<table>
<thead>
<tr>
<th>Title</th>
<th>Balanced $C_4$- Quatrefoil Designs (Theory and Applications of Combinatorial Designs with Related Field)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ushio, Kazuhiko</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1465: 88-97</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/48018">http://hdl.handle.net/2433/48018</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Balanced $C_4$-Quatrefoil Designs

近畿大学・理工学部 潮 和彌 (Kazuhiko Ushio)
Department of Informatics
Faculty of Science and Technology
Kinki University

Abstract

In graph theory, the decomposition problems of graphs are very important topics. Various types of decompositions of many graphs can be seen in the literature of graph theory.

We show that the necessary and sufficient condition for the existence of a balanced $C_4$-quatrefoil decomposition of the complete multi-graph $\lambda K_n$ is $\lambda(n - 1) \equiv 0 \pmod{32}$ and $n \geq 13$.

This decomposition is to be known as a balanced $C_4$-quatrefoil design.

Key words: Balanced $C_4$-quatrefoil decomposition, Complete multi-graph, Graph theory

MSC2000 classification: 05B30, 05C70

1. Introduction

Let $K_n$ denote the complete graph of $n$ vertices. The complete multi-graph $\lambda K_n$ is the complete graph $K_n$ in which every edge is taken $\lambda$ times. Let $C_4$ be the 4-cycle (or the cycle on 4 vertices). The $C_4$-quatrefoil is a graph of 4 edge-disjoint $C_4$'s with a common vertex and the common vertex is called the center of the $C_4$-quatrefoil.

When $\lambda K_n$ is decomposed into edge-disjoint sum of $C_4$-quatrefoils, we say that $\lambda K_n$ has a $C_4$-quatrefoil decomposition. Moreover, when every vertex of $\lambda K_n$ appears in the same number of $C_4$-quatrefoils, we say that $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition and this number is called the replication number. This balanced $C_4$-quatrefoil decomposition of $\lambda K_n$ is to be known as a balanced $C_4$-quatrefoil design.

In this paper, it is shown that the necessary and sufficient condition for the existence of such a balanced $C_4$-quatrefoil decomposition of $\lambda K_n$ is $\lambda(n - 1) \equiv 0 \pmod{32}$ and $n \geq 13$.

It is a well-known result that $K_n$ has a $C_3$ decomposition if and only if $n \equiv 1$ or $3 \pmod{6}$. This decomposition is known as a Steiner triple system. See Colbourn and Rosa[2] and Wallis[14]. Horák and Rosa[3] proved that $K_n$ has a $C_3$-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as a $C_3$-bowtie system.

For combinatorial designs, see [1,4,5,14]. Another type of foil-decompositions, see [6-13].

2. Balanced $C_4$-quatrefoil decomposition of $\lambda K_n$

We use the following notation.

Notation. We consider the vertex set $V$ of $\lambda K_n$ as $V = \{1, 2, \ldots, n\}$. The vertex operations $i + x$ in the following theorems are taken modulo $n$ with residues $1, 2, \ldots, n$. We denote a $C_4$-quatrefoil with $4$ $C_4$'s $(1, 2, 3, 4)$, $(1, 5, 6, 7)$, $(1, 8, 9, 10)$, $(1, 11, 12, 13)$ passing through $1 - 2 - 3 - 4 - 1 - 5 - 6 - 7$...
$1 - 8 - 9 - 10 - 1 - 11 - 12 - 13 - 1$ by \{(1, 2, 3, 4), (1, 5, 6, 7), (1, 8, 9, 10), (1, 11, 12, 13)\}.

We have the following theorem.

**Theorem 1.** If $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition, then $\lambda(n - 1) \equiv 0 \mod{32}$ and $n \geq 13$.

**Proof.** Suppose that $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition. Since a $C_4$-foil is a subgraph of $\lambda K_n$, $n \geq 13$. Let $b$ be the number of $C_4$-quatrefoils and $r$ be the replication number. Then $b = \lambda n(n - 1)/32$ and $r = 13\lambda(n - 1)/32$. Among $r$ $C_4$-quatrefoils having a vertex $v$ of $\lambda K_n$, let $r_1$ and $r_2$ be the numbers of $C_4$-quatrefoils in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $8r_1 + 2r_2 = \lambda(n - 1)$. From these relations, $r_1 = \lambda(n - 1)/32$ and $r_2 = 12\lambda(n - 1)/32$. Thus, $\lambda(n - 1) \equiv 0 \mod{32}$.

