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Kyoto University
Optimal non-projective ternary linear codes

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Abstract

We prove the existence of a $[406, 6, 270]_3$ code and the nonexistence of linear codes with parameters $[458, 6, 304]_3$, $[467, 6, 310]_3$, $[471, 6, 313]_3$, $[522, 6, 347]_3$. These yield that $n_3(6, d) = g_3(6, d)$ for $268 \leq d \leq 270$, $n_3(6, d) = g_3(6, d) + 1$ for $d \in \{280 - 282, 304 - 306, 313 - 315, 347, 348\}$, $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $298 \leq d \leq 301$ and $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $310 \leq d \leq 312$, where $n_q(k, d)$ denotes the minimum length $n$ for which an $[n, k, d]_q$ code exists and $g_q(k, d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$.

1. Introduction

Let $V(n, q)$ denote the vector space of $n$-tuples over $\text{GF}(q)$, the Galois field of order $q$. A $q$-ary linear code $C$ of length $n$ and dimension $k$ is a $k$-dimensional subspace of $V(n, q)$. The Hamming distance $d(x, y)$ between two vectors $x, y \in V(n, q)$ is the number of nonzero coordinate positions in $x - y$. Now the minimum distance of a linear code $C$ is defined by $d(C) = \min\{d(x, y) \mid x, y \in C, x \neq y\}$ which is equal to the minimum weight of $C$ defined by $wt(C) = \min\{wt(x) \mid x \in C, x \neq 0\}$, where $0$ is the all-0-vector and $wt(x) = d(x, 0)$ is the weight of $x$. A $q$-ary linear code of length $n$, dimension $k$ and minimum distance $d$ is referred to as an $[n, k, d]_q$ code. The weight distribution of $C$ is the
list of numbers $A_i$ which is the number of codewords of $C$ with weight $i$. A $k \times n$ matrix having as rows the vectors of a basis of $C$ is called a generator matrix of $C$.

A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length $n$ for which an $[n, k, d]_q$ code exists ([13]). An $[n, k, d]_q$ code is called optimal if $n = n_q(k, d)$. There is a natural lower bound on $n_q(k, d)$, the so-called Griesmer bound ([8],[25]):

$$n_q(k, d) \geq g_q(k, d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\left\lceil x \right\rceil$ denotes the smallest integer greater than or equal to $x$. The values of $n_q(k, d)$ are determined for all $d$ only for some small values of $q$ and $k$. For ternary linear codes, $n_3(k, d)$ is known for $k \leq 5$ for all $d$. As for the case $k = 6$, the value of $n_3(6, d)$ is unknown for many integers $d$ ([1],[4],[5],[9],[10], [17],[20],[22]). See [2] or [24] for the updated table of $n_q(k, d)$ for some small $q$. A linear code $C$ with a generator matrix $G$ is called projective if any two columns of $G$ are independent, equivalently, if the dual code of $C$ has the minimum distance $> 2$.

We concentrate ourselves to find optimal ternary linear codes of dimension 6 with the minimum distance $d > 243$, which are necessarily non-projective. For $d \geq 244$, it is only known ([20]) that $n_3(6, d) = g_3(6, d) + 1$ for $349 \leq d \leq 351$ and that $n_3(6, d) = g_3(6, d)$ for $d \geq 352$. The existence of an $[n_1, k, d_1]_q$ code and an $[n_2, k, d_2]_q$ code trivially implies the existence of an $[n_1 + n_2, k, d_1 + d_2]_3$ code. For example, one can get a $[372, 6, 246]_3$ code from a $[56, 6, 36]_3$ code and a $[316, 6, 210]_3$ code. Similarly one can get $[g_3(6, d), 6, d]_3$ codes for $d \in \{244 - 252, 271 - 279, 322 - 330, 334 - 336\}$, $[g_3(6, d) + 1, 6, d]_3$ codes for $d \in \{253 - 270, 331 - 333, 337 - 351\}$ and $[g_3(6, d) + 2, 6, d]_3$ codes for $280 \leq d \leq 315$ from the known $n_3(6, d)$ table. We also have $[g_3(6, d), 6, d]_3$ codes for $316 \leq d \leq 321$ by Theorem 2.1 in [13] and a $[474, 6, 315]_3$ code by Theorem 4.5 in [12] from a $[158, 5, 105]_3$ code. On the other hand, the nonexistence of $[n, 5, d]_3$ codes for $(n, d) \in \{(143,94), (144,95), (145,96), (147,97), (148,98), (149,99)\}$ implies $n_3(6, d) \geq g_3(6, d) + 1$ for $280 \leq d \leq 297$, for the residual code (see [13]) of each $[g_3(6, d), 6, d]_3$ code with respect to a codeword with weight $d$ cannot exist. Hence we obtain the following.

