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<tr>
<td>Author(s)</td>
<td>Takenaka, Mito; Okamoto, Kei; Maruta, Tatsuya</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1465: 107-118</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/48020">http://hdl.handle.net/2433/48020</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Optimal non-projective ternary linear codes

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Abstract

We prove the existence of a $[406, 6, 270]_3$ code and the nonexistence of linear codes with parameters $[458, 6, 304]_3$, $[467, 6, 310]_3$, $[471, 6, 313]_3$, $[522, 6, 347]_3$. These yield that $n_3(6, d) = g_3(6, d)$ for $268 \leq d \leq 270$, $n_3(6, d) = g_3(6, d) + 1$ for $d \in \{280 - 282, 304 - 306, 313 - 315, 347, 348\}$, $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $298 \leq d \leq 301$ and $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $310 \leq d \leq 312$, where $n_q(k, d)$ denotes the minimum length $n$ for which an $[n, k, d]_q$ code exists and $g_q(k, d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$.

1. Introduction

Let $V(n, q)$ denote the vector space of $n$-tuples over $GF(q)$, the Galois field of order $q$. A $q$-ary linear code $C$ of length $n$ and dimension $k$ is a $k$-dimensional subspace of $V(n, q)$. The Hamming distance $d(x, y)$ between two vectors $x, y \in V(n, q)$ is the number of nonzero coordinate positions in $x - y$. Now the minimum distance of a linear code $C$ is defined by $d(C) = \min\{d(x, y) \mid x, y \in C, x \neq y\}$ which is equal to the minimum weight of $C$ defined by $wt(C) = \min\{wt(x) \mid x \in C, x \neq 0\}$, where $0$ is the all-0-vector and $wt(x) = d(x, 0)$ is the weight of $x$. A $q$-ary linear code of length $n$, dimension $k$ and minimum distance $d$ is referred to as an $[n, k, d]_q$ code. The weight distribution of $C$ is the
list of numbers \( A_i \) which is the number of codewords of \( C \) with weight \( i \). A \( k \times n \) matrix having as rows the vectors of a basis of \( C \) is called a generator matrix of \( C \).

A fundamental problem in coding theory is to find \( n_q(k, d) \), the minimum length \( n \) for which an \( [n, k, d]_q \) code exists ([13]). An \( [n, k, d]_q \) code is called optimal if \( n = n_q(k, d) \). There is a natural lower bound on \( n_q(k, d) \), the so-called Griesmer bound ([8],[25]):

\[
 n_q(k, d) \geq g_q(k, d) = \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the smallest integer greater than or equal to \( x \). The values of \( n_q(k, d) \) are determined for all \( d \) only for some small values of \( q \) and \( k \). For ternary linear codes, \( n_3(k, d) \) is known for \( k \leq 5 \) for all \( d \). As for the case \( k = 6 \), the value of \( n_3(6, d) \) is unknown for many integers \( d \) ([1],[4],[5],[9],[10],[17],[20],[22]). See [2] or [24] for the updated table of \( n_q(k, d) \) for some small \( q \). A linear code \( C \) with a generator matrix \( G \) is called projective if any two columns of \( G \) are independent, equivalently, if the dual code of \( C \) has the minimum distance \( > 2 \).

