<table>
<thead>
<tr>
<th>種目</th>
<th>項目</th>
</tr>
</thead>
<tbody>
<tr>
<td>項目</td>
<td>項目</td>
</tr>
<tr>
<td>項目</td>
<td>項目</td>
</tr>
<tr>
<td>項目</td>
<td>項目</td>
</tr>
<tr>
<td>項目</td>
<td>項目</td>
</tr>
<tr>
<td>項目</td>
<td>項目</td>
</tr>
</tbody>
</table>

タイトル: The transitivity of Conway's $M_13$(Theory and Applications of Combinatorial Designs with Related Field)

著者: Nakashima, Yasuhiro

引用: 数理解析研究所講究録 (2006), 1465: 126-129

発行日: 2006-01

URL: http://hdl.handle.net/2433/48022

種目: Departmental Bulletin Paper

公開許諾: 公開

発行者: KYOTO UNIVERSITY
The transitivity of Conway's $M_{13}$

東北大学大学院・情報科学研究科
中嶋 康博 (Yasuhiro Nakashima)
Graduates School of Information Sciences, Tohoku University

1 Introduction

Mathieu groups $M_{11}, M_{12}, M_{23}$ and $M_{24}$ are the only nontrivial 4-transitive permutation groups, and $M_{12}, M_{24}$ are the only nontrivial 5-transitive permutation groups. Conway introduced a set of permutations $M_{13}$ on 13 letters, which contains Mathieu group $M_{12}$, and he claims that $M_{13}$ is 6-transitive in some sense.

Martin and Sagan [2] generalized the concept of transitivity for a set of permutations. For a partition $\lambda$ of a positive integer $n$, a subset $D$ of the symmetric group $S_n$ is said to be $\lambda$-transitive if there exists $r > 0$ such that for any partitions $P, Q$ of shape $\lambda$, $\# \{\tau \in D \mid P^\tau = Q\} = r$. In particular, $D$ is $(n-t, 1, \ldots, 1)$-transitive if and only if $D$ is $t$-transitive.

Conway's $M_{13}$ is not $(7, 1, 1, 1, 1, 1, 1)$-transitive according to this definition, so Martin and Sagan raised a question to determine the full transitivity of $M_{13}$.

In this paper, we give a recursive definition of the elements of $M_{13}$, and answer the question of Martin and Sagan.

2 Construction of $M_{13}$

Let $\Omega := \{0, 1, \ldots, 11, \infty\}$ be the set of points of a projective plane $P$ of order 3, and $\Omega_{12} := \Omega - \{\infty\}$. For $a \in \Omega_{12}$ we define a permutation on $\Omega$ by $\sigma(a) := (\infty a)(bc)$ where $\{\infty, a, b, c\}$ is the line of $P$, determined by $a$, $\infty$, and set $\sigma(\emptyset) = \sigma(\infty) = \text{id}_{\Omega}$. Recursively, for an integer $k$ such that $k \geq 2$ and $a_1, a_2, \ldots, a_k \in \Omega$ such that $a_1 \neq a_2 \neq \cdots \neq a_k$, define

$$\sigma(a_1, a_2, \ldots, a_k) := \tau(\infty a_k^\tau)(b^\tau c^\tau)$$

where $\tau = \sigma(a_1, \ldots, a_{k-1})$ and $\{a_{k-1}, a_k, b, c\}$ is the line determined by $a_{k-1}, a_k$.

Note that $\sigma(a)$ is the move $a|bc$, and a triangular permutation in the sense of [1] is of the form $\sigma(a, b, \infty)$. The sets $M_{13}, M_{12}$ are defined as

$$M_{13} := \{\sigma(a_1, \ldots, a_k) \mid k \in \mathbb{N}, a_i \in \Omega, a_i \neq a_{i+1} (1 \leq i \leq k - 1)\},$$
$$M_{12} := \{\tau \in M_{13} \mid \infty^\tau = \infty\}.$$

The next proposition is useful to describe the elements of $M_{13}$. 