**Theorem 1.1.**

1. $n_3(6, d) = g_3(6, d)$ for $d \in \{244 - 252, 271 - 279, 316 - 330, 334 - 336\}$ and for $d \geq 352$.
2. $n_3(6, d) = g_3(6, d) + 1$ for $349 \leq d \leq 351$.
3. $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $d \in \{253 - 270, 313 - 315, 331 - 333, 337 - 348\}$.
4. $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $280 \leq d \leq 297$.
5. $g_3(6, d) \leq n_3(6, d) \leq g_3(6, d) + 2$ for $298 \leq d \leq 312$.

We improve Theorem 1.1 for $d \in \{268 - 270, 280 - 282, 298 - 301, 304 - 306, 310 - 315, 347, 348\}$ as follows.
Theorem 1.2. (1) $n_3(6, d) = g_3(6, d)$ for $268 \leq d \leq 270$.
(2) $n_3(6, d) = g_3(6, d) + 1$ for $d \in \{280 - 282, 304 - 306, 313 - 315, 347, 348\}$.
(3) $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $298 \leq d \leq 301$.
(4) $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $310 \leq d \leq 312$.

To prove Theorem 1.2, we need to show the following theorems.

Theorem 1.3. There exist a $[406, 6, 270]_3$ code.

Theorem 1.4. There exists no $[g_3(6, d), 6, d]_3$ code for $d = 304, 310, 313, 347$.

We prove Theorem 1.4 in Section 4 and Theorems 1.3 and 1.2 in Section 5.

2. Preliminaries

We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension $r$ over GF$(q)$. A $j$-flat is a projective subspace of dimension $j$ in $\mathrm{PG}(r, q)$. 0-flats, 1-flats, 2-flats, 3-flats, $(r - 2)$-flats and $(r - 1)$-flats are called points, lines, planes, solids, secundums and hyperplanes respectively. We denote by $\mathcal{F}_j$ the set of $j$-flats of $\mathrm{PG}(r, q)$ and denote by $\theta_j$ the number of points in a $j$-flat, i.e.

$$\theta_j = (q^{j+1} - 1)/(q - 1).$$

Let $C$ be an $[n, k, d]_q$ code which does not have any coordinate position in which all the codewords have a zero entry. The columns of a generator matrix of $C$ can be considered as a multiset of $n$ points in $\Sigma = \mathrm{PG}(k - 1, q)$ denoted also by $C$. We see linear codes from this geometrical point of view. An $i$-point is a point of $\Sigma$ which has multiplicity $i$ in $C$. Denote by $\gamma_0$ the maximum multiplicity of a point from $\Sigma$ in $C$ and let $C_i$ be the set of $i$-points in $\Sigma$, $0 \leq i \leq \gamma_0$. For any subset $S$ of $\Sigma$ we define the multiplicity of $S$ with respect to $C$, denoted by $m_C(S)$, as

$$m_C(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where $|T|$ denotes the number of points in $T$ for a subset $T$ of $\Sigma$. When the code is projective, i.e. when $\gamma_0 = 1$, the multiset $C$ forms an $n$-set in $\Sigma$ and the above $m_C(S)$ is equal to $|C \cap S|$. A line $l$ with $t = m_C(l)$ is called a $t$-line. A $t$-plane, a $t$-solid and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that

