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On the isoperimetric inequality for mappings with remainder term

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Abstract. We prove a version of the isoperimetric inequality for mappings with remainder term. Let $S = (32\pi)^{1/3}$ and $Q(u) = \int_{\mathbf{R}^2} u \cdot u_{x_1} \wedge u_{x_2} dx$ for a mapping $u : \mathbf{R}^2 \to \mathbf{R}^3$ in a function space $\overline{\mathcal{W}}$ defined below. Then the classical isoperimetric inequality for mappings says that $S|Q(u)|^{2/3} \leq \int_{\mathbf{R}^2} |\nabla u|^2 dx$ holds for any $u \in \overline{\mathcal{W}}$. Let $\mathcal{M}$ be a manifold of functions in $\overline{\mathcal{W}}$ for which we have equality in the above isoperimetric inequality.

Following an argument by Bianchi and Egnell [2] for the case of Sobolev inequality and using a crucial estimate proved by Isobe [8], we show that for some positive constant $C > 0$,

$$\int_{\mathbf{R}^2} |\nabla u|^2 dx - S|Q(u)|^{2/3} \geq Cd(u, \mathcal{M})^2$$

holds for any $u \in \overline{\mathcal{W}}$. Here $d(u, \mathcal{M})$ denotes the distance of u from $\mathcal{M}$ in $\overline{\mathcal{W}}$.

Keywords: Isoperimetric inequality for mappings; Non-degeneracy of critical manifold; Concentration-Compactness Principle.

1. Introduction. The classical isoperimetric inequality on the Euclidean plane says that among the simple closed plane curves of length L, circles (of arbitrary center) have the largest enclosed area. Analytically, this fact is expressed as

$$L^2 \geq 4\pi A$$

where L is the length of a simple closed curve C on the plane and A is the area of the domain enclosed by C, and equality holds if and only if C is a circle. This classical inequality is one of the most famous inequalities in the field of geometric analysis and now numerous proofs have been known.

On the other hand, the following version of isoperimetric inequality, called as *Bonnesen-style isoperimetric inequality* ([11]) is probably less known than the usual one. This has the form

$$L^2 - 4\pi A \geq B,$$

where the Bonnesen term B (sometimes called the "Bonus" term) is a nonnegative quantity which reflects a geometry of the curve C and vanishes if and only if C is a circle. Thus, Bonnesen-style isoperimetric inequality always implies the classical isoperimetric inequality and, in some sense, the term B describes "the deviation of a given curve C from a circle". In an elegant expository paper [11], various forms of the term B are considered: for example, $B = (2\pi R - L)^2$ or $B = (\pi R^2 - A)^2/R^2$ is known, where R is a circumradius of a given curve C.

Now, classical isoperimetric inequality in $\mathbf{R^N}$ states that

$$N\omega_N^{1/N} Vol(E)^{(N-1)/N} \le Area(\partial E)$$

where $E \subset \mathbf{R^N}$ is a Caccioppoli set, ∂E denotes its boundary and ω_N is the volume of the unit ball in $\mathbf{R^N}$.

Note that for an appropriate regular function $u : \mathbf{R^2} \to \mathbf{R^3}$,

$$V(u) = \frac{1}{3}\int_{\mathbf{R^2}} u \cdot u_{x_1} \wedge u_{x_2} dx$$

can be considered as "volume" of the set enclosed by the graph of u and

$$A(u) = \int_{\mathbf{R^2}} |u_{x_1} \wedge u_{x_2}| dx$$

as "area" of the boundary of the set enclosed by the graph of u. In these notations, the isoperimetric inequality should be the following form:

$$3\omega_3^{1/3} V(u)^{2/3} \le A(u), \quad 3\omega_3^{1/3} = (36\pi)^{1/3}.$$

If u is a conformal map, that is, $|u_{x_1}|^2 - |u_{x_2}|^2 = u_{x_1} \cdot u_{x_2} = 0$, then $A(u)$ is exactly the usual Dirichlet integral

$$D(u) = \frac{1}{2}\int_{\mathbf{R^2}} |\nabla u|^2 dx.$$

Then by rewriting the constant, we obtain

$$(32\pi)^{1/3} \left| \int_{\mathbf{R^2}} u \cdot u_{x_1} \wedge u_{x_2} dx \right|^{2/3} \le \int_{\mathbf{R^2}} |\nabla u|^2 dx$$

for an appropriate mapping u from $\mathbf{R^2}$ to $\mathbf{R^3}$.

