11).
(ii) There exists a solution of (30).

Since the first two terms of $(U_{in}^{\pm}, V_{in}^{\pm})$ are determined, we choose appropriate q and $\hat{q}$ so that $(U_{in}^{\pm}, V_{in}^{\pm})$ are an upper and lower solutions.

6.2 Upper and lower solutions away from the interface

In this subsection we will construct upper and lower solutions away from the interface modifying the first term of outer expansion.

Let g be a smooth function satisfying

$$g(s) = 0 \text{ if } s < 0, \quad g(s) = 1 \text{ if } s > 1$$

$$g'(0) = g'(1) = 0, \quad g'(s) \geq 0 \text{ for } 0 \leq s \leq 1$$

and set

$$\lambda_1(s) = g\left(\frac{s}{\epsilon} + \tilde{R}|\log\epsilon|\right), \quad \lambda_2(s) = g\left(-\frac{s}{\epsilon} - \tilde{R}|\log\epsilon|\right).$$

Moreover let δ satisfy $0 < \delta \ll d^*$ and define

$$H(s) = \begin{cases} -\beta\epsilon|\log\epsilon|(s+\delta)^2 + \beta\delta\tilde{R}\epsilon^2|\log\epsilon|^2 + \dfrac{\beta\delta^2}{\tilde{R}}\epsilon|\log\epsilon|, \\ \qquad\qquad (-\delta - \tilde{R}\epsilon|\log\epsilon| \leq s \leq -\tilde{R}\epsilon|\log\epsilon|), \\ \beta\delta\tilde{R}\epsilon^2|\log\epsilon|^2 + \dfrac{\beta\delta^2}{\tilde{R}}\epsilon|\log\epsilon|, \qquad (s \leq -\delta - \tilde{R}\epsilon|\log\epsilon|). \end{cases}$$

Now we will define upper and lower solutions in the following form:

$$U^+_{out}(x,t) = \begin{cases} u^*(x,t) + \epsilon|\log\epsilon|\alpha\exp(Lt) - H(\tilde{d}(x,t)), & d(x,t) \le -\tilde{R}\epsilon|\log\epsilon| \\ (1-\lambda_1(\tilde{d}(x,t)))U^+_\epsilon + \lambda_1(\tilde{d}(x,t))\epsilon^4, & d(x,t) > \tilde{R}\epsilon|\log\epsilon| \end{cases}$$

$$V^+_{out}(x,t) = \begin{cases} 0, & d(x,t) \le -\tilde{R}\epsilon|\log\epsilon|, \\ v^*(x,t) - \epsilon|\log\epsilon|\alpha\exp(Lt) + H(-\tilde{d}(x,t)), & d(x,t) > \tilde{R}\epsilon|\log\epsilon| \end{cases}$$

$$U^-_{out}(x,t) = \begin{cases} u^*(x,t) - \epsilon|\log\epsilon|\alpha\exp(Lt) + H(\tilde{d}(x,t)), & d(x,t) \le -\tilde{R}\epsilon|\log\epsilon| \\ 0, & d(x,t) > \tilde{R}\epsilon|\log\epsilon| \end{cases}$$

$$V^-_{out}(x,t) = \begin{cases} (1-\lambda_2(\tilde{d}(x,t)))W^-_\epsilon + \lambda_2(\tilde{d}(x,t))\epsilon^4, & d(x,t) \le -\tilde{R}\epsilon|\log\epsilon| \\ v^*(x,t) + \epsilon|\log\epsilon|\alpha\exp(Lt) - H(-\tilde{d}(x,t)), & d(x,t) > \tilde{R}\epsilon|\log\epsilon|. \end{cases}$$

Here $\alpha, \beta, \tilde{R}$ are positive constants to be specified appropriately.

$(U^\pm_{out}, V^\pm_{out})$ are chosen so as to satisfy the following condition.

- $(U^\pm_{out}, V^\pm_{out})$ is an upper and a lower solution for $|d(x,t)| > \tilde{R}\epsilon|\log\epsilon|$.
- The entire upper and lower solution given by (31) below is not smooth for $|d(x,t)| = \tilde{R}\epsilon|\log\epsilon|$. (We need to care about the derivative of $(U^\pm_{in}, V^\pm_{in})$ and $(U^\pm_{out}, V^\pm_{out})$ at $|d(x,t)| = \tilde{R}\epsilon|\log\epsilon|$.) $(U^\pm_{out}, V^\pm_{out})$ are determined so that $(u^\pm, v^\pm)$ given below become an upper and a lower solutions.
- $(U^\pm_{out}, V^\pm_{out})$ has the following estimate.

$$(U^\pm_{out}, V^\pm_{out}) = (u^*, v^*) + O(\epsilon|log\epsilon|).$$

6.3 Entire solution for the motion of interface

The entire solution is given by

$$(u^\pm, v^\pm) = \begin{cases} (U^\pm_{in}, V^\pm_{in}) & |d(x,t)| \le \tilde{R}\epsilon|\log\epsilon|, \\ (U^\pm_{out}, V^\pm_{out}) & |d(x,t)| > \tilde{R}\epsilon|\log\epsilon|. \end{cases} \tag{31}$$

Let $(u^*_m.v^*_m, R_m)$ be a solution to (2) with an arbitrary initial data $(u^*_m(x,0).v^*_m(x,0))$. Assume