$$n = m_C(\Sigma),$$

$$n - d = \max\{m_C(\pi) \mid \pi \in \mathcal{F}_{r-2}\}.$$
Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n, k, d]_q$ code in the natural manner. For an $m$-flat $\Pi$ in $\Sigma$ we define

$$\gamma_j(\Pi) = \max\{m_C(\Delta) \mid \Delta \subseteq \Pi, \Delta \in \mathcal{F}_j\}, \ 0 \leq j \leq m.$$ 

We denote simply by $\gamma_j$ instead of $\gamma_j(\Sigma)$. Clearly we have $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$.

**Lemma 2.1 ([22]).** (1) Let $\Pi$ be an $(s-1)$-flat in $\Sigma$, $2 \leq s \leq k-1$, with $m_C(\Pi) = w$. For any $(s-2)$-flat $\delta$ in $\Pi$, we have

$$m_C(\delta) \leq \gamma_{s-1} - \frac{n - w}{\theta_{k-s} - 1}.$$ 

In particular for $0 \leq j \leq k-3$,

$$\gamma_j \leq \gamma_{j+1} - \frac{n - \gamma_{j+1}}{\theta_{k-2-j} - 1}.$$ 

(2) Let $\delta_1$ and $\delta_2$ be distinct $t$-flats in a fixed $(t+1)$-flat $\Delta$ in $\Sigma$, $1 \leq t \leq k-2$. Then

$$m_C(\delta_1) + m_C(\delta_2) \geq m_C(\Delta) - (q-1)\gamma_t + q \cdot m_C(\delta_1 \cap \delta_2).$$

When $\mathcal{C}$ attains the Griesmer bound, $\gamma_j$’s are uniquely determined as follows.

**Lemma 2.2 ([19]).** Let $\mathcal{C}$ be an $[n, k, d]_q$ code attaining the Griesmer bound. Then it holds that

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \leq j \leq k-1.$$ 

By Lemma 2.2 every $[n, k, d]_q$ code attaining the Griesmer bound is projective if $d \leq q^{k-1}$. Denote by $a_i$ the number of hyperplanes $\Pi$ in $\Sigma$ with $m_C(\Pi) = i$ and by $\lambda_s$ the number of $s$-points in $\Sigma$. Note that we have $\lambda_2 = \lambda_0 + n - \theta_{k-1}$ when $\gamma_0 = 2$. The list of $a_i$’s is called the spectrum of $\mathcal{C}$. Simple counting arguments yield the following.

**Lemma 2.3.** (1) $\sum_{i=0}^{\gamma_{k-3}} a_i = \theta_{k-1}$. (2) $\sum_{i=1}^{\gamma_{k-2}} i a_i = n \theta_{k-2}$.

(3) $\sum_{i=2}^{\gamma_{k-2}} i(i-1) a_i = n(n-1) \theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} s(s-1) \lambda_s$.

**Lemma 2.4 ([22]).** Let $\Pi$ be an $i$-hyperplane through a $t$-secundum $\delta$ with $t = \gamma_{k-3}(\Pi)$. Then
(1) \[ t \leq \gamma_{k-2} - n - \frac{i}{q} = \frac{i + q\gamma_{k-2} - n}{q}. \]

(2) \[ a_i = 0 \text{ if an } [i, k-1, d_0]_q \text{ code with } d_0 \geq i - \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor \text{ does not exist, where } \left\lfloor x \right\rfloor \text{ denotes the largest integer less than or equal to } x. \]

(3) \[ t = \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor \text{ if an } [i, k-1, d_1]_q \text{ code with } d_1 \geq i - \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor + 1 \text{ does not exist.} \]

(4) Let \[ c_j \] be the number of \( j \)-hyperplanes through \( \delta \) other than \( \Pi \). Then the following equality holds:
\[
\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \quad (2.1)
\]

(5) For a \( \gamma_{k-2} \)-hyperplane \( \Pi_0 \) with spectrum \( (\tau_0, \cdots, \tau_{\gamma_3}) \), \( \tau_t > 0 \) holds if \( i + q\gamma_{k-2} - n - qt < q \).