We concentrate ourselves to find optimal ternary linear codes of dimension 6 with the minimum distance \( d > 243 \), which are necessarily non-projective. For \( d \geq 244 \), it is only known ([20]) that \( n_3(6, d) = g_3(6, d) + 1 \) for \( 349 \leq d \leq 351 \) and that \( n_3(6, d) = g_3(6, d) \) for \( d \geq 352 \). The existence of an \( [n_1, k, d_1]_3 \) code and an \( [n_2, k, d_2]_3 \) code trivially implies the existence of an \( [n_1 + n_2, k, d_1 + d_2]_3 \) code. For example, one can get a \([372, 6, 246]_3\) code from a \([56, 6, 36]_3\) code and a \([316, 6, 210]_3\) code. Similarly one can get \([g_3(6, d), 6, d]_3\) codes for \( d \in \{244 - 252, 271 - 279, 322 - 330, 334 - 336\} \), \([g_3(6, d) + 1, 6, d]_3\) codes for \( d \in \{253 - 270, 331 - 333, 337 - 351\} \) and \([g_3(6, d) + 2, 6, d]_3\) codes for \( 280 \leq d \leq 315 \) from the known \( n_3(6, d) \) table. We also have \([g_3(6, d), 6, d]_3\) codes for \( 316 \leq d \leq 321 \) by Theorem 2.1 in [13] and a \([474, 6, 315]_3\) code by Theorem 4.5 in [12] from a \([158, 5, 105]_3\) code. On the other hand, the nonexistence of \([n, 5, d]_3\) codes for \( (n, d) \in \{(143,94), (144,95), (145,96), (147,97), (148,98), (149,99)\} \) implies \( n_3(6, d) \geq g_3(6, d) + 1 \) for \( 280 \leq d \leq 297 \), for the residual code (see [13]) of each \([g_3(6, d), 6, d]_3\) code with respect to a codeword with weight \( d \) cannot exist. Hence we obtain the following.

**Theorem 1.1.**

(1) \( n_3(6, d) = g_3(6, d) \) for \( d \in \{244 - 252, 271 - 279, 316 - 330, 334 - 336\} \) and for \( d \geq 352 \).

(2) \( n_3(6, d) = g_3(6, d) + 1 \) for \( 349 \leq d \leq 351 \).

(3) \( n_3(6, d) = g_3(6, d) \) or \( g_3(6, d) + 1 \) for \( d \in \{253 - 270, 313 - 315, 331 - 333, 337 - 348\} \).

(4) \( n_3(6, d) = g_3(6, d) + 1 \) or \( g_3(6, d) + 2 \) for \( 280 \leq d \leq 297 \).

(5) \( g_3(6, d) \leq n_3(6, d) \leq g_3(6, d) + 2 \) for \( 298 \leq d \leq 312 \).

We improve Theorem 1.1 for \( d \in \{268 - 270, 280 - 282, 298 - 301, 304 - 306, 310 - 315, 347, 348\} \) as follows.
Theorem 1.2. (1) \( n_3(6, d) = g_3(6, d) \) for \( 268 \leq d \leq 270 \).
(2) \( n_3(6, d) = g_3(6, d) + 1 \) for \( d \in \{280 - 282, 304 - 306, 313 - 315, 347, 348\} \).
(3) \( n_3(6, d) = g_3(6, d) \) or \( g_3(6, d) + 1 \) for \( 298 \leq d \leq 301 \).
(4) \( n_3(6, d) = g_3(6, d) + 1 \) or \( g_3(6, d) + 2 \) for \( 310 \leq d \leq 312 \).

To prove Theorem 1.2, we need to show the following theorems.

Theorem 1.3. There exist a \([406, 6, 270]_3\) code.

Theorem 1.4. There exists no \([g_3(6, d), 6, d]_3\) code for \( d = 304, 310, 313, 347 \).

We prove Theorem 1.4 in Section 4 and Theorems 1.3 and 1.2 in Section 5.

2. Preliminaries

We denote by \( \text{PG}(r, q) \) the projective geometry of dimension \( r \) over \( \text{GF}(q) \). A \( j \)-flat is a projective subspace of dimension \( j \) in \( \text{PG}(r, q) \). 0-flats, 1-flats, 2-flats, 3-flats, \( (r - 2) \)-flats and \( (r - 1) \)-flats are called points, lines, planes, solids, secundums and hyperplanes respectively. We denote by \( \mathcal{F}_j \) the set of \( j \)-flats of \( \text{PG}(r, q) \) and denote by \( \theta_j \) the number of points in a \( j \)-flat, i.e.

\[ \theta_j = \frac{q^{j+1} - 1}{q - 1}. \]