Assumption 5 *Assumption 2 and Assumption 3 hold with replacing (u^*, v^*, R) by (u^*_m, v^*_m, R_m).*

Set $\Gamma_m = \partial R_m$, and let $\tilde{d}_m$ be difined by (15) with Γ replaced by Γ_m. They give the following result:

Theorem 3 *(Iida-Karali-Mimura-Nakashima-Yanagida [7]) For any sufficiently large $\gamma > 0$ and any sufficiently small $\sigma > 0$, there exist $C_5, C_6, C_7 > 0$ such that for sufficiently small $\epsilon > 0$ and the initial data satisfying*

$$|u_\epsilon(x,0) - u_m^*(x,0)| < C_5\epsilon|\log\epsilon|, \qquad |v_\epsilon(x,0) - v_m^*(x,0)| < C_5\epsilon|\log\epsilon|. \tag{32}$$

$$\begin{aligned} &|u_\epsilon(x,0)| < C_5\exp(-\frac{\sigma|\tilde{d}_m(x,0)|}{\epsilon}), \\ &\qquad\qquad for \quad \{x \in \Omega\backslash\overline{R_m(0)}\ ;\ |\tilde{d}_m(x,0)| > \gamma\epsilon|\log\epsilon|\}, \\ &|v_\epsilon(x,0)| < C_5\exp(-\frac{\sigma|\tilde{d}_m(x,0)|}{\epsilon}), \\ &\qquad\qquad for \quad \{x \in R_m(0)\ ;\ |\tilde{d}_m(x,0)| > \gamma\epsilon|\log\epsilon|\}, \end{aligned} \tag{33}$$

it holds that

$$|u_\epsilon(x,t) - u_m^*(x,t)| < C_6\epsilon|\log\epsilon|, \qquad |v_\epsilon(x,t) - v_m^*(x,t)| < C_6\epsilon|\log\epsilon|.$$

$$\begin{aligned} &|u_\epsilon(x,t)| < C_6\exp(-\frac{\sigma|\tilde{d}_m(x,t)|}{\epsilon}), \\ &\qquad\qquad for \quad \{x \in \Omega\backslash\overline{R_m(t)}\ ;\ |\tilde{d}_m(x,t)| > C_7\epsilon|\log\epsilon|\}, \\ &|v_\epsilon(x,t)| < C_6\exp(-\frac{\sigma|\tilde{d}_m(x,t)|}{\epsilon}) \quad , \\ &\qquad\qquad for \quad \{x \in R_m(t)\ ;\ |\tilde{d}_m(x,t)| > C_7\epsilon|\log\epsilon|\}. \end{aligned}$$

7 Proof of Theorem 1

Combining the estimate in Theorem 2 and expressions of $(u^\pm, v^\pm)$, we have

$$u^-(x,\epsilon^2) \le U^-(x,\epsilon^2) \le U^+(x,\epsilon^2) \le u^+(x,\epsilon^2),$$

$$v^-(x,\epsilon^2) \ge V^-(x,\epsilon^2) \ge V^+(x,\epsilon^2) \ge v^+(x,\epsilon^2).$$

This and Theorems 2 and 3 implies that for arbitrarily chosen initial data satisfying Assumption 1, the solution of (3) stays between (U^-, V^-) and (U^+, V^+) for $t \in (0, \epsilon^2]$, and stays between (u^-, v^-) and (u^+, v^+) for $t \in [\epsilon^2, T]$. Using the estimate in Theorem 3, the proof is completed.

References

[1] J.R. CANNON AND C.D. HILL, *On the movement of a chemical reaction interface*, Indiana Univ. Math. J., **20** (1970), pp.429–454.

[2] E.N. DANCER AND Y. DU, *Competing species equations with diffusion, large interactions, and jumping nonlinearities*, J. Differential Equations, **114** (1994), pp.434–475.

[3] L.C. EVANS, *A convergence theorem for a chemical diffusion-reaction system*, Houston J. Math., **6** (1980), pp.259–267.

[4] D. HILHORST, R. VAN DER HOUT AND L.A. PELETIER, *The fast reaction limit for a reaction-diffusion system*, J. Math. Anal. Appl., **199** (1996), pp.349–373.

[5] D. HILHORST, R. VAN DER HOUT AND L.A. PELETIER, *Diffusion in the presence of fast reaction: the case of a general monotone reaction term*, J. Math. Sci. Univ. Tokyo **4** (1997), no. 3, 469–517.

[6] E. N. DANCER, D. HILHORST, M. MIMURA AND L.A. PELETIER, *Spatial segregation limit of a competition-diffusion system*, European J. Appl. Math. **10** (1999), no. 2, 97–115.

[7] M. IIDA, G. KARALI, M. MIMURA, K. NAKASHIMA AND E, YANAGIDA, A free boundary problem as a singular limit of a competition-diffusion system, in preparation.

[8] M. MIMURA, Y. YAMADA AND S. YOTSUTANI, *A free boundary problem in ecology*, Japan J. Appl. Math., **2** (1985), pp. 151–186.

pp. 477–498.

[9] K. NAKASHIMA, T. WAKASA, Generation of interface in competition-diffusion system with large interaction, in preparation.

[10] K. OGAWA, Master Thesis (in Japanese), Tokyo Institute of Technology (1996).

[11] Y. TONEGAWA, *On the regularity of a chemical reaction interface*, Comm. Partial Differential Equations **23** (1998), no7-8, 1181-1207.