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Sparseness of triple systems: a survey

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Abstract

In 1976, Paul Erdős conjectured that there is an integer $v_0(r)$ such that for every $v > v_0(r)$ and $v \equiv 1,3 \pmod 6$, there exists a Steiner triple system of order v containing no i blocks on $i+2$ points for every $1 < i \leq r$. Such an STS is said to be r-sparse. This article surveys recent developments on the existence of r-sparse trile systems and related designs.

1 Introduction

A *Steiner triple system* S of *order* v, briefly $\mathrm{STS}(v)$, is an ordered pair $(V,\mathcal{B})$, where V is a finite set of v elements called *points*, and $\mathcal{B}$ is a set of 3-element subsets of V called *blocks*, such that each unordered pair of distinct elements of V is contained in exactly one block of $\mathcal{B}$. It is well-known that an $\mathrm{STS}(v)$ exists if and only if $v \equiv 1,3 \pmod 6$; such orders are called *admissible*.

Let $\mathrm{G}^{(3)}(n;m)$ denote a 3-uniform hypergraph of n vertices and m edges, that is, 3-tuples. Since an $\mathrm{STS}(v)$ contains exactly $v(v-1)/6$ triples, it can be considered to be a special $\mathrm{G}^{(3)}(v;v(v-1)/6)$. In 1976, Erdős [10] conjectured that for $r \geq 4$, there is an integer $v_0(r)$ such that for every $v > v_0(r)$, $v \equiv 1,3 \pmod 6$, there exists a Steiner triple system on v elements containing no $\mathrm{G}^{(3)}(k+2;k)$ for every $1 < k \leq r$. Such an STS is said to be *r-sparse*. Since the same pair of points appear twice in every $\mathrm{G}^{(3)}(k+2;k)$ for $1 < k \leq 3$, every $\mathrm{STS}(v)$ is 3-sparse. Obviously, every r-sparse $\mathrm{STS}(v)$, $r > 2$, is also $(r-1)$-sparse.

The Erdős r-sparse conjecture, and especially the problem of characterizing those v for which there exists an r-sparse $\mathrm{STS}(v)$ have been studied for a long time. One direction is regarding the r-sparse conjecture as an extremal problem on hypergraphs. In fact, Erdős posed the conjecture as a problem related to extremal set theory on hypergraphs. Brown, Erdős and Sós [2] proved:

Theorem 1.1 (Brown, Erdős and Sós) [2] *Let $\mathcal{L}(k,l)$ be the family of all nonisomorphic 3-uniform hypergraphs with l edges on k vertices and let $ex(n,\mathcal{L}(k,l))$ be the largest positive integer m such that there exists a triple system with m triples on n vertices containing no member of $\mathcal{L}(k,l)$. Then,*

$$ex(n,\mathcal{L}(k+2,k)) \le \frac{1}{3}\cdot(n\cdot\lfloor\frac{k}{k+1}\cdot(n-1)\rfloor+1).$$

Let $\mathcal{V}(k+2,k)=\bigcup_{j=2}^{k}\mathcal{L}(j+2,j)$. By probabilistic methods, Lefmann, Phelps and Rödl [22] showed that for every positive integer k, $k\ge 2$, there exists c_k such that $ex(n,\mathcal{V}(k+2,k))\ge c_k\cdot n^2$. They also gave the following theorem.

Theorem 1.2 (Lefmann, Phelps and Rödl) [22] *There exists a positive constant $c>0$, such that every Steiner triple system of order v contains a $\mathrm{G}^{(3)}(k+2;k)$ for some $k\le c\cdot\frac{\log v}{\log\log v}$.*

On the other hand, a lot of construction techniques for r-sparse STSs of particular small r and related triple systems have been developed. A $\mathrm{G}^{(3)}(k;l)$ appearing in a triple system is often called a "configuration" in recent related papers and so we shall use the same term here.