**Note.** The condition $\lambda(n - 1) \equiv 0 \mod{32}$ and $n \geq 13$ in Theorem 1 can be classified as follows:

(1) $\lambda \geq 1, n \equiv 1 \mod{32}, n \geq 33$
(2) $\lambda \equiv 0 \mod{2}, n \equiv 1 \mod{16}, n \geq 17$
(3) $\lambda \equiv 0 \mod{4}, n \equiv 1 \mod{8}, n \geq 17$
(4) $\lambda \equiv 0 \mod{8}, n \equiv 1 \mod{4}, n \geq 13$
(5) $\lambda \equiv 0 \mod{16}, n \equiv 1 \mod{2}, n \geq 13$
(6) $\lambda \equiv 0 \mod{32}, n \geq 13$

We have the following theorem.

**Theorem 2.** If $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition, then $(s\lambda)K_n$ has a balanced $C_4$-quatrefoil decomposition for every $s$.

**Proof.** Obvious. Repeat $s$ times the balanced $C_4$-quatrefoil decomposition of $\lambda K_n$.

**Definition.** The $C_4$-$t$-foil is a graph of $t$ edge-disjoint $C_4$'s with a common vertex and the $C_4$-$t$-foiloid is a multi-graph of $t$ $C_4$'s with a common vertex.

For example, \{(1, 2, 3, 4), (1, 5, 6, 7), (1, 8, 9, 10), (1, 11, 12, 13), (1, 14, 15, 16)\} is a $C_4$-5-foil.
\{(1, 2, 3, 4), (1, 5, 6, 7), (1, 8, 9, 10), (1, 2, 3, 5), (1, 6, 8, 10)\} is a $C_4$-5-foiloid.

So the appearance number of each vertex in the $C_4$-$t$-foil is 1. But the appearance number of each vertex in the $C_4$-$t$-foiloid is not always 1.

**Notation.** The $t$ $C_4$'s in a $C_4$-$t$-foil or a $C_4$-$t$-foiloid are denoted in order like sequence.

For example, put a $C_4$-3-foiloid $B = \{(1, 2, 3, 4), (1, 3, 5, 6), (1, 4, 6, 7)\}$. Then $(1, 2, 3, 4)$ is the first $C_4$, $(1, 3, 5, 6)$ is the second $C_4$, and $(1, 4, 6, 7)$ is the third $C_4$ of $B$.

Put a $C_4$-3-foiloid $D = \{(1, 3, 4, 5), (1, 4, 5, 6), (1, 5, 6, 7)\}$. Then we denote $B \cup B = \{(1, 2, 3, 4), (1, 3, 5, 6), (1, 4, 6, 7), (1, 2, 3, 4), (1, 3, 5, 6), (1, 4, 6, 7)\}$, and $B \cup D = \{(1, 2, 3, 4), (1, 3, 5, 6), (1, 4, 6, 7), (1, 3, 4, 5), (1, 4, 5, 6), (1, 5, 6, 7)\}$, and $D \cup B = \{(1, 3, 4, 5), (1, 4, 5, 6), (1, 5, 6, 7), (1, 2, 3, 4), (1, 3, 5, 6), (1, 4, 6, 7)\}$.

We have the following theorems.

**Theorem 3.** When $n \equiv 1 \mod{32}$ and $n \geq 33$, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition for every $\lambda$.

**Proof.** Put $n = 32t + 1$. Construct $n$ $C_4$-$t$-foils as follows:
Decompose each $C_4$-foil into $t$ $C_4$-quatrefoils. Then they comprise a balanced $C_4$-quatrefoil decomposition of $K_n$. Applying Theorem 2, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

Example 3.1. Balanced $C_4$-quatrefoil decomposition of $K_{33}$.
\[
\{(i, i+1, i+14, i+5), (i, i+2, i+16, i+6), (i, i+3, i+18, i+7), (i, i+4, i+20, i+8)\} (i = 1, 2, ..., 33).
\]

Example 3.2. Balanced $C_4$-quatrefoil decomposition of $K_{65}$.
\[
\{(i, i+1, i+9, i+12), (i, i+2, i+20, i+15), (i, i+3, i+30, i+18), (i, i+4, i+40, i+21)\}
\{(i, i+5, i+34, i+13), (i, i+6, i+36, i+14), (i, i+7, i+38, i+15), (i, i+8, i+40, i+16)\}
\{(i, i+9, i+46, i+17), (i, i+10, i+48, i+18), (i, i+11, i+50, i+19), (i, i+12, i+52, i+20)\}
\{(i, i+13, i+54, i+21), (i, i+14, i+56, i+22), (i, i+15, i+58, i+23), (i, i+16, i+60, i+24)\}
\{(i = 1, 2, ..., 65)\}.
\]

Example 3.3. Balanced $C_4$-quatrefoil decomposition of $K_{97}$.
\[
\{(i, i+1, i+38, i+13), (i, i+2, i+40, i+14), (i, i+3, i+42, i+15), (i, i+4, i+44, i+16)\}
\{(i, i+5, i+46, i+17), (i, i+6, i+48, i+18), (i, i+7, i+50, i+19), (i, i+8, i+52, i+20)\}
\{(i, i+9, i+54, i+21), (i, i+10, i+56, i+22), (i, i+11, i+58, i+23), (i, i+12, i+60, i+24)\}
\{(i = 1, 2, ..., 97)\}.
\]

Theorem 4. When $\lambda \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{16}$ and $n \geq 17$, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

Proof. We consider 2 cases.