Let \( C \) be an \([n, k, d]_q\) code which does not have any coordinate position in which all the codewords have a zero entry. The columns of a generator matrix of \( C \) can be considered as a multiset of \( n \) points in \( \Sigma = \text{PG}(k - 1, q) \) denoted also by \( C \). We see linear codes from this geometrical point of view. An \( i \)-point is a point of \( \Sigma \) which has multiplicity \( i \) in \( C \). Denote by \( \gamma_0 \) the maximum multiplicity of a point from \( \Sigma \) in \( C \) and let \( C_i \) be the set of \( i \)-points in \( \Sigma \), \( 0 \leq i \leq \gamma_0 \). For any subset \( S \) of \( \Sigma \) we define the multiplicity of \( S \) with respect to \( C \), denoted by \( m_C(S) \), as

\[ m_C(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|, \]

where \(|T|\) denotes the number of points in \( T \) for a subset \( T \) of \( \Sigma \). When the code is projective, i.e. when \( \gamma_0 = 1 \), the multiset \( C \) forms an \( n \)-set in \( \Sigma \) and the above \( m_C(S) \) is equal to \(|C \cap S|\). A line \( l \) with \( t = m_C(l) \) is called a \( t \)-line. A \( t \)-plane, a \( t \)-solid and so on are defined similarly. Then we obtain the partition \( \Sigma = \bigcup_{i=0}^{\gamma_0} C_i \) such that

\[ n = m_C(\Sigma), \]
\[ n - d = \max\{m_C(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \]
Conversely such a partition $\Sigma = \bigcup_{i=0}^{n_0} C_i$ as above gives an $[n, k, d]_q$ code in the natural manner. For an $m$-flat $\Pi$ in $\Sigma$ we define

$$\gamma_j(\Pi) = \max\{m_C(\Delta) \mid \Delta \subset \Pi, \Delta \in \mathcal{F}_j\}, 0 \leq j \leq m.$$  

We denote simply by $\gamma_j$ instead of $\gamma_j(\Sigma)$. Clearly we have $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$.

**Lemma 2.1 ([22]).** (1) Let $\Pi$ be an $(s - 1)$-flat in $\Sigma$, $2 \leq s \leq k - 1$, with $m_C(\Pi) = w$. For any $(s - 2)$-flat $\delta$ in $\Pi$, we have

$$m_C(\delta) \leq \gamma_{s-1} - \frac{n - w}{\theta_{k-s} - 1}.$$  

In particular for $0 \leq j \leq k - 3$,

$$\gamma_j \leq \gamma_{j+1} - \frac{n - \gamma_{j+1}}{\theta_{k-2-j} - 1}.$$  

(2) Let $\delta_1$ and $\delta_2$ be distinct $t$-flats in a fixed $(t + 1)$-flat $\Delta$ in $\Sigma$, $1 \leq t \leq k - 2$. Then

$$m_C(\delta_1) + m_C(\delta_2) \geq m_C(\Delta) - (q - 1)\gamma_t + q \cdot m_C(\delta_1 \cap \delta_2).$$

When $C$ attains the Griesmer bound, $\gamma_j$'s are uniquely determined as follows.

**Lemma 2.2 ([19]).** Let $C$ be an $[n, k, d]_q$ code attaining the Griesmer bound. Then it holds that

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \leq j \leq k - 1.$$  

By Lemma 2.2 every $[n, k, d]_q$ code attaining the Griesmer bound is projective if $d \leq q^{k-1}$. Denote by $a_i$ the number of hyperplanes $\Pi$ in $\Sigma$ with $m_C(\Pi) = i$ and by $\lambda_s$ the number of $s$-points in $\Sigma$. Note that we have $\lambda_2 = \lambda_0 + n - \theta_{k-1}$ when $\gamma_0 = 2$. The list of $a_i$'s is called the spectrum of $C$. Simple counting arguments yield the following.

**Lemma 2.3.** (1) $\sum_{i=0}^{\gamma_k-3} a_i = \theta_{k-1}$.  

(2) $\sum_{i=1}^{\gamma_k-2} ia_i = n\theta_{k-2}$.  

(3) $\sum_{i=2}^{\gamma_k-2} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} s(s-1)\lambda_s$.  