The code obtained by deleting the same coordinate from each codeword of \( C \) is called a punctured code of \( C \). If there exists an \([n + 1, k, d + 1]_q\) code \( C' \) which gives \( C \) as a punctured code, \( C \) is called extendable (to \( C' \)) and \( C' \) is an extension of \( C \).

Let \( C \) be an \([n, k, d]_q\) code with \( k \geq 3 \), \( \gcd(q, d) = 1 \). Define
\[
\Phi_0 = \frac{1}{q-1} \sum_{q \mid i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i \equiv n \text{ (mod } q)} A_i,
\]
where the notation \( x \mid y \) means that \( x \) is a divisor of \( y \). The pair \( (\Phi_0, \Phi_1) \) is called the diversity of \( C \) ([21]).

**Theorem 2.5** ([14]). Let \( C \) be an \([n, k, d]_q\) code with diversity \( (\Phi_0, \Phi_1) \), \( \gcd(q, d) = 1 \), \( k \geq 3 \). Then \( C \) is extendable if \( \Phi_1 = 0 \).

See [23] for the extendability of ternary linear codes in detail. Note that \( a_i = A_{n-i}/(q-1) \) for \( 0 \leq i \leq \gamma_{k-2} \). Hence the above diversity is given as
\[
\Phi_0 = \sum_{i \equiv n \text{ (mod } 3)} a_i, \quad \Phi_1 = \sum_{i \not\equiv n, n-d \text{ (mod } 3)} a_i.
\]

The following is known as the Ward's divisibility theorem.

**Theorem 2.6** ([26]). Let \( C \) be an \([n, k, d]_p\) code, \( p \) a prime, attaining the Griesmer bound. If \( p^e \mid d \), then \( p^e \) is a divisor of all nonzero weights of \( C \).
3. The spectra of some ternary linear codes of dimension \( k \leq 5 \)

We supply the results about the possibilities of spectra for some ternary linear codes of dimension \( k \leq 5 \) which we need to prove Theorem 1.4 in the next section.

An \( f \)-set \( F \) in \( \text{PG}(r, q) \) satisfying

\[
 m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{r-1}\}
\]

is called an \( \{f, m; r, q\}\)-minihyper. When an \([n, k, d]_q\) code is projective (i.e. \( \gamma_0 = 1 \)), the set of 0-points \( C_0 \) forms a \( \{\theta_{k-1} - n, \theta_{k-2} - (n - d); k - 1, q\}\)-minihyper, where \( \theta_j = (q^{j+1} - 1)/(q - 1) \). The following lemma can be obtained from the classification of some minihypers by Hamada [11].

**Lemma 3.1.** (1) The spectrum of a \([80, 5, 53]_3\) code is \((a_0, a_{26}, a_{27}) = (1, 40, 80)\).
(2) The spectrum of a \([81, 5, 54]_3\) code is \((a_0, a_{27}) = (1, 120)\).
(3) The spectrum of a \([104, 5, 69]_3\) code is \((a_{26}, a_{32}, a_{35}) = (4, 13, 104)\).
(4) The spectrum of a \([107, 5, 71]_3\) code is \((a_{26}, a_{27}, a_{35}, a_{36}) = (1, 3, 39, 78)\).
(5) The spectrum of a \([108, 5, 72]_3\) code is \((a_{27}, a_{36}) = (4, 117)\).
(6) The spectrum of a \([113, 5, 75]_3\) code is \((a_{32}, a_{35}, a_{39}) = (1, 24, 96)\).
(7) The spectrum of a \([116, 5, 77]_3\) code is \((a_{35}, a_{36}, a_{38}, a_{39}) = (4, 9, 36, 72)\).
(8) The spectrum of a \([117, 5, 78]_3\) code is \((a_{36}, a_{39}) = (13, 108)\).