Case 1. $n \equiv 1 \pmod{32}$ and $n \geq 33$. By Theorem 3 and Theorem 2, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

Case 2. $n \equiv 17 \pmod{32}$ and $n \geq 17$. Put $n = 32t + 17$. Construct $n C_4$-(8t + 4)-foils as follows:
\[
\{(i, i+16t+9, i+24t+14, i+24t+13), (i, i+16t+11, i+24t+18, i+24t+15), (i, i+16t+13, i+24t+22, i+24t+17), ..., (i, i+20t+9, i+32t+14, i+28t+13)\} \cup \{(i, i+20t+11, i+1, i+28t+15), (i, i+20t+13, i+5, i+28t+17), (i, i+20t+15, i+9, i+28t+19), ..., (i, i+24t+11, i+8t+1, i+32t+15)\}
\cup \{(i, i+16t+10, i+24t+16, i+2), (i, i+16t+12, i+24t+20, i+4), (i, i+16t+14, i+24t+24, i+6), ..., (i, i+20t+10, i+32t+16, i+4t+2)\} \cup \{(i, i+20t+12, i+3, i+4t+4), (i, i+20t+14, i+7, i+4t+6), (i, i+20t+16, i+11, i+4t+8), ..., (i, i+24t+12, i+8t+3, i+8t+4)\}
\{(i = 1, 2, ..., n)\}.
\]

Decompose each $C_4$-(8t + 4)-foil into $(2t + 1)$ $C_4$-quatrefoils. Then they comprise a balanced $C_4$-quatrefoil decomposition of $2K_n$. Applying Theorem 2, $\lambda K_n$ has a balanced $C_4$-trefoil decomposition.

Example 4.1. Balanced $C_4$-quatrefoil decomposition of $2K_{17}$.
\[
\{(i, i+9, i+14, i+13), (i, i+11, i+1, i+15), (i, i+10, i+16, i+2), (i, i+12, i+3, i+4)\} (i = 1, 2, ..., 17).
\]

Example 4.2. Balanced $C_4$-quatrefoil decomposition of $2K_{49}$.
\[
\{(i, i+25, i+38, i+37), (i, i+27, i+42, i+39), (i, i+29, i+46, i+41), (i, i+31, i+1, i+43)\}
\{(i, i+33, i+5, i+45), (i, i+35, i+9, i+47), (i, i+26, i+40, i+2), (i, i+28, i+44, i+4)\}
\{(i, i+30, i+48, i+6), (i, i+32, i+3, i+8), (i, i+34, i+7, i+10), (i, i+26, i+11, i+12)\}
\{(i = 1, 2, ..., 49)\}.
\]

Example 4.3. Balanced $C_4$-quatrefoil decomposition of $2K_{81}$.
\[
\{(i, i+41, i+62, i+61), (i, i+43, i+66, i+63), (i, i+45, i+70, i+65), (i, i+47, i+74, i+67)\}
\{(i, i+49, i+78, i+69), (i, i+51, i+1, i+71), (i, i+53, i+5, i+73), (i, i+55, i+9, i+75)\}
\{(i, i+57, i+13, i+77), (i, i+59, i+17, i+79), (i, i+42, i+64, i+2), (i, i+44, i+68, i+4)\}
\{(i, i+46, i+72, i+6), (i, i+48, i+76, i+8), (i, i+50, i+80, i+10), (i, i+52, i+3, i+12)\}
\]
We use the following permutation for a $C_{4}$-foil or a $C_{4}$-foiloid.

**Notation.** Put $B = \{C_{4}^{(1)}, \ldots, C_{4}^{(j)}, \ldots, C_{4}^{(i)}, \ldots, C_{4}^{(t)}\}$ be a $C_{4}$-foil or a $C_{4}$-foiloid and $\sigma = (i, j)$ be a permutation. Then $\sigma B = \{C_{4}^{(1)}, \ldots, C_{4}^{(j)}, \ldots, C_{4}^{(i)}, \ldots, C_{4}^{(t)}\}$.

**Theorem 5.** When $\lambda \equiv 0 \pmod{4}$, $n \equiv 1 \pmod{8}$ and $n \geq 17$, $\lambda K_{n}$ has a balanced $C_{4}$-quatrefoil decomposition.

**Proof.** We consider 2 cases.

**Case 1.** $n \equiv 1 \pmod{16}$ and $n \geq 17$. By Theorem 4 and Theorem 2, $\lambda K_{n}$ has a balanced $C_{4}$-quatrefoil decomposition.