A (k,l)-*configuration* in an STS is a set of l blocks whose union contains precisely k points. An STS is r-sparse if and only it contains no $(k+2,k)$-configuration for every $1<k\le r$. Most of constructions for r-sparse triple systems and related designs mainly concern with two particular configurations, Pasches and mitres. The unique $(6,4)$-configuration, called the *Pasch configuration*, is described by six distinct points on four blocks $\{a,b,c\}$, $\{a,d,e\}$, $\{f,b,d\}$ and $\{f,c,e\}$. One of two $(7,5)$-configurations is called the *mitre*, described by seven distinct points on five blocks $\{a,b,e\}$, $\{a,c,f\}$, $\{a,d,g\}$, $\{b,c,d\}$ and $\{e,f,g\}$; a is referred to as the *centre* or *central element* of the mitre and the unique pair of blocks with no common point, that is, $\{b,c,d\}$ and $\{e,f,g\}$, is referred to as the *parallel blocks*. The other $(7,5)$-configuration, the *mia*, is obtained by joining two noncollinear points in a Pasch configuration: $\{a,b,c\}$, $\{a,d,e\}$, $\{f,b,d\}$, $\{f,c,e\}$ and $\{g,c,d\}$. An STS is said to be *anti-Pasch* or *anti-mitre* if it contains

no Pasch configuration or mitre configuration, respectively. In particular, an anti-Pasch STS does not contain a mia configuration. Hence, an STS is 5-sparse if it is both anti-Pasch and anti-mitre.

As well as in combinatorial design theory, 4- and 5-sparse triple systems are also important in some applications to information theory (see, for example, Chee, Colbourn and Ling [5], Johnson and Weller [21], Vasic, Kurtas and Kuznetsov [29] and Vasic and Milenkovic [30]), and hence constructions for an r-sparse STS are studied extensively from both sides.

This article briefly surveys recent developments on the existence of r-sparse trile systems and related designs. In section 2, we briefly give a histrical survey on 4-sparse STSs and related designs. Anti-mitre and 5-sparse STSs are considered in section 3. In section 4, we list recent results on an STS with higher sparseness. Mentioned are existence of a 6-spasrse STS, a triple system with highest sparseness at the time of writing, and nonexistence of an STS with high sparseness having particular automorphisms. Proofs for some unpublished theorems shall be provided in future papers.

2 4-sparse systems

In this section, we give a brief survey on the developments on the Erdős r-sparse conjecture for $r = 4$. We also remark about 4-sparse STS with additional properties.

It is known that the unique STS(7), and both nonisomorphic STS(13), contain Pasch configurations, while the unique STS(9) is anti-Pasch. Also, it is known that a class of the Netto system is 4-sparse (see Netto [25] and Robinson [26]).

Lemma 2.1 (Robinson [26]) *There exists a 4-sparse* $\mathrm{STS}(p^\alpha)$ *for prime* $p \equiv 19 \pmod{24}$*, and* α *a nonnegative integer.*

It is well-known that the points and lines of $AG(n,3)$, the n-demensional affine space, forms the elements and triples of a 4-sparse $\mathrm{STS}(3^n)$.

Brouwer [1] gave a more general construction for prime powers and proved the following theorem:

Theorem 2.2 (Brouwer [1]) *For* $q \equiv 1 \pmod 6$, $q = p^\alpha$, $p \notin \{7,13\}$ *a prime, there is a 4-sparse* $\mathrm{STS}(q)$ *whenever* $p \equiv 1,3 \pmod 8$ *or* $\alpha \equiv 0 \pmod 2$.

The first results on 4-sparse systems for non-prime powers are due to Brouwer [1] and Doyen [9]. They observed that the Bose construction for triple systems over the additive group of $\boldsymbol{Z}_v$ can generate 4-sparse STS(v) under a certain restriction. Brouwer [1] and Griggs, Murphy and Phelan [19] extended this result and constructed a 4-sparse STS(v) for all $v \equiv 3 \pmod 6$.

Theorem 2.3 (Brouwer [1] and Griggs, Murphy and Phelan [19]) *For all* $v \equiv 3 \pmod 6$, *there exists a* 4*-sparse* STS(v).

Also, Brouwer [1] refined the Erdős r-sparse conjecture for the case $r = 4$ to assert that a 4-sparse STS(v) exists for all $v \equiv 1$ or $3 \pmod 6$ except $v = 7$ and 13. Many partial results had been developed for this conjecture (see Colbourn and Rosa [7]). In particular, by developing