Since a \([\theta_{k-1} - e, k, q^{k-1} - e]_3\) code \((0 \leq e \leq 2)\) is projective, the set of 0-points \( C_0 \) consists of \( e \) points. Hence the following lemma follows.

**Lemma 3.2.** Assume \( k \geq 3 \) and put \( u = \theta_{k-2} \).
(1) The spectrum of a \([\theta_{k-1} - 2, k, q^{k-1} - 2]_3\) code is

\[
(a_{u-2}, a_{u-1}, a_u) = (\theta_{k-3}, (\theta_{k-1} - \theta_{k-3})/2, (\theta_{k-1} - \theta_{k-3})/2).
\]
(2) The spectrum of a \([\theta_{k-1} - 1, k, q^{k-1} - 1]_3\) code is \((a_{u-1}, a_u) = (\theta_{k-2}, q^{k-1})\).
(3) The spectrum of a \([\theta_{k-1}, k, q^{k-1}]_3\) code is \(a_u = \theta_{k-1}\).

The following lemma relies upon the classification of some optimal ternary linear codes of small length by van Eupen and Lisoněk [7].

**Lemma 3.3 ([7]).** (1) The spectrum of a \([8, 3, 5]_3\) code is \((a_0, a_2, a_3) = (1, 4, 8)\).
(2) The spectrum of a \([9, 3, 6]_3\) code is \((a_0, a_3) = (1, 12)\).
(3) The spectrum of a \([14, 3, 9]_3\) code is either \((a_4, a_5) = (9, 4)\), \((a_2, a_5) = (3, 10)\) or \((a_3, a_4, a_5) = (3, 3, 7)\).
(4) The spectrum of a \([18, 3, 12]_3\) code is \((a_0, a_6) = (1, 12)\) or \((a_3, a_6) = (2, 11)\).
The spectrum of a $[20, 3, 13]_3$ code satisfies $a_i = 0$ for all $i \notin \{2, 3, 4, 5, 6, 7\}$.

(6) The spectrum of a $[10, 4, 6]_3$ code is $(a_1, a_4) = (10, 30)$.

(7) The spectrum of a $[19, 4, 12]_3$ code is $(a_1, a_4, a_7) = (1, 9, 30)$.

(8) The spectrum of a $[27, 4, 35]_3$ code is $(a_0, a_9) = (1, 39)$.

(9) The spectrum of a $[36, 4, 24]_3$ code is $(a_8, a_9) = (1, 39)$.

(10) The spectrum of a $[35, 4, 23]_3$ code is $(a_8, a_9, a_{12}) = (1, 3, 12, 24)$.

(11) The spectrum of a $[36, 4, 24]_3$ code is $(a_9, a_{12}) = (4, 36)$.

Lemma 3.4. The spectrum of a $[41, 4, 27]_3$ code satisfies $a_i = 0$ for all $i \notin \{11, 12, 13, 14\}$.

Lemma 3.5. (1) The spectrum of a $[52, 4, 34]_3$ code satisfies

$$a_i = 0 \quad \text{for all } i \notin \{0, 7, 8, 9, 16, 17, 18\}.$$

(2) The spectrum of a $[53, 4, 35]_3$ code is one of the following:

(a) $(a_0, a_{17}, a_{18}) = (1, 13, 26)$,  
(b) $(a_8, a_{9}, a_{11}, a_{12}) = (1, 3, 12, 24)$,  
(c) $(a_9, a_{17}, a_{18}) = (2, 13, 25)$.

Lemma 3.6. The spectrum of a $[59, 4, 39]_3$ code satisfies $a_i = 0$ for all $i \notin \{8, 11, 14, 17, 20\}$.

