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The transitivity of Conway's $M_13$(Theory and Applications of Combinatorial Designs with Related Field)

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The transitivity of Conway’s $M_{13}$

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1 Introduction

Mathieu groups $M_{11}, M_{12}, M_{23}$ and $M_{24}$ are the only nontrivial 4-transitive permutation groups, and $M_{12}, M_{24}$ are the only nontrivial 5-transitive permutation groups. Conway introduced a set of permutations $M_{13}$ on 13 letters, which contains Mathieu group $M_{12}$, and he claims that $M_{13}$ is 6-transitive in some sense.

Martin and Sagan [2] generalized the concept of transitivity for a set of permutations. For a partition $\lambda$ of a positive integer $n$, a subset $D$ of the symmetric group $S_n$ is said to be $\lambda$-transitive if there exists $r > 0$ such that for any partitions $P, Q$ of shape $\lambda$, $\#\{\tau \in D \mid P^\tau = Q\} = r$. In particular, $D$ is $(n-t, 1, \ldots, 1)$-transitive if and only if $D$ is $t$-transitive.

Conway's $M_{13}$ is not $(7, 1, 1, 1, 1, 1, 1)$-transitive according to this definition, so Martin and Sagan raised a question to determine the full transitivity of $M_{13}$.

In this paper, we give a recursive definition of the elements of $M_{13}$, and answer the question of Martin and Sagan.

2 Construction of $M_{13}$

Let $\Omega := \{0, 1, \ldots, 11, \infty\}$ be the set of points of a projective plane $P$ of order 3, and $\Omega_{12} := \Omega - \{\infty\}$. For $a \in \Omega_{12}$ we define a permutation on $\Omega$ by $\sigma(a) := (\infty a)(b c)$ where $\{\infty, a, b, c\}$ is the line of $P$, determined by $a, \infty$, and set $\sigma(\emptyset) = \sigma(\infty) = \text{id}_\Omega$. Recursively, for an integer $k$ such that $k \geq 2$ and $a_1, a_2, \ldots, a_k \in \Omega$ such that $a_1 \neq a_2 \neq \cdots \neq a_k$, define

$$
\sigma(a_1, a_2, \ldots, a_k) := \tau(\infty a_k^\tau)(b^\tau c^\tau)
$$

where $\tau = \sigma(a_1, \ldots, a_{k-1})$ and $\{a_{k-1}, a_k, b, c\}$ is the line determined by $a_{k-1}, a_k$. Note that $\sigma(a)$ is the move $a|bc$, and a triangular permutation in the sense of [1] is of the form $\sigma(a, b, \infty)$. The sets $M_{13}, M_{12}$ are defined as

$$
M_{13} := \{\sigma(a_1, \ldots, a_k) \mid k \in \mathbb{N}, a_i \in \Omega, a_i \neq a_{i+1} \ (1 \leq i \leq k-1)\},
$$

$$
M_{12} := \{\tau \in M_{13} \mid \infty^\tau = \infty\}.
$$

The next proposition is useful to describe the elements of $M_{13}$. 
Proposition 1. Let \( a_1, \ldots, a_k, b_1, \ldots, b_l \in \Omega \) be such that \( a_1 \neq \ldots \neq a_k \neq b_1 \neq \ldots \neq b_l \). Then
\[
\sigma(a_1, \ldots, a_k, b_1, \ldots, b_l) = \sigma(b_1, \ldots, b_l) \cdot \sigma(a_1, \ldots, a_k, \infty).
\]

Proof. We prove by induction on \( l \). Let
\[
\rho = \sigma(b_1, \ldots, b_{l-1}),
\]
\[
\pi = \sigma(a_1, \ldots, a_k, \infty),
\]
so \( \sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_{l-1}) = \rho \pi \) by the inductive hypothesis. Suppose that the line determined by \( b_{l-1}, b_l \) is \( \{b_{l-1}, b_l, c, d\} \). Then
\[
\sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_l) = \rho \pi (\infty b_l^\rho \pi) (c^\rho \pi \infty) = \pi (\infty b_l^\rho \pi) (c^\rho \pi \infty) = \pi (\infty b_l^\rho \pi) (c^\rho \pi \infty) = \pi (c^\rho \pi \infty) = \pi (c^\rho \pi \infty).
\]

The following propositions are obvious.

Proposition 2. If \( i \) is an integer such that \( 1 \leq i \leq k \) and \( x \in \Omega - \{a_i\} \), then
\[
\sigma(a_1, \ldots, a_k) = \sigma(a_1, \ldots, a_i, x, a_i, a_{i+1}, \ldots, a_k).
\]

Proposition 3. For \( a, b \in \Omega_{12} \) such that \( \{a, b, \infty\} \) is contained in a line,
\[
\sigma(a, \infty) = \sigma(a, b, \infty) = \text{id}_\Omega.
\]

With these propositions, we prove the following theorem.

Theorem 4. \( M_{12} \) is the group generated by triangular permutations, and
\[
M_{13} = \bigotimes_{a \in \Omega} \sigma(a)M_{12}.
\]

Proof. Let \( \alpha = \sigma(a_1, \ldots, a_k) \). Then \( \alpha \in M_{12} \) if and only if \( a_k = \infty \). For \( i \) in \( \{1, \ldots, k-1\} \), if \( a_i \neq \infty \) then we insert \( \infty, a_i \) between \( a_i \) and \( a_{i+1} \) by Proposition 2. So by Propositions 1 and 3, \( \alpha \) is written as a product of triangular permutations.