Proposition 1. Let $a_1, \ldots, a_k, b_1, \ldots, b_l \in \Omega$ be such that $a_1 \neq \ldots \neq a_k \neq \infty \neq b_1 \neq \ldots \neq b_l$. Then

$$\sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_l) = \sigma(b_1, \ldots, b_l) \cdot \sigma(a_1, \ldots, a_k, \infty).$$

Proof. We prove by induction on $l$. Let

$$\rho = \sigma(b_1, \ldots, b_{l-1}), \quad \pi = \sigma(a_1, \ldots, a_k, \infty),$$

so $\sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_{l-1}) = \rho \pi$ by the inductive hypothesis. Suppose that the line determined by $b_{l-1}, b_l$ is $\{b_{l-1}, b_l, c, d\}$. Then

$$\sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_l) = \rho \pi (\infty b_l^\rho) (c^\rho d^\rho) \pi (\infty^\pi b_l^\rho \pi) (c^\tau d^\tau) \pi (\infty b_l^\rho \pi) (c^\rho d^\rho) = \sigma(a_1, \ldots, a_k, \infty).$$

The following propositions are obvious.

Proposition 2. If $i$ is an integer such that $1 \leq i \leq k$ and $x \in \Omega - \{a_i\}$, then

$$\sigma(a_1, \ldots, a_k) = \sigma(a_1, \ldots, a_i, x, a_i, a_{i+1}, \ldots, a_k).$$

Proposition 3. For $a, b \in \Omega_{12}$ such that $\{a, b, \infty\}$ is contained in a line,

$$\sigma(a, \infty) = \sigma(a, b, \infty) = \text{id}_\Omega.$$

With these propositions, we prove the following theorem.

Theorem 4. $M_{12}$ is the group generated by triangular permutations, and

$$M_{13} = \coprod_{a \in \Omega} \sigma(a) M_{12}.$$

Proof. Let $\alpha = \sigma(a_1, \ldots, a_k)$. Then $\alpha \in M_{12}$ if and only if $a_k = \infty$. For $i$ in $\{1, \ldots, k-1\}$, if $a_i \neq \infty$ then we insert $\infty, a_i$ between $a_i$ and $a_{i+1}$ by Proposition 2. So by Propositions 1 and 3, $\alpha$ is written as a product of triangular permutations.

If $a_k = \infty$, then $\alpha \in M_{12}$. Otherwise, Proposition 2 implies

$$\alpha = \sigma(a_1, \ldots, a_k, \infty, a_k)$$

so $\alpha \in \sigma(a_k) M_{12}$ by Proposition 1.
3 Transitivity of $M_{13}$

An integer tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called a partition of a positive integer $n$ if $\lambda_i \geq \lambda_{i+1} \geq 0$ and $\sum_{i=1}^{k} \lambda_i = n$. A partition $P = (P_1, \ldots, P_k)$ of the set $\Omega_n := \{1, \ldots, n\}$ is said to have shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$, if $|P_i| = \lambda_i$.

Definition 5. Let $n$ be an integer and $D$ be a set of permutations on $\Omega_n$. For a partition $\lambda$ of $n$, we say that $D$ is $\lambda$-transitive if there exists $r > 0$ such that for any partitions $P, Q$ of shape $\lambda$, $\|\{\tau \in D \mid P^\tau = Q\} = r$.

For example, a permutation group $G$ is $t$-transitive on $\Omega_n$ if and only if $G$ is $(n-t, 1^t)$-transitive, where $1^t$ means $\underbrace{1, \ldots, 1}_{t}$.

We first prove the following general result.

Lemma 6. For each $i \in \Omega_{n+1} = \{1, \ldots, n+1\}$, let $a_i$ be a permutation on $\Omega_{n+1}$ such that $i^{a_i} = n+1$, and $G$ be a permutation group on $\Omega_n = \{1, \ldots, n\}$. If $G$ is $(n-t, 1^t)$-transitive on $\Omega_n$ and $D = \cup a_i G$, then $D$ is a $(n-t+1, 1^t)$-transitive set on $\Omega_{n+1}$.