**Lemma 2.4 ([22]).** Let $\Pi$ be an $i$-hyperplane through a $t$-secundum $\delta$ with $t = \gamma_{k-3}(\Pi)$. Then
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(1) \[ t \leq \gamma_{k-2} - \frac{n - i}{q} = \frac{i + q\gamma_{k-2} - n}{q}. \]

(2) \[ a_i = 0 \] if an \([i, k - 1, d_0]_q\) code with \(d_0 \geq i - \lfloor \frac{i + q\gamma_{k-2} - n}{q} \rfloor\) does not exist, where \(\lfloor x \rfloor\) denotes the largest integer less than or equal to \(x\).

(3) \[ t = \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor \] if an \([i, k - 1, d_1]_q\) code with \(d_1 \geq i - \lfloor \frac{i + q\gamma_{k-2} - n}{q} \rfloor + 1\) does not exist.

(4) Let \(c_j\) be the number of \(j\)-hyperplanes through \(\delta\) other than \(\Pi\). Then the following equality holds:

\[ \sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \] \hspace{1cm} (2.1)

(5) For a \(\gamma_{k-2}\)-hyperplane \(\Pi_0\) with spectrum \((\tau_0, \cdots, \tau_{\gamma_{k-2}})\), \(\tau_t > 0\) holds if \(i + q\gamma_{k-2} - n - qt < q\).

The code obtained by deleting the same coordinate from each codeword of \(C\) is called a punctured code of \(C\). If there exists an \([n + 1, k, d + 1]_q\) code \(C'\) which gives \(C\) as a punctured code, \(C\) is called extendable (to \(C')\) and \(C'\) is an extension of \(C\).

Let \(C\) be an \([n, k, d]_q\) code with \(k \geq 3\), \(\gcd(q, d) = 1\). Define

\[ \Phi_0 = \frac{1}{q - 1} \sum_{q|\iota, \iota \neq 0} A_{i}, \quad \Phi_1 = \frac{1}{q - 1} \sum_{i \neq 0, d \mod q} A_{i}, \]

where the notation \(x|y\) means that \(x\) is a divisor of \(y\). The pair \((\Phi_0, \Phi_1)\) is called the diversity of \(C\) (\(\text{[21]}\)).

**Theorem 2.5** (\(\text{[14]}\)). Let \(C\) be an \([n, k, d]_q\) code with diversity \((\Phi_0, \Phi_1)\), \(\gcd(q, d) = 1\), \(k \geq 3\). Then \(C\) is extendable if \(\Phi_1 = 0\).

See \(\text{[23]}\) for the extendability of ternary linear codes in detail. Note that \(a_i = A_{n-i}/(q-1)\) for \(0 \leq i \leq \gamma_{k-2}\). Hence the above diversity is given as

\[ \Phi_0 = \sum_{i \equiv n \mod 3} a_i, \quad \Phi_1 = \sum_{i \not\equiv n, n-d \mod 3} a_i. \]

The following is known as the Ward's divisibility theorem.

**Theorem 2.6** (\(\text{[26]}\)). Let \(C\) be an \([n, k, d]_p\) code, \(p\) a prime, attaining the Griesmer bound. If \(p^e|d\), then \(p^e\) is a divisor of all nonzero weights of \(C\).
3. The spectra of some ternary linear codes of dimension $k \leq 5$

We supply the results about the possibilities of spectra for some ternary linear codes of dimension $k \leq 5$ which we need to prove Theorem 1.4 in the next section.

An $f$-set $F$ in $\text{PG}(n, q)$ satisfying

$$m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{n-1}\}$$

is called an $\{f, m; n, q\}$-minihyper. When an $[n, k, d]_q$ code is projective (i.e. $\gamma_0 = 1$), the set of 0-points $C_0$ forms a $(\theta_{k-1} - n, \theta_{k-2} - (n - d); k - 1, q)$-minihyper, where $\theta_j = (q^{j+1} - 1)/(q - 1)$. The following lemma can be obtained from the classification of some minihypers by Hamada [11].