If \( a_k = \infty \), then \( \alpha \in M_{12} \). Otherwise, Proposition 2 implies
\[
\alpha = \sigma(a_1, \ldots, a_k, \infty, a_k)
\]
so \( \alpha \in \sigma(a_k)M_{12} \) by Proposition 1.
3 Transitivity of $M_{13}$

An integer tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called a partition of a positive integer $n$ if $\lambda_i \geq \lambda_{i+1} \geq 0$ and $\sum_{i=1}^{k} \lambda_i = n$. A partition $P = (P_1, \ldots, P_k)$ of the set $\Omega_n := \{1, \ldots, n\}$ is said to have shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$, if $|P_i| = \lambda_i$.

**Definition 5.** Let $n$ be an integer and $D$ be a set of permutations on $\Omega_n$. For a partition $\lambda$ of $n$, we say that $D$ is $\lambda$-transitive if there exists $r > 0$ such that for any partitions $P, Q$ of shape $\lambda$, $|\{\tau \in D \mid P^\tau = Q\}| = r$.

For example, a permutation group $G$ is $t$-transitive on $\Omega_n$ if and only if $G$ is $(n-t, 1^t)$-transitive, where $1^t$ means $1, \ldots, 1$. We first prove the following general result.

**Lemma 6.** For each $i \in \Omega_{n+1} = \{1, \ldots, n+1\}$, let $a_i$ be a permutation on $\Omega_{n+1}$ such that $i^{a_i} = n+1$, and $G$ be a permutation group on $\Omega_n = \{1, \ldots, n\}$. If $G$ is $(n-t, 1^t)$-transitive on $\Omega_n$ and $D = \bigcup a_iG \in \Omega_{n+1}$, then $D$ is a $(n-t+1, 1^t)$-transitive set on $\Omega_{n+1}$.

**Proof.** For $t$-tuples $X = (x_1, \ldots, x_t), Y = (y_1, \ldots, y_t)$ of distinct elements of $\Omega_{n+1}$, we define

$$D^X_Y := \{\tau \in D \mid X^\tau = Y\}.$$  

First, we suppose that $Y$ contains $n+1$, for example, $y_1 = n+1$. Then $D^X_Y \subset a_1G$, and $\{x_1^{a_1} \mid 2 \leq k \leq t\}, \{y_k \mid 2 \leq k \leq t\} \subset \Omega_n$. By the $(n-(t-1), 1^{t-1})$-transitivity of $G$ on $\Omega_n$,

$$|D^X_Y| = \#\{g \in G \mid (x_k^{a_1})^g = y_k \ (2 \leq k \leq t)\} \leq \frac{|G|}{n \cdot (n-1) \cdots (n-(t-2))}.$$  

Next, we assume that $n+1$ does not appear in $Y$. For an integer $i$ such that $1 \leq i \leq t$, if $a_iG \cap D^X_Y$ then $i \notin X$ and $\{x_i^{a_i}, \ldots, x_t^{a_i}\}, Y \subset \Omega_n$, so by the $(n-t, 1^t)$-transitivity of $G$ on $\Omega_n$,

$$|D^X_Y| = \sum_{i \in \Omega_{n+1} - X} \#\{g \in G \mid (X^{a_i})^g = Y\} \leq \frac{|G|}{n \cdot (n-1) \cdots (n-(t-1))}.$$  

$\square$
By this lemma, we see that $M_{13}$ is $(8,1^5)$-transitive. We will show that $M_{13}$ is not $(7,6)$-transitive.

If $M_{13}$ is $(7,6)$-transitive, then for any 6-element sets $P, Q$ of $\Omega_{13}$,
\[
\| \{ \tau \in M_{13} \mid P^\tau = Q \} = \frac{|G|}{\binom{13}{6}} = 720.
\]
It is known that $M_{12}$ leaves the set of hexads invariant (see [1] for details). We define $H := \{ h^\sigma(a) \mid a \in \Omega, \ h : \text{hexad} \}$. If $h = \{1,2,3,4,5,6\}$ then $h^\sigma(7) = h^\sigma(8)$ and
\[
|H| < |\Omega| \cdot 132 = \binom{13}{6},
\]
so there is a 6-element set $P$ that is not contained in $H$. Taking $Q$ as a hexad, we obtain
\[
\| \{ \tau \in M_{13} \mid P^\tau = Q \} = 0.
\]
Therefore $M_{13}$ is not $(7,6)$-transitive.

We need to introduce the dominance order on partitions of $n$, in order to state a result of Martin and Sagan [2]. For two integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_l)$, we define
\[
\lambda \triangleleft \mu \iff \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i, \text{ for any positive integer } j
\]
where $\lambda_i = 0$ for $i \geq k$ and $\mu_i = 0$ for $i \geq l$.

**Theorem 7 (Martin and Sagan [2]).** If a set $D$ is $\lambda$-transitive and $\lambda \triangleleft \mu$, then $D$ is $\mu$-transitive.

Using this theorem, we can determine the transitivity of $M_{13}$.

**Theorem 8.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of 13. Then $M_{13}$ is $\lambda$-transitive if and only if $\lambda_1 \geq 8$.

**Proof.** If $\lambda_1 \geq 8$ then $\lambda \geq (8,1^5)$. Since $M_{13}$ is $(8,1^5)$-transitive, $M_{13}$ is also $\lambda$-transitive by Theorem 7. Suppose $M_{13}$ is $\lambda$-transitive for some $\lambda$ with $\lambda_1 \leq 7$. Then $M_{13}$ is $(7,6)$-transitive by Theorem 7 again since $\lambda \triangleleft (7,6)$. This is a contradiction. \( \square \)

**References**