In this short note, we concern a sharp version of the classical isoperimetric inequality for mappings from $\mathbf{R^2}$ to $\mathbf{R^3}$, which can be considered as a Bonnesen-style isoperimetric inequality for mappings. Full details of the proofs omitted in this note can be seen in [14].

2. Result. First, let us introduce the following function spaces:

$$\begin{aligned} \mathcal{W} &:= \left\{ u \in L^1_{loc}(\mathbf{R^2};\mathbf{R^3}) : \int_{\mathbf{R^2}} |\nabla u|^2 dx + \int_{\mathbf{R^2}} \frac{|u|^2}{(1+|x|^2)^2} dx < \infty \right\}, \\ \overline{\mathcal{W}} &:= \left\{ u \in \mathcal{W} : \int_{\mathbf{R^2}} \frac{u}{(1+|x|^2)^2} dx = 0 \right\}. \end{aligned}$$

Let $\Pi : \mathbf{S^2} \to \mathbf{R^2} \cup \{\infty\}$ denote the stereographic projection from the north pole and let

$$\Pi^{-1}(x_1, x_2) = \frac{2}{1+|x|^2}\begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbf{S^2}$$

be its inverse, then the space $\mathcal{W}$ can be written as

$$\mathcal{W} = \left\{ u \in L^1_{loc}(\mathbf{R^2}; \mathbf{R^3}) : u \circ \Pi \in H^1(\mathbf{S^2}; \mathbf{R^3}) \right\}.$$

Thus, identified u as $u \circ \Pi$, $\mathcal{W}$ is exactly the function space $H^1(\mathbf{S^2}; \mathbf{R^3})$ and $\overline{\mathcal{W}}$ is a set of mappings in $H^1(\mathbf{S^2}; \mathbf{R^3})$ with null average on the sphere. Note that by Poincaré inequality, $(u, v)_{\overline{\mathcal{W}}} := \int_{\mathbf{R^2}} \nabla u \cdot \nabla v dx$ defines a scalar product on $\overline{\mathcal{W}}$. From now on, we set $\overline{u} := u - \frac{1}{\pi}\int_{\mathbf{R^2}} \frac{u}{(1+|x|^2)^2}dx \in \overline{\mathcal{W}}$ for $u \in \mathcal{W}$.

Let Q denote the oriented volume functional

$$Q(u) := \int_{\mathbf{R^2}} u \cdot u_{x_1} \wedge u_{x_2} dx$$

where $u_{x_i} = \frac{\partial}{\partial x_i} u$ and $\wedge$ denotes the vector product in $\mathbf{R^3}$.

The following inequality is referred to as the classical *isoperimetric inequality for mappings*:

$$S|Q(u)|^{2/3} \le \int_{\mathbf{R^2}} |\nabla u|^2 dx \quad \text{for } \forall u \in \overline{\mathcal{W}}. \tag{1}$$

Here

$$S = \inf_{u \in \overline{\mathcal{W}}, Q(u) \neq 0} \frac{\int_{\mathbf{R^2}} |\nabla u|^2 dx}{|Q(u)|^{2/3}} = (32\pi)^{1/3}$$

denotes the best (largest) possible value for which the classical isoperimetric inequality (1) holds true.