Lemma 3.7. The spectrum of a $[122, 5, 81]_3$ code satisfies $a_i = 0$ for all $i \notin \{38, 39, 40, 41\}$.

The following lemma is due to Landjev [18].

Lemma 3.8 ([18]). (1) The spectrum of a $[50, 4, 33]_3$ code is one of the following:

(a) $(a_8, a_{14}, a_{17}) = (2, 4, 34)$,  
(b) $(a_{11}, a_{14}, a_{17}) = (2, 6, 32)$,  
(c) $(a_{14}, a_{17}) = (11, 30)$.

(2) Every $[49, 4, 32]_3$ code is extendable, so $a_i = 0$ for all $i \notin \{7, 8, 10, 11, 13, 14, 16, 17\}$.

Lemma 3.9. The spectrum of a $[154, 5, 102]_3$ code satisfies $a_i = 0$ for all $i \notin \{25, 46, 49, 52\}$.

Lemma 3.10. (1) The spectrum of a $[158, 5, 105]_3$ code is $(a_{26}, a_{50}, a_{55}) = (2, 13, 106)$.

(2) Every $[157, 5, 104]_3$ code is extendable.

We omit the proof of Lemmas 3.1–3.10 here.

Lemma 3.11. (1) The spectrum of a $[176, 5, 117]_3$ code is either

(a) $(a_{31}, a_{50}, a_{58}) = (1, 8, 112)$ or

(b) $(a_{41}, a_{50}, a_{58}) = (a, 11 - 2a, 110 + a)$ for some $a$ with $0 \leq a \leq 5$.

(2) Every $[175, 5, 116]_3$ code is extendable.
Proof. (1) See [20].
(2) Let $C$ be a $[175, 5, 116]_3$ code. Then $\gamma_3$-solid has no $j$-solid for $j < 8$ by Lemma 3.6, so $a_i = 0$ for all $i < 22$ by Lemma 2.1. Hence, by Lemma 2.4, we have

$$a_i = 0 \text{ for all } i \not\in \{31, 32, 40, 41, 49, 50, 58, 59\},$$

which implies that $C$ is extendable by Theorem 2.5.

4. Proof of Theorem 1.4

Theorem 4.1. There exists no $[458, 6, 304]_3$ code.

Proof. Let $C$ be a $[458, 6, 304]_3$ code. Then a $\gamma_4$-hyperplane has no $j$-solid for $j < 25$ by Lemma 3.9, so $a_i = 0$ for all $i < 71$ by Lemma 2.1. Hence $a_i = 0$ for all $i \not\in \{80, 81, 104, 107, 108, 113, 116, 117, 119 - 122, 134, 135, 136, 152, 153, 154\}$

by Lemma 2.4. Now, let $\Pi$ be a 104-hyperplane. Then the spectrum of $\Pi$ is $(\tau_{26}, \tau_{32}, \tau_{35}) = (4, 13, 104)$ by Lemma 3.1(3), which contradicts Lemma 3.9 (a $\gamma_4$-hyperplane has no $j$-solid for $j = 26, 32, 35$). Hence $a_{104} = 0$. Similarly, we get $a_{107} = a_{108} = a_{113} = a_{122} = 0$ by Lemmas 3.1(4)(5)(6), 3.7, 3.9. Hence

$$a_i = 0 \text{ for all } i \not\in \{80, 81, 116, 117, 119 - 121, 134 - 136, 152 - 154\}.$$

Next, let $\Pi_0$ be a 154-hyperplane. Since (2.1) with $i = 154$ has no solution for $t = 25$ and for $t = 49$, the spectrum of $\Pi_0$ satisfies $a_i = 0$ for all $i \not\in \{46, 52\}$ by Lemma 3.9. Let $\Delta$ be a 52-solid in $\Pi_5$. Applying Lemma 2.4 to $\Pi_0$, (2.1) with $i = 52$ has no solution for $t = 0, 7, 8, 9, 17$. Hence the spectrum of $\Delta$ satisfies $a_i = 0$ for all $i \not\in \{16, 18\}$ by Lemma 3.5(1). Let $\delta$ be a 16-plane in $\Delta$. Applying Lemma 2.4 to $\Delta$, (2.1) with $i = 16$ has no solution for $t = 0, 1, 2, 3, 5$. Hence the spectrum of $\delta$ satisfies $a_i = 0$ for all $i \not\in \{4, 6\}$. But there exists no $[16, 3, 10]_3$ code with such spectrum (see [7]), a contradiction. This completes the proof.