**Lemma 3.1.** (1) The spectrum of a $[80, 5, 53]_3$ code is $(a_0, a_{26}, a_{27}) = (1, 40, 80)$.
(2) The spectrum of a $[81, 5, 54]_3$ code is $(a_0, a_{27}) = (1, 120)$.
(3) The spectrum of a $[104, 5, 69]_3$ code is $(a_{26}, a_{32}, a_{36}) = (4, 13, 104)$.
(4) The spectrum of a $[107, 5, 71]_3$ code is $(a_{26}, a_{27}, a_{35}, a_{36}) = (1, 3, 39, 78)$.
(5) The spectrum of a $[108, 5, 72]_3$ code is $(a_{36}, a_{39}) = (4, 117)$.
(6) The spectrum of a $[113, 5, 75]_3$ code is $(a_{32}, a_{35}, a_{39}) = (1, 24, 96)$.
(7) The spectrum of a $[116, 5, 77]_3$ code is $(a_{35}, a_{36}, a_{39}) = (4, 9, 36, 72)$.
(8) The spectrum of a $[117, 5, 78]_3$ code is $(a_{36}, a_{39}) = (13, 108)$.

Since a $[\theta_{k-1} - e, k, q^{k-1} - e]_3$ code $(0 \leq e \leq 2)$ is projective, the set of 0-points $C_0$ consists of $e$ points. Hence the following lemma follows.

**Lemma 3.2.** Assume $k \geq 3$ and put $u = \theta_{k-2}$.
(1) The spectrum of a $[\theta_{k-1} - 2, k, q^{k-1} - 2]_3$ code is $(a_{u-2}, a_{u-1}, a_u) = (\theta_{k-3}, (\theta_{k-1} - \theta_{k-3})/2, (\theta_{k-1} - \theta_{k-3})/2)$.
(2) The spectrum of a $[\theta_{k-1} - 1, k, q^{k-1} - 1]_3$ code is $(a_{u-1}, a_u) = (\theta_{k-2}, q^{k-1})$.
(3) The spectrum of a $[\theta_{k-1}, k, q^{k-1}]_3$ code is $a_u = \theta_{k-1}$.

The following lemma relies upon the classification of some optimal ternary linear codes of small length by van Eupen and Lisoněk [7].

**Lemma 3.3 ([7]).** (1) The spectrum of a $[8, 3, 5]_3$ code is $(a_0, a_2, a_3) = (1, 4, 8)$.
(2) The spectrum of a $[9, 3, 6]_3$ code is $(a_0, a_3) = (1, 12)$.
(3) The spectrum of a $[14, 3, 9]_3$ code is either $(a_4, a_5) = (9, 4), (a_2, a_5) = (3, 10)$ or $(a_3, a_4, a_5) = (3, 3, 7)$.
(4) The spectrum of a $[18, 3, 12]_3$ code is $(a_0, a_6) = (1, 12)$ or $(a_3, a_6) = (2, 11)$. 
(5) The spectrum of a $[20, 3, 13]_3$ code satisfies $a_i = 0$ for all $i \not\in \{2, 3, 4, 5, 6, 7\}$.

(6) The spectrum of a $[10, 4, 6]_3$ code is $(a_1, a_4) = (10, 30)$.

(7) The spectrum of a $[19, 4, 12]_3$ code is $(a_1, a_4, a_7) = (1, 9, 30)$.

(8) The spectrum of a $[27, 4, 35]_3$ code is $(a_0, a_9) = (1, 39)$.

(9) The spectrum of a $[32, 4, 21]_3$ code is $(a_8, a_12) = (1, 32)$.

(10) The spectrum of a $[35, 4, 23]_3$ code is $(a_8, a_9, a_{12}) = (1, 3, 12, 24)$.

(11) The spectrum of a $[36, 4, 24]_3$ code is $(a_9, a_{12}) = (4, 36)$.

Lemma 3.4. The spectrum of a $[41, 4, 27]_3$ code satisfies $a_i = 0$ for all $i \not\in \{11, 12, 13, 14\}$.