By simple calculation, we see the function $\overline{U_{\lambda,a}} \in \overline{\mathcal{W}}$ where

$$U_{\lambda,a}(x) = \frac{2\lambda}{\lambda^2 + |x-a|^2}\begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ -\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

attains the infimum value S for any $\lambda > 0$ and $a = (a_1, a_2) \in \mathbf{R^2}$. Furthermore, if we set the 7-dimensional manifold

$$\mathcal{M} := \left\{ c\overline{RU_{\lambda,a}} : c \in \mathbf{R} \setminus \{0\}, R \in SO(3), \lambda > 0, a \in \mathbf{R^2} \right\} \subset \overline{\mathcal{W}} \setminus \{0\}$$

where $SO(3) = \{R : 3 \times 3\text{matrix}, R^t = R^{-1}, \det(R) = 1\}$, then by a classification theorem of Brezis and Coron ([3] Lemma A.1), we see that this manifold consists of all mappings that achieve the best isoperimetric constant in (1):

$$\mathcal{M} = \left\{ u \in \overline{\mathcal{W}} \setminus \{0\} : \int_{\mathbf{R^2}} |\nabla u|^2 dx = S|Q(u)|^{2/3} \right\}.$$

Geometrically this fact is clear: the extremal functions for (1) are the conformal parameterizations of spheres of arbitrary radius and arbitrary center in $\mathbf{R}^3$.

Now, main theorem in this note is the following.

Theorem. *There exists a positive constant $C > 0$ such that*

$$\int_{\mathbf{R}^2} |\nabla u|^2 dx - S|Q(u)|^{2/3} \geq Cd(u, \mathcal{M})^2$$

holds for any $u \in \overline{\mathcal{W}}$. Here $d(u, \mathcal{M})$ denotes the distance of u from $\mathcal{M}$ in $\overline{\mathcal{W}}$;

$$d(u, \mathcal{M}) = \inf\{||u - v||_{\overline{\mathcal{W}}} : v \in \mathcal{M}\}.$$

In the proof, we follow the argument of Bianchi-Egnell [2] and Bartsch-Weth-Willem [1], in which the Sobolev inequality with remainder term was studied.

3. Key lemmas. Key ingredients in the proof of the theorem are the followings.

(1) Non-degeneracy of critical manifold (Isobe [8]).

(2) Relative compactness of the minimizing sequence for S up to translation and dilation.

On (1), we set a 6-dimensional submanifold in $\mathcal{M}$

$$\mathcal{Z} := \left\{\pm\overline{RU_{\lambda,a}} : R \in SO(3), \lambda > 0, a \in \mathbf{R}^2\right\}.$$

This is a manifold which consists of the conformal parameterizations of unit spheres with null average.

Next lemma is equivalent to the fact that $\mathcal{Z}$ is a non-degenerate critical manifold in $\overline{\mathcal{W}}$ of the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 dx + \frac{2}{3}Q(u), \quad u \in \overline{\mathcal{W}},$$

that is,

$$T_{\overline{RU_{\lambda,a}}}\mathcal{Z} = \ker D^2E(\overline{RU_{\lambda,a}})$$

holds for any $\overline{RU_{\lambda,a}} \in \mathcal{Z}$. Here $T_{\overline{RU_{\lambda,a}}}\mathcal{Z}$ is a tangent space of $\mathcal{Z}$ at $\overline{RU_{\lambda,a}}$ identified as the subspace of $\overline{\mathcal{W}}$ and is given explicitly as

$$T_{\overline{RU_{\lambda,a}}}\mathcal{Z} = \text{span}\left\{\overline{R\frac{\partial U_{\lambda,a}}{\partial a_i}}(i = 1, 2), \overline{R\frac{\partial U_{\lambda,a}}{\partial \lambda}}, \overline{R\xi_i U_{\lambda,a}}(i = 1, 2, 3), \right\},$$

where $\langle \xi_1, \xi_2, \xi_3 \rangle$ is a basis of the Lie algebra of $SO(3)$:

$$\xi_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \xi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Lemma 1.(Isobe) *There exists a constant $C_1 > 0$ such that*