**Case 2.** $n \equiv 9 \pmod{16}$ and $n \geq 25$. Put $n = 16t + 9$. Construct $n C_{4^{-}}(8t + 4)$-foiloids as follows:

$B_{t} = \{(i, i + 1 + 8t + 6, i + 8t + 5), (i, i + 2, i + 8t + 8, i + 8t + 6), (i, i + 3, i + 8t + 10, i + 8t + 7), \ldots\} \cup \{(i, i + 4t + 3, i + 1, i + 12t + 7), (i, i + 4t + 4, i + 3, i + 12t + 8), (i, i + 4t + 5, i + 5, i + 12t + 9), \ldots\} \ (i = 1, 2, \ldots, n)$.  

Put $D_{t} = \sigma B_{t}$ and $\sigma = (2, 8t + 3)$. Decompose each $D_{t}$ into $(2t + 1)$ $C_{4}$-foiloids. Then they comprise a balanced $C_{4}$-quatrefoil decomposition of $4K_{n}$. Applying Theorem 2, $\lambda K_{n}$ has a balanced $C_{4}$-quatrefoil decomposition.

**Example 5.1.** Balanced $C_{4}$-quatrefoil decomposition of $4K_{25}$.

$\{(i, i + 1 + 14, i + 13), (i, i + 1 + 14, i + 23), (i, i + 3, i + 18, i + 15), (i, i + 4, i + 20, i + 16)\}$  

$\{(i, i + 5, i + 22, i + 17), (i, i + 6, i + 24, i + 18), (i, i + 7, i + 19), (i, i + 8, i + 3, i + 20)\}$  

$\{(i, i + 9, i + 5, i + 21), (i, i + 10, i + 7, i + 22), (i, i + 2, i + 16, i + 14), (i, i + 12, i + 11, i + 24)\}$  

$(i = 1, 2, \ldots, 25)$.  

**Example 5.2.** Balanced $C_{4}$-quatrefoil decomposition of $4K_{41}$.

$\{(i, i + 1 + 22, i + 21), (i, i + 19, i + 17, i + 39), (i, i + 3, i + 26, i + 23), (i, i + 4, i + 28, i + 24)\}$  

$\{(i, i + 5, i + 25), (i, i + 6, i + 32, i + 26), (i, i + 7, i + 34, i + 27), (i, i + 8, i + 36, i + 28)\}$  

$\{(i, i + 9, i + 38, i + 29), (i, i + 10, i + 40, i + 30), (i, i + 11, i + 31), (i, i + 12, i + 3, i + 32)\}$  

$\{(i, i + 13, i + 5, i + 33, (i, i + 14, i + 17, i + 34), (i, i + 15, i + 9, i + 35), (i, i + 16, i + 11, i + 36)\}$  

$\{(i, i + 17, i + 13, i + 37), (i, i + 18, i + 15, i + 38), (i, i + 2, i + 24, i + 22), (i, i + 20, i + 19, i + 40)\}$  

$(i = 1, 2, \ldots, 41)$.  

**Example 5.3.** Balanced $C_{4}$-quatrefoil decomposition of $4K_{57}$.

$\{(i, i + 1 + 30, i + 29), (i, i + 27, i + 25, i + 55), (i, i + 3, i + 34, i + 31), (i, i + 4, i + 36, i + 32)\}$  

$\{(i, i + 5, i + 38, i + 33), (i, i + 6, i + 40, i + 34), (i, i + 7, i + 42, i + 35), (i, i + 8, i + 44, i + 36)\}$  

$\{(i, i + 9, i + 46, i + 37), (i, i + 10, i + 48, i + 38), (i, i + 11, i + 50, i + 39), (i, i + 12, i + 52, i + 40)\}$  

$\{(i, i + 13, i + 54, i + 41, (i, i + 14, i + 56, i + 42), (i, i + 15, i + 1, i + 43), (i, i + 16, i + 3, i + 44)\}$  

$\{(i, i + 17, i + 5, i + 45), (i, i + 18, i + 7, i + 46), (i, i + 19, i + 9, i + 47), (i, i + 20, i + 11, i + 48)\}$  

$\{(i, i + 21, i + 13, i + 49), (i, i + 22, i + 15, i + 50), (i, i + 23, i + 17, i + 51), (i, i + 24, i + 19, i + 52)\}$  

$\{(i, i + 25, i + 21, i + 53), (i, i + 26, i + 23, i + 54), (i, i + 2, i + 32, i + 30), (i, i + 28, i + 27, i + 56)\}$  

$(i = 1, 2, \ldots, 57)$.  

**Theorem 6.** When $\lambda \equiv 0 \pmod{8}$, $n \equiv 1 \pmod{4}$ and $n \geq 13$, $\lambda K_{n}$ has a balanced $C_{4}$-quatrefoil decomposition.