Lemma 3.5. (1) The spectrum of a $[52, 4, 34]_3$ code satisfies

$$a_i = 0 \quad \text{for all} \ i \not\in \{0, 7, 8, 9, 16, 17, 18\}.$$ 

(2) The spectrum of a $[53, 4, 35]_3$ code is one of the following:

(a) $(a_0, a_{17}, a_{18}) = (1, 13, 26)$,  
(b) $(a_8, a_9, a_{11}, a_{12}) = (1, 3, 12, 24)$.

Lemma 3.6. The spectrum of a $[59, 4, 39]_3$ code satisfies $a_i = 0$ for all $i \not\in \{8, 11, 14, 17, 20\}$.

Lemma 3.7. The spectrum of a $[122, 5, 81]_3$ code satisfies $a_i = 0$ for all $i \not\in \{38, 39, 40, 41\}$.

The following lemma is due to Landjev [18].

Lemma 3.8 ([18]). (1) The spectrum of a $[50, 4, 33]_3$ code is one of the following:

(a) $(a_8, a_{14}, a_{17}) = (2, 4, 34)$,  
(b) $(a_{11}, a_{14}, a_{17}) = (2, 6, 32)$,  
(c) $(a_{14}, a_{17}) = (11, 30)$.

(2) Every $[49, 4, 32]_3$ code is extendable, so $a_i = 0$ for all $i \not\in \{7, 8, 10, 11, 13, 14, 16, 17\}$.

Lemma 3.9. The spectrum of a $[154, 5, 102]_3$ code satisfies $a_i = 0$ for all $i \not\in \{25, 46, 49, 52\}$.

Lemma 3.10. (1) The spectrum of a $[158, 5, 105]_3$ code is $(a_{26}, a_{50}, a_{53}) = (2, 13, 106)$.

(2) Every $[157, 5, 104]_3$ code is extendable.

We omit the proof of Lemmas 3.1–3.10 here.

Lemma 3.11. (1) The spectrum of a $[176, 5, 117]_3$ code is either

(a) $(a_{21}, a_{50}, a_{59}) = (1, 8, 112)$ or

(b) $(a_{41}, a_{50}, a_{59}) = (a, 11 - 2a, 110 + a)$ for some $a$ with $0 \leq a \leq 5$.

(2) Every $[175, 5, 116]_3$ code is extendable.
Proof. (1) See [20].
(2) Let $C$ be a $[175, 5, 116]_3$ code. Then $\gamma_3$-solid has no $j$-solid for $j < 8$ by Lemma 3.6, so $a_i = 0$ for all $i < 22$ by Lemma 2.1. Hence, by Lemma 2.4, we have
\[
a_i = 0 \text{ for all } i \notin \{31, 32, 40, 41, 49, 50, 58, 59\},
\]
which implies that $C$ is extendable by Theorem 2.5.

4. Proof of Theorem 1.4

Theorem 4.1. There exists no $[458, 6, 304]_3$ code.

Proof. Let $C$ be a $[458, 6, 304]_3$ code. Then a $\gamma_4$-hyperplane has no $j$-solid for $j < 25$ by Lemma 3.9, so $a_i = 0$ for all $i < 71$ by Lemma 2.1. Hence $a_i = 0$ for all $i \notin \{80, 81, 104, 107, 108, 113, 116, 117, 119 - 122, 134, 135, 136, 152, 153, 154\}$. by Lemma 2.4. Now, let $\Pi$ be a 104-hyperplane. Then the spectrum of $\Pi$ is $(\tau_{26}, \tau_{32}, \tau_{35}) = (4, 13, 104)$ by Lemma 3.1(3), which contradicts Lemma 3.9 (a $\gamma_4$-hyperplane has no $j$-solid for $j = 26, 32, 35$). Hence $a_{104} = 0$. Similarly, we get $a_{107} = a_{108} = a_{113} = a_{122} = 0$ by Lemmas 3.1(4)(5)(6), 3.7, 3.9. Hence
\[
a_i = 0 \text{ for all } i \notin \{80, 81, 116, 117, 119 - 121, 134 - 136, 152 - 154\}.
\]