$$\int_{\mathbf{R}^2} |\nabla\phi|^2 dx + 4\int_{\mathbf{R}^2} \overline{RU_{\lambda,a}} \cdot \phi_{x_1} \wedge \phi_{x_2} dx \geq C_1 \int_{\mathbf{R}^2} |\nabla\phi|^2 dx$$

holds for any $\overline{RU_{\lambda,a}} \in \overline{\mathcal{W}}$ and any $\phi \in \overline{\mathcal{W}}$ with $\phi \perp span\{\overline{RU_{\lambda,a}}\} \oplus T_{\overline{RU_{\lambda,a}}}\mathcal{Z}$.

The non-degeneracy of the critical manifold is a key step in performing a variational perturbative method first introduced by A. Ambrosetti in the study of H-system; see [5] and the reference therein. Also see [7] for an alternative proof.

On (2), Concentration-Compactness argument of P.- L.Lions ([9] [10]) applies to

$$I = \inf\{-|Q(v)| : v \in \overline{\mathcal{W}}, \int_{\mathbf{R}^2} |\nabla v|^2 dx = 1\} < 0,$$

thus we get the next lemma.

Lemma 2. *Let $(u^n) \subset \overline{\mathcal{W}}$ be any minimizing sequence for I. Then there exist $a_n \in \mathbf{R}^2$ and $\lambda_n \in \mathbf{R}_+$ such that the new minimizing sequence defined by*

$$\tilde{u}^n(\cdot) = \overline{u^n(\frac{\cdot - a_n}{\lambda_n})}$$

is relatively compact in $\overline{\mathcal{W}}$. In particular, there exists a minimizer for I in $\overline{\mathcal{W}}$.

From this lemma, we obtain the relative compactness of the minimizing sequence for S up to translation and dilation.

Note that one would apply the general Concentration-Compactness argument to the minimization problem

$$S = \inf\{\int_{\mathbf{R}^2} |\nabla u|^2 dx : u \in \overline{\mathcal{W}}, |Q(u)| = 1\}.$$

However, since $u \cdot u_{x_1} \wedge u_{x_2}$ is not absolutely integrable on $\mathbf{R}^2$ for $u \in \overline{\mathcal{W}}$, we cannot define $\mu(E) = |\int_E u \cdot u_{x_1} \wedge u_{x_2} dx|$ for $E \subset \mathbf{R}^2$ as a probability measure on $\mathbf{R}^2$.

In the proof of Lemma 2, the 2nd Concentration-Compactness Lemma (CCL II) for the best constant of the isoperimetric inequality ([13]) plays an important role.

Concentration-Compactness Lemma II for the isoperimetric inequality.

Let $(v^n) \subset \overline{\mathcal{W}}$ satisfy the followings:

- *$v^n \rightharpoonup v^0$ weakly in $\overline{\mathcal{W}}$ for some v^0,*
- *$|\nabla v^n|^2 dx \overset{*}{\rightharpoonup} \mu$ weakly in $\mathcal{M}(\mathbf{R}^2)$, where μ is a nonnegative finite Radon measure on $\mathbf{R}^2$,*
- *$T^n \to T$ in $\mathcal{D}'(\mathbf{R}^2)$ for some distribution T, where $T^n \in \mathcal{D}'(\mathbf{R}^2)$ is defined as*

$$T^n(\varphi) = \int_{\mathbf{R}^2} (\varphi v^n) \cdot v^n_{x_1} \wedge v^n_{x_2} dx, \quad \forall \varphi \in \mathcal{D}(\mathbf{R}^2).$$

Then we have T *is a finite signed measure on* $\mathbf{R}^2$ *and there exist at most countable set (possibly empty)* J*, distinct points* $\{x_j\}_{j\in J} \subset \mathbf{R}^2$*, nonnegative numbers* $\{\mu_j\}_{j\in J}$*, real numbers* $\{\nu_j\}_{j\in J}$ *such that*