**Proof.** We consider 3 cases.
Case 1. \( n \equiv 1 \pmod{8} \) and \( n \geq 17 \). By Theorem 5 and Theorem 2, \( \lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition.

Case 2.1. \( n = 13 \). (Example 6.1. Balanced \( C_4 \)-quatrefoil decomposition of \( 8K_{13} \).) Construct a balanced \( C_4 \)-quatrefoil decomposition of \( 8K_{13} \) as follows:

\[
\{(i, i+1, i+2, i+3), (i, i+3, i+5, i+7), (i, i+6, i+8, i+10), (i, i+13, i+11, i+9)\}
\]

Applying Theorem 2, \( \lambda K_{13} \) has a balanced \( C_4 \)-quatrefoil decomposition.

Case 2.2. \( n \equiv 5 \pmod{8} \) and \( n \geq 21 \). Put \( n = 8t + 5 \). Construct \( n C_4 \)-foils as follows:

\[
B_i = \{(i, i+1, i+2t+4, i+2t+6), (i, i+3, i+4t+3, i+4t+6), \ldots, (i, i+2t+1, i+8t+4, i+6t+3)\} \cup \{(i, i+2t+2, i+1, i+6t+4), (i, i+2t+3, i+3, \ldots)\}
\]

Put \( D_i = B_i \cup B_i \). Then each \( D_i \) is a \( C_4 \)-foiloid. Put \( E_i = \sigma_1 \sigma_2 \sigma_3 \sigma_4 D_i \), where \( \sigma_1 = (2, 5) \), \( \sigma_2 = (4t, 4t+2) \), \( \sigma_3 = (4t+3, 4t+5) \). Decompose each \( E_i \) into \( (2t+1) C_4 \)-foils. Then they comprise a balanced \( C_4 \)-quatrefoil decomposition of \( 8K_n \). Applying Theorem 2, \( \lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition.

Example 6.2. Balanced \( C_4 \)-quatrefoil decomposition of \( 8K_{21} \).

\[
\{(i, i+2, i+3, i+5, i+10, i+11), (i, i+3, i+5, i+10, i+11), (i, i+4, i+18, i+14)\}
\]

Example 6.3. Balanced \( C_4 \)-quatrefoil decomposition of \( 8K_{29} \).

\[
\{(i, i+2, i+16, i+15), (i, i+5, i+24, i+19), (i, i+3, i+20, i+17), (i, i+4, i+22, i+18)\}
\]

Theorem 7. When \( \lambda \equiv 0 \pmod{16} \), \( n \equiv 1 \pmod{2} \) and \( n \geq 13 \), \( \lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition.

Proof. We consider 7 cases.

Case 1. \( n \equiv 1 \pmod{4} \) and \( n \geq 13 \). By Theorem 6 and Theorem 2, \( \lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition.

Case 2.1. \( n = 15 \). (Example 7.1. Balanced \( C_4 \)-quatrefoil decomposition of \( 16K_{15} \).) Construct a balanced \( C_4 \)-quatrefoil decomposition of \( 16K_{15} \) as follows:

\[
\{(i, i+1, i+2, i+9), (i, i+4, i+10, i+7), (i, i+13, i+11, i+6)\}
\]

\[
\{(i, i+1, i+14, i+8), (i, i+2, i+9, i+4), (i, i+10, i+7, i+3), (i, i+5, i+11, i+12)\}
\]

\[
\{(i, i+3, i+1, i+2), (i, i+5, i+8, i+12), (i, i+13, i+6, i+11), (i, i+9, i+4, i+10)\}
\]

\[
\{(i, i+13, i+14, i+14), (i, i+6, i+11, i+5), (i, i+4, i+3, i+10), (i, i+12, i+7, i+2)\}
\]

\[
\{(i, i+7, i+3, i+1), (i, i+8, i+12, i+14), (i, i+2, i+9, i+13), (i, i+10, i+6, i+4)\}
\]

\[
\{(i, i+9, i+13, i+1), (i, i+3, i+10, i+6), (i, i+14, i+8, i+5), (i, i+11, i+12, i+7)\}
\]
\{ (i, i + 1, i + 2, i + 4), (i, i + 3, i + 4, i + 7), (i, i + 1, i + 3, i + 8), (i, i + 1, i + 3, i + 7) \}

Applying Theorem 2, \( \Lambda K_{15} \) has a balanced \( C_4 \)-quatrefoil decomposition.

**Case 2.2.** \( n = 19 \). (Example 7.2. Balanced \( C_4 \)-quatrefoil decomposition of \( 16K_{19} \))

Construct a balanced \( C_4 \)-quatrefoil decomposition of \( 16K_{19} \) as follows:

\{ (i, i + 1, i + 2, i + 4), (i, i + 1, i + 3, i + 3), (i, i + 1, i + 3, i + 7) \}

Applying Theorem 2, \( \Lambda K_{19} \) has a balanced \( C_4 \)-quatrefoil decomposition.