Theorem 4.2. There exists no $[467, 6, 310]_3$ code.

Proof. Let $C$ be a $[467, 6, 310]_3$ code. Then a $\gamma_4$-hyperplane has no $j$-solid for $j < 25$ by Lemma 3.10, so $a_i = 0$ for all $i < 71$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \not\in \{74, 80, 81, 104, 107, 108, 113, 116, 117, 119 - 122, 146, 152 - 157\}$$

by Lemma 2.4. Let $\Pi$ be a 108-hyperplane. Then the spectrum of $\Pi$ is $(\tau_{27}, \tau_{36}) = (4, 117)$ by Lemma 3.1(5), which contradicts Lemma 3.10 (a $\gamma_4$-hyperplane has no 27- nor 36-solid).
Hence \(a_{108} = 0\). Similarly, we get \(a_{81} = a_{113} = a_{116} = a_{117} = a_{120} = a_{121} = a_{122} = 0\) by Lemmas 3.1(2)(6)(7)(8), 3.2, 3.7, 3.10. Hence

\[
a_i = 0 \quad \text{for all} \quad i \notin \{74, 80, 104, 107, 146, 152 - 157\}.
\]

Suppose \(a_{80} > 0\) and let \(\Pi\) be a 80-hyperplane. Setting \((i, t) = (80, 27)\), (2.1) has no solution since \(c_{157} = 0\) (a 157-hyperplane has no 27-solid), which contradicts the spectrum of \(\Pi\) (Lemma 3.1(1)). Hence \(a_{80} = 0\). Similarly we get \(a_{104} = a_{107} = 0\) by Lemmas 2.1, 2.4, 3.1(3)(4). Hence

\[
a_i = 0 \quad \text{for all} \quad i \notin \{74, 146, 152 - 157\},
\]

Now, let \(\Pi_0\) be a 157-hyperplane with the spectrum \((\tau_{25}, \tau_{26}, \cdots, \tau_{53})\). Then \(\tau_{25} + \tau_{26} = 2\) by Lemma 3.10. Since all the solutions of (2.1) for \(i = 157\) are \((c_{74}, c_{154}, c_{157}) = (1, 1, 1)\) or \((c_{74}, c_{155}, c_{156}) = (1, 1, 1)\) for \(t = 25\); \((c_{74}, c_{157}) = (1, 2)\) for \(t = 26\), and so on, we obtain

\[
a_{74} \geq \tau_{25} + \tau_{26} = 2.
\]

On the other hand, it holds that \(a_{74} \leq 1\) by Lemma 2.1, a contradiction. This completes the proof.

**Theorem 4.3.** There exists no \([471, 6, 313]_3\) code.

**Proof.** Let \(C\) be a \([471, 6, 313]_3\) code. Then a \(\gamma_4\)-hyperplane has no \(j\)-solid for \(j < 26\) by Lemma 3.10(1), so \(a_i = 0\) for all \(i < 75\) by Lemma 2.1. Hence

\[
a_i = 0 \quad \text{for all} \quad i \notin \{81, 108, 117, 120, 121, 156 - 158\}
\]

by Lemma 2.4. Now, let \(\Pi\) be a 158-hyperplane. Then the spectrum of \(\Pi\) is \((\tau_{26}, \tau_{50}, \tau_{53}) = (2, 13, 106)\) by Lemma 3.10(1), but (2.1) has no solution for \((i, t) = (158, 50)\), a contradiction. This completes the proof.