Proof. For $t$-tuples $X = (x_1, \ldots, x_t), Y = (y_1, \ldots, y_t)$ of distinct elements of $\Omega_{n+1}$, we define

$D_Y^X := \{\tau \in D \mid X^\tau = Y\}$.

First, we suppose that $Y$ contains $n+1$, for example, $y_1 = n+1$. Then $D_Y^X \subset a_{y_1} G$, and $\{x_k^{a_{y_1}} \mid 2 \leq k \leq t\}, \{y_k \mid 2 \leq k \leq t\} \subset \Omega_n$. By the $(n-(t-1), 1^{t-1})$-transitivity of $G$ on $\Omega_n$,

$|D_Y^X| = \#\{g \in G \mid (x_k^{a_{y_1}})^g = y_k \ (2 \leq k \leq t)\} = |G|/n \cdot (n-1) \cdots (n-(t-2))$.

Next, we assume that $n+1$ does not appear in $Y$. For an integer $i$ such that $1 \leq i \leq t$, if $a_i G \cap D_Y^X \subset a_i G$ then $i \notin X$ and $\{x_1^{a_i}, \ldots, x_t^{a_i}\}, Y \subset \Omega_n$, so by the $(n-t, 1^t)$-transitivity of $G$ on $\Omega_n$,

$|D_Y^X| = \sum_{i \in \Omega_{n+1} - X} \#\{g \in G \mid (X^{a_i})^g = Y\}

= |\Omega_{n+1} - X| \cdot |G|/n \cdot (n-1) \cdots (n-(t-1)) = \frac{|G|}{n \cdot (n-1) \cdots (n-(t-2))}$.

$\square$
By this lemma, we see that $M_{13}$ is $(8, 1^5)$-transitive. We will show that $M_{13}$ is not $(7, 6)$-transitive.

If $M_{13}$ is $(7, 6)$-transitive, then for any 6-element sets $P, Q$ of $\Omega_{13}$,

$$\sharp\{\tau \in M_{13} \mid P^\tau = Q\} = \frac{|G|}{\binom{13}{6}} = 720.$$ 

It is known that $M_{12}$ leaves the set of hexads invariant (see [1] for details). We define $H := \{h^{\sigma(a)} \mid a \in \Omega, \ h: \text{hexad}\}$. If $h = \{1, 2, 3, 4, 5, 6\}$ then $h^{\sigma(7)} = h^{\sigma(8)}$ and

$$|H| < |\Omega| \cdot 132 = \binom{13}{6},$$

so there is a 6-element set $P$ that is not contained in $H$. Taking $Q$ as a hexad, we obtain

$$\sharp\{\tau \in M_{13} \mid P^\tau = Q\} = 0.$$ 

Therefore $M_{13}$ is not $(7, 6)$-transitive.

We need to introduce the dominance order on partitions of $n$, in order to state a result of Martin and Sagan [2]. For two integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_l)$, we define

$$\lambda \triangleleft \mu \iff \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i,$$

for any positive integer $j$, where $\lambda_i = 0$ for $i \geq k$ and $\mu_i = 0$ for $i \geq l$.

**Theorem 7 (Martin and Sagan [2]).** If a set $D$ is $\lambda$-transitive and $\lambda \triangleleft \mu$, then $D$ is $\mu$-transitive.

Using this theorem, we can determine the transitivity of $M_{13}$.

**Theorem 8.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of 13. Then $M_{13}$ is $\lambda$-transitive if and only if $\lambda_1 \geq 8$.

**Proof.** If $\lambda_1 \geq 8$ then $\lambda \geq (8, 1^5)$. Since $M_{13}$ is $(8, 1^5)$-transitive, $M_{13}$ is also $\lambda$-transitive by Theorem 7. Suppose $M_{13}$ is $\lambda$-transitive for some $\lambda$ with $\lambda_1 \leq 7$. Then $M_{13}$ is $(7, 6)$-transitive by Theorem 7 again since $\lambda \leq (7, 6)$. This is a contradiction. 

**References**