**Case 2.3.** \( n = 23 \). (Example 7.3. Balanced \( C_4 \)-quatrefoil decomposition of \( 16K_{23} \))

Construct a balanced \( C_4 \)-quatrefoil decomposition of \( 16K_{23} \) as follows:

\{ (i, i + 1, i + 2, i + 4), (i, i + 1, i + 3, i + 3), (i, i + 1, i + 3, i + 7) \}

Applying Theorem 2, \( \Lambda K_{23} \) has a balanced \( C_4 \)-quatrefoil decomposition.

**Case 2.4.1.** \( n \equiv 3 \pmod{12} \) and \( n \geq 27 \). Put \( n = 12t + 3 \). Construct \( n \) \( C_4 \)-2t-foils \( X_i \), \( n \) \( C_4 \)-2t-foils \( Z_i \) as follows:

\{ (i, i + 1, i + 2, i + 4), (i, i + 1, i + 3, i + 3), (i, i + 1, i + 3, i + 7) \}

Applying Theorem 2, \( \Lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition.
$X_i = \{(i, i+1, i+6t+4, i+6t+3)\} \cup \{(i, i+4, i+7, i+3), (i, i+7, i+13, i+6), (i, i+10, i+19, i+9), \ldots\}$

$Y_i = \{(i, i+2, i+3, i+1), (i, i+5, i+9, i+4), (i, i+8, i+15, i+7), \ldots\}$

$Z_i = \{(i, i+3, i+5, i+2), (i, i+6, i+11, i+5), (i, i+9, i+17, i+8), \ldots\}$

$\ldots$  

Put $B_4 = X_1 \cup X_2 \cup X_3 \cup X_4 \cup Y_1 \cup Y_2 \cup Y_3 \cup Z_1 \cup Z_2 \cup Z_3$. Then each $B_i$ is a $C_4-(2t+12)$-foiloid. When $n=31$, put $D_4 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 B_4$, where $\sigma_1 = (3, 42)$, $\sigma_2 = (8, 47)$, $\sigma_3 = (13, 52)$, $\sigma_4 = (18, 57)$.  

When $n = 12t + 7$ and $n \geq 43$, put $D_4 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 B_4$, where $\sigma_1 = (3, 16t + 10)$, $\sigma_2 = (2t+4, 18t+11)$, $\sigma_3 = (4t+5, 20t+12)$, $\sigma_4 = (6t+6, 22t+13)$, $\sigma_5 = (7, 16t+14)$, $\sigma_6 = (2t+8, 18t+15)$, $\sigma_7 = (4t+9, 20t+16)$, $\sigma_8 = (6t+10, 22t+17)$.

Decompose each $D_i$ into $(6t+3)$ $C_4$-quatrefoils. Then they comprise a balanced $C_4$-quatrefoil decomposition of $16K_n$. Applying Theorem 2, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

**Case 2.4.3.** $n \equiv 11 \pmod{12}$ and $n \geq 35$. Put $n = 12t + 11$. Construct $n$ $C_4-(2t+2)$-foiloids $X_i$, $n$ $C_4-(2t+2)$-foiloids $Y_i$, and $n$ $C_4-(2t+1)$-foiloids $Z_i$ as follows:

$X_i = \{(i, i+1, i+6t+6, i+6t+5)\} \cup \{(i, i+4, i+7, i+3), (i, i+7, i+13, i+6), (i, i+10, i+19, i+9), \ldots\}$

$Y_i = \{(i, i+2, i+3, i+1), (i, i+5, i+9, i+4), (i, i+8, i+15, i+7), \ldots\}$

$Z_i = \{(i, i+3, i+5, i+2), (i, i+6, i+11, i+5), (i, i+9, i+17, i+8), \ldots\}$

Applying Theorem 2, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

**Theorem 8.** When $\lambda \equiv 0 \pmod{32}$ and $n \geq 13$, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

**Proof.** We consider 5 cases.

**Case 1.** $n \equiv 1 \pmod{2}$ and $n \geq 13$. By Theorem 7 and Theorem 2, $\lambda K_n$ has a balanced $C_4$-quatrefoil decomposition.

**Case 2.1.** $n = 14$. (Example 8.1. Balanced $C_4$-quatrefoil decomposition of $32K_{14}$.)

Construct a balanced $C_4$-quatrefoil decomposition of $32K_{14}$ as follows:

$\{(i, i+1, i+8, i+2), (i, i+4, i+7, i+6), (i, i+5, i+9, i+11), (i, i+10, i+13, i+12)\}$

$\{(i, i+1, i+8, i+2), (i, i+5, i+9, i+11), (i, i+10, i+6, i+4), (i, i+13, i+12, i+3)\}$

$\{(i, i+1, i+8, i+9), (i, i+3, i+5, i+2), (i, i+6, i+4, i+7), (i, i+10, i+13, i+12)\}$

$\{(i, i+3, i+8), \ldots\}$

Applying Theorem 2, $\lambda K_{14}$ has a balanced $C_4$-quatrefoil decomposition.