**Theorem 4.4.** There exists no \([522, 6, 347]_3\) code.

**Proof.** Let \(C\) be a \([522, 6, 347]_3\) code. Then a \(\gamma_4\)-hyperplane has no \(j\)-solid for \(j < 31\) by Lemma 3.11, so \(a_i = 0\) for all \(i < 90\) by Lemma 2.1. Hence

\[
a_i = 0 \quad \text{for all} \quad i \notin \{90, 91, 108, 117 - 122, 162, 171, 172, 174, 175\},
\]

by Lemma 2.4. Let \(\Pi\) be a \(\gamma_4\)-hyperplane. Then (2.1) for \(i = 175\) has no solution for \(t = 49, 50\), which contradicts that the spectrum of \(\Pi\) satisfies \(\tau_{49} + \tau_{50} > 0\) by Lemma 3.11. This completes the proof.
5. Proof of Theorem 1.2

A linear code $C$ is $w$-weight if $C$ has exactly $w$ non-zero weights $i$ with $A_i > 0$. The method finding another code (called projective dual in [16]) from a given 2-weight code was first found by van Eupen and Hill [6], see also [3]. We consider the projective dual of a 3-weight code with $\gamma_0 = 2$. Recall that $\lambda_i$ stands for the number of $i$-points in $\Sigma = \text{PG}(k-1, q)$ defined from $C$. Considering $(n - d - m)$-hyperplanes, $(n - d - m)$-hyperplanes and $(n - d)$-hyperplanes of $\Sigma$ as 2-points, 1-points and 0-points respectively in the dual space $\Sigma^\star$ of $\Sigma$, we obtain the following lemma.

Lemma 5.1. Let $C$ be a 3-weight $[n, k, d]_q$ code with $q = p^h$, $p$ prime, $\gamma_0 = 2$, whose spectrum is $(a_{n-d-2m}, a_{n-d-m}, a_{n-d}) = (\alpha, \beta, \theta_{k-1} - \alpha - \beta)$, where $m = p^r$ for some $1 \leq r < h(k-2)$ satisfying $m | d$ and $\lambda_i > 0$ ($0 \leq i \leq 2$). Then there exists a 3-weight $[n^\star, k, d^\star]_q$ code $C^\star$ with $n^\star = 2\alpha + \beta$, $d^\star = 2\alpha + \beta - nt + \frac{d}{m}\theta_{k-2}$ whose spectrum is $(a_{n^\star-d^\star-2t}, a_{n^\star-d^\star-t}, a_{n^\star-d^\star}) = (\lambda_2, \lambda_1, \lambda_0)$, where $t = p^{h(k-2)-r}$.

Proof of Theorem 1.3. Let $C$ be a $[14, 6, 6]_3$ code with a generator matrix

\[
\begin{array}{cccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 \\
\end{array}
\]

Then the spectrum of $C$ is $(a_2, a_5, a_8) = (93, 220, 51)$ and we have $(\lambda_2, \lambda_1, \lambda_0) = (1, 12, 351)$. Applying Lemma 5.1 we get a $[406, 6, 270]_3$ code $C^\star$ with the spectrum $(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$.

Lemma 5.2 ([15]). Let $C_1$ and $C_2$ be $[n_1, k, d_1]_q$ and $[n_2, k-1, d_2]_q$ codes respectively and assume that $C_1$ contains a codeword of weight at least $d_1 + d_2$. Then there exists an $[n_1 + n_2, k, d_1 + d_2]_q$ code.

Applying Lemma 5.2 to a $[406, 6, 270]_3$ code as $C_1$ and $[20, 5, 12]_3, [47, 5, 30]_3, [49, 5, 31]_3, [55, 5, 36]_3$ codes as $C_2$, we get $[426, 6, 282]_3, [453, 6, 300]_3, [455, 6, 301]_3, [461, 6, 306]_3$ codes respectively. Hence Theorem 1.2 follows from Theorems 1.1 and 1.4.

References