(1) $\mu \geq |\nabla v^0|^2 dx + \sum_{j\in J} \mu_j \delta_{x_j}$,

(2) $T = T^0 + \sum_{j\in J} \nu_j \delta_{x_j}$ *in* $\mathcal{D}'(\mathbf{R}^2)$*, where* T^0 *is defined as* T *through* v^0,

(3) $S|\nu_j|^{2/3} \leq \mu_j$ $(j \in J)$,

(4) $S|T(\mathbf{R}^2)|^{2/3} \leq \mu(\mathbf{R}^2)$,

(5) If $v^0 \equiv 0$ *and* $\mu(\mathbf{R}^2) = S|T(\mathbf{R}^2)|^{2/3}$*, then card*$(J) = 1$ *and there exists some* x_0 *in* $\mathbf{R}^2$ *such that* $T = C\delta_{x_0}, \mu = SC^{2/3}\delta_{x_0}$ *for some* $C \in \mathbf{R} \setminus \{0\}$.

Another proof of the relative compactness of the minimizing sequence for the best isoperimetric constant (without using the general Concentration-Compactness Principle) can be seen in [6] Lemma 1.1.

4. Outline of the proof of theorem. We set

$$\Psi(u) := \frac{\int_{\mathbf{R}^2} |\nabla u|^2 dx - S|Q(u)|^{2/3}}{d(u,\mathcal{M})^2}, \quad u \in \left(\overline{\mathcal{W}} \setminus \{0\}\right) \setminus \mathcal{M}.$$

It is enough to show that Ψ is bounded from below by some positive constant independent of $u \in \overline{\mathcal{W}}$.

First, we consider the local behavior of Ψ near $\mathcal{M}$. By the non-degeneracy inequality Lemma 1, we can prove

Lemma 3. *There exists a positive constant* $C_2 > 0$ *such that*

$$\liminf_{n\to\infty} \Psi(u_n) \geq C_2$$

holds true for any sequence $(u_n) \subset \left(\overline{\mathcal{W}} \setminus \{0\}\right) \setminus \mathcal{M}$*, bounded away from* 0 *with* $d(u_n, \mathcal{M}) \to 0$.

Now, we prove the theorem by contradiction.

Proof of the main theorem. Suppose the contrary. Then there exists a sequence $(u_n) \subset \overline{\mathcal{W}} \setminus \mathcal{M}, u_n \neq 0$ satisfying $\lim_{n\to\infty} \Psi(u_n) = 0$. We may assume $||u_n||_{\overline{\mathcal{W}}} = 1$ by homogeneity of Ψ, and by subtracting a subsequence if necessary, we may further assume that $d(u_n, \mathcal{M}) \to L$ for some $L \in [0, 1 + 8\pi]$.

If $L = 0$, we have a contradiction from Lemma 3. Hence $L > 0$. In particular, we know

$$\int_{\mathbf{R}^2} |\nabla u_n|^2 dx - S|Q(u_n)|^{2/3} \to 0$$

as $n \to \infty$. That is, (u_n) is a minimizing sequence of S.

Then by the relative compactness of the minimizing sequence up to dilation and translation, which is assured by Lemma 2, we can find a sequence $(\lambda_n, a_n) \in \mathbf{R}_+ \times \mathbf{R}^2$ such that $v_n(\cdot) := u_n(\frac{\cdot - a_n}{\lambda_n})$ is a relatively compact minimizing sequence in $\overline{\mathcal{W}}$.

Thus, by choosing a suitable subsequence again, we have $v_n \to v$ in $\overline{\mathcal{W}}$ and v satisfies $\int_{\mathbf{R}^2} |\nabla v|^2 dx = S|Q(v)|^{2/3}$. Thus we have $v \in \mathcal{M}$, therefore

$$d(u_n, \mathcal{M}) = d(v_n, \mathcal{M}) \to 0,$$

which is a desired contradiction to the fact $L > 0$. □

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