Next, let $\Pi_0$ be a 154-hyperplane. Since (2.1) with $i = 154$ has no solution for $t = 25$ and for $t = 49$, the spectrum of $\Pi_0$ satisfies $a_i = 0$ for all $i \notin \{46, 52\}$ by Lemma 3.9. Let $\Delta$ be a 52-solid in $\Pi_0$. Applying Lemma 2.4 to $\Pi_0$, (2.1) with $i = 52$ has no solution for $t = 0, 7, 8, 9, 17$. Hence the spectrum of $\Delta$ satisfies $a_i = 0$ for all $i \notin \{16, 18\}$ by Lemma 3.5(1). Let $\delta$ be a 16-plane in $\Delta$. Applying Lemma 2.4 to $\Delta$, (2.1) with $i = 16$ has no solution for $t = 0, 1, 2, 3, 5$. Hence the spectrum of $\delta$ satisfies $a_i = 0$ for all $i \notin \{4, 6\}$. But there exists no $[16, 3, 10]_3$ code with such spectrum (see [7]), a contradiction. This completes the proof.

Theorem 4.2. There exists no $[467, 6, 310]_3$ code.

Proof. Let $C$ be a $[467, 6, 310]_3$ code. Then a $\gamma_4$-hyperplane has no $j$-solid for $j < 25$ by Lemma 3.10, so $a_i = 0$ for all $i < 71$ by Lemma 2.1. Hence
\[
a_i = 0 \text{ for all } i \notin \{74, 80, 81, 104, 107, 108, 113, 116, 117, 119 - 122, 146, 152 - 157\}
\]
by Lemma 2.4. Let $\Pi$ be a 108-hyperplane. Then the spectrum of $\Pi$ is $(\tau_{37}, \tau_{38}) = (4, 117)$ by Lemma 3.1(5), which contradicts Lemma 3.10 (a $\gamma_4$-hyperplane has no 27-nor 36-solid).
Hence $a_{108} = 0$. Similarly, we get $a_{81} = a_{113} = a_{116} = a_{117} = a_{120} = a_{121} = a_{122} = 0$
by Lemmas 3.1(2)(6)(7)(8), 3.2, 3.7, 3.10. Hence
\[ a_i = 0 \text{ for all } i \notin \{74, 80, 104, 107, 113, 116, 117, 120, 121, 122\}. \]

Suppose $a_{80} > 0$ and let $\Pi$ be a 80-hyperplane. Setting $(i, t) = (80, 27)$, (2.1) has no
solution since $c_{157} = 0$ (a 157-hyperplane has no 27-solid), which contradicts the spectrum
of $\Pi$ (Lemma 3.1(1)). Hence $a_{80} = 0$. Similarly we get $a_{104} = a_{107} = 0$ by Lemmas 2.5,
3.1(3) (4). Hence $a_i = 0$ for all $i \notin \{74, 146, 152 - 157\}$.

Now, let $\Pi_0$ be a 158-hyperplane. Then the spectrum of $\Pi$ is $(\tau_{26}, \tau_{50}, \tau_{53}) = (2, 13, 106)$
by Lemma 3.10(1), but (2.1) has no solution for $(i, t) = (158, 50)$, a contradiction. This completes
the proof.

**Theorem 4.3.** There exists no $[471, 6, 313]_3$ code.

**Proof.** Let $C$ be a $[471, 6, 313]_3$ code. Then a $\gamma_4$-hyperplane has no $j$-solid for $j < 26$ by
Lemma 3.10(1), so $a_i = 0$ for all $i < 75$ by Lemma 2.1. Hence
\[ a_i = 0 \text{ for all } i \notin \{81, 108, 117, 120, 121, 156 - 158\} \]
by Lemma 2.4. Now, let $\Pi$ be a 158-hyperplane. Then the spectrum of $\Pi$ is $(\tau_{26}, \tau_{50}, \tau_{53}) = (2, 13, 106)$
by Lemma 3.10(1), but (2.1) has no solution for $(i, t) = (158, 50)$, a contradiction. This completes the proof.