**Case 2.2.** $n = 16$. (Example 8.2. Balanced $C_4$-quatrefoil decomposition of $32K_{16}$.)

[Continued...]
Construct a balanced $C_4$-quatrefoil decomposition of $32K_{18}$ as follows:

$$\{(i, i+1, i+2, i+9), (i, i+3, i+6, i+11), (i, i+4, i+8, i+12), (i, i+5, i+10, i+13)\}$$
$$\{(i, i+1, i+2, i+9), (i, i+3, i+6, i+11), (i, i+4, i+8, i+12), (i, i+5, i+7, i+14, i+15)\}$$
$$\{(i, i+1, i+2, i+9), (i, i+3, i+6, i+11), (i, i+4, i+8, i+12), (i, i+5, i+7, i+14, i+15)\}$$
$$\{(i, i+2, i+4, i+10), (i, i+9, i+1, i+8), (i, i+12, i+7, i+3), (i, i+14, i+11, i+5)\}$$
$$\{(i, i+2, i+4, i+10), (i, i+9, i+1, i+8), (i, i+12, i+7, i+3), (i, i+15, i+13, i+6)\}$$
$$\{(i, i+2, i+4, i+10), (i, i+9, i+1, i+8), (i, i+14, i+11, i+5), (i, i+15, i+13, i+6)\}$$
$$\{(i, i+2, i+4, i+10), (i, i+9, i+1, i+8), (i, i+14, i+11, i+5), (i, i+15, i+13, i+6)\}$$

Applying Theorem 2, $\lambda K_{18}$ has a balanced $C_4$-quatrefoil decomposition.

**Case 2.3.** $n = 18$. (Example 8.3. Balanced $C_4$-quatrefoil decomposition of $32K_{18}$.)

Construct a balanced $C_4$-quatrefoil decomposition of $32K_{18}$ as follows:

$$\{(i+2, i+4, i+8, i+10), (i, i+9, i+11, i+12), (i, i+5, i+7, i+14)\}$$
$$\{(i+2, i+4, i+8, i+10), (i, i+9, i+11, i+12), (i, i+5, i+7, i+14)\}$$
$$\{(i+2, i+4, i+8, i+10), (i, i+9, i+11, i+12), (i, i+5, i+7, i+14)\}$$
$$\{(i+2, i+4, i+8, i+10), (i, i+9, i+11, i+12), (i, i+5, i+7, i+14)\}$$

**Case 2.4.** $n \equiv 0 \pmod{2}$ and $n \geq 20$. Put $n = 2t$. Construct a $C_4-(2t-1)$-foils as follows:

$$B_t = \{(i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i+8, i+9, i+10, i+11, i+12)\}$$
$$\cup \{(i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i+8, i+9, i+10, i+11, i+12)\}$$
$$\cup \{(i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i+8, i+9, i+10, i+11, i+12)\}$$
$$\cup \{(i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i+8, i+9, i+10, i+11, i+12)\}$$

Put $D_t = B_t \cup B_t \cup B_t \cup B_t$. Then each $D_t$ is a $C_4-(8t-4)$-foiloid.

When $n = 20$, put $E_t = \sigma D_t$ and $\sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4$, where $\sigma_1 = (1,5)$, $\sigma_2 = (4,76)$, $\sigma_3 = (8,9)$, $\sigma_4 = (11,13)$, $\sigma_5 = (16,19)$, $\sigma_6 = (20,21)$, $\sigma_7 = (27,30)$, $\sigma_8 = (39,41)$, $\sigma_9 = (47,50)$, $\sigma_{10} = (53,59)$, $\sigma_{11} = (64,66)$, $\sigma_{12} = (68,69)$.

When $n = 22$, first put $E_t = \sigma D_t$ and $\sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12}$, where $\sigma_1 = (1,84)$, $\sigma_2 = (2,5)$, $\sigma_3 = (8,10)$, $\sigma_4 = (11,15)$, $\sigma_5 = (16,20)$, $\sigma_6 = (23,25)$, $\sigma_7 = (31,34)$, $\sigma_8 = (51,54)$,
\( \sigma_9 = (57, 61), \sigma_{10} = (64, 65), \sigma_{11} = (72, 73), \sigma_{12} = (75, 77), \sigma_{13} = (80, 83) \). Next, put \( E''_i = \sigma_{14}E'_i \) and \( \sigma_{14} = (48, 51) \). Last, put \( E_i = \sigma_{15}E''_i \) and \( \sigma_{15} = (43, 48) \).