**Theorem 4.4.** There exists no $[522, 6, 347]_3$ code.

**Proof.** Let $C$ be a $[522, 6, 347]_3$ code. Then a $\gamma_4$-hyperplane has no $j$-solid for $j < 31$ by
Lemma 3.11, so $a_i = 0$ for all $i < 90$ by Lemma 2.1. Hence
\[ a_i = 0 \text{ for all } i \notin \{90, 91, 108, 117 - 122, 162, 171, 172, 174, 175\}, \]
by Lemma 2.4. Let $\Pi$ be a $\gamma_4$-hyperplane. Then (2.1) for $i = 175$ has no solution for $t = 49, 50$, which contradicts that the spectrum of $\Pi$ satisfies $\tau_{49} + \tau_{50} > 0$ by Lemma 3.11.
This completes the proof.
5. Proof of Theorem 1.2

A linear code \( C \) is \( w \)-weight if \( C \) has exactly \( w \) non-zero weights \( i \) with \( A_i > 0 \). The method finding another code (called projective dual in [16]) from a given \( 2 \)-weight code was first found by van Eupen and Hill [6], see also [3]. We consider the projective dual of a \( 3 \)-weight code with \( \gamma_0 = 2 \). Recall that \( \lambda_i \) stands for the number of \( i \)-points in \( \Sigma = \text{PG}(k-1, q) \) defined from \( C \).

Considering \( (n-d-2m) \)-hyperplanes, \( (n-d-m)- \) hyperplanes and \( (n-d) \)-hyperplanes of \( \Sigma \) as \( 2 \)-points, \( 1 \)-points and \( 0 \)-points respectively in the dual space \( \Sigma^* \) of \( \Sigma \), we obtain the following lemma.

**Lemma 5.1.** Let \( C \) be a \( 3 \)-weight \([n, k, d]_q \) code with \( q = p^h \), \( p \) prime, \( \gamma_0 = 2 \), whose spectrum is \( (a_{n-d-2m}, a_{n-d-m}, a_{n-d}) = (\alpha, \beta, \theta_{k-1} - \alpha - \beta) \), where \( m = p^r \) for some \( 1 \leq r < h(k-2) \) satisfying \( m | d \) and \( \lambda_i > 0 \) \( (0 \leq i \leq 2) \). Then there exists a \( 3 \)-weight \([n^*, k, d^*]_q \) code \( C^* \) with \( n^* = 2\alpha + \beta \), \( d^* = 2\alpha + \beta - nt + \frac{d}{m}\theta_{k-2} \) whose spectrum is \( (a_{n^*-2t}, a_{n^*-t}, a_{n^*-d}) = (\lambda_2, \lambda_1, \lambda_0) \), where \( t = p^{h(k-2)-r} \).

**Proof of Theorem 1.3.** Let \( C \) be a \([14, 6, 6]_3 \) code with a generator matrix

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

Then the spectrum of \( C \) is \( (a_2, a_5, a_8) = (93, 220, 51) \) and we have \( (\lambda_2, \lambda_1, \lambda_0) = (1, 12, 351) \). Applying Lemma 5.1 we get a \([406, 6, 270]_3 \) code \( C^* \) with the spectrum \( (a_{82}, a_{109}, a_{136}) = (1, 12, 351) \).

**Lemma 5.2** ([15]). Let \( C_1 \) and \( C_2 \) be \([n_1, k, d_1]_q \) and \([n_2, k-1, d_2]_q \) codes respectively and assume that \( C_1 \) contains a codeword of weight at least \( d_1 + d_2 \). Then there exists an \([n_1 + n_2, k, d_1 + d_2]_q \) code.

Applying Lemma 5.2 to a \([406, 6, 270]_3 \) code as \( C_1 \) and \([20, 5, 12]_3 \), \([47, 5, 30]_3 \), \([49, 5, 31]_3 \), \([55, 5, 36]_3 \) codes as \( C_2 \), we get \([426, 6, 282]_3 \), \([453, 6, 300]_3 \), \([455, 6, 301]_3 \), \([461, 6, 306]_3 \) codes respectively. Hence Theorem 1.2 follows from Theorems 1.1 and 1.4.

**References**