When \( n = 24 \), put \( E_i = \sigma D_i \) and \( \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \sigma_{13} \), where \( \sigma_1 = (1, 92), \sigma_2 = (2, 7), \sigma_3 = (11, 14), \sigma_4 = (20, 23), \sigma_5 = (24, 25), \sigma_6 = (32, 34), \sigma_7 = (35, 39), \sigma_8 = (47, 49), \sigma_9 = (56, 57), \sigma_{10} = (59, 61), \sigma_{11} = (64, 65), \sigma_{12} = (70, 73), \sigma_{13} = (79, 82) \).

When \( n = 26 \), put \( E_i = \sigma D_i \) and \( \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \sigma_{13} \), where \( \sigma_1 = (1, 100), \sigma_2 = (2, 5), \sigma_3 = (11, 14), \sigma_4 = (20, 24), \sigma_5 = (27, 29), \sigma_6 = (36, 37), \sigma_7 = (39, 41), \sigma_8 = (51, 53), \sigma_9 = (60, 62), \sigma_{10} = (63, 67), \sigma_{11} = (70, 73), \sigma_{12} = (76, 77), \sigma_{13} = (87, 90) \).

When \( n \equiv 4 \) (mod 8) and \( n \geq 28 \), put \( E_i = \sigma D_i \) and \( \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \), where \( \sigma_1 = (2, 8t - 4), \sigma_2 = (t - 2, t - 1), \sigma_3 = (t + 1, t + 3), \sigma_4 = (2t - 4, 2t - 1), \sigma_5 = (2t, 2t + 1), \sigma_6 = (3t - 3, 3t), \sigma_7 = (4t - 1, 4t + 1), \sigma_8 = (5t - 3, 5t), \sigma_9 = (6t - 8, 6t - 7), \sigma_{10} = (6t - 1, 6t + 1), \sigma_{11} = (7t - 6, 7t - 4), \sigma_{12} = (7t - 2, 7t - 1) \).

When \( n \equiv 6 \) (mod 8) and \( n \geq 30 \), put \( E_i = \sigma D_i \) and \( \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \), where \( \sigma_1 = (2, 8t - 4), \sigma_2 = (t - 3, t - 1), \sigma_3 = (t + 4), \sigma_4 = (2t - 6, 2t - 5), \sigma_5 = (2t + 1, 2t + 3), \sigma_6 = (3t - 2, 3t + 1), \sigma_7 = (4t - 1, 4t + 1), \sigma_8 = (5t - 4, 5t - 1), \sigma_9 = (6t - 6, 6t - 3), \sigma_{10} = (6t - 2, 6t - 1), \sigma_{11} = (7t - 5, 7t - 4), \sigma_{12} = (7t - 2, 7t) \).

When \( n \equiv 0 \) (mod 8) and \( n \geq 32 \), put \( E_i = \sigma D_i \) and \( \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \), where \( \sigma_1 = (2, 8t - 4), \sigma_2 = (t - 1, t + 2), \sigma_3 = (2t - 4, 2t - 1), \sigma_4 = (2t, 2t + 1), \sigma_5 = (3t - 4, 3t - 2), \sigma_6 = (3t - 3, 3t + 3), \sigma_7 = (4t - 1, 4t + 1), \sigma_8 = (5t - 4, 5t - 3), \sigma_9 = (5t - 1, 5t + 1), \sigma_{10} = (6t - 8, 6t - 7), \sigma_{11} = (6t - 1, 6t + 1), \sigma_{12} = (7t - 5, 7t - 2) \).

When \( n \equiv 2 \) (mod 8) and \( n \geq 34 \), put \( E_i = \sigma D_i \) and \( \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \), where \( \sigma_1 = (2, 8t - 4), \sigma_2 = (t - 2, t + 1), \sigma_3 = (2t - 6, 2t - 5), \sigma_4 = (2t + 1, 2t + 3), \sigma_5 = (3t - 3, 3t - 2), \sigma_6 = (3t, 3t + 2), \sigma_7 = (4t - 1, 4t + 1), \sigma_8 = (5t - 5, 5t - 3), \sigma_9 = (5t - 2, 5t + 2), \sigma_{10} = (6t - 6, 6t - 3), \sigma_{11} = (6t - 2, 6t - 1), \sigma_{12} = (7t - 4, 7t - 1) \).

Decompose each \( E_i \) into \((2t - 1)\) \( C_4 \)-quatrefoils. Then they comprise a balanced \( C_4 \)-quatrefoil decomposition of \( 32K_n \). Applying Theorem 2, \( \lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition.

Therefore, we have the following main theorem and its corollary.

**Main Theorem.** \( \lambda K_n \) has a balanced \( C_4 \)-quatrefoil decomposition if and only if \( \lambda(n - 1) \equiv 0 \) (mod 32) and \( n \geq 13 \).

**Corollary.** \( K_n \) has a balanced \( C_4 \)-quatrefoil decomposition if and only if \( n \equiv 1 \) (mod 32), \( n \geq 33 \).

**References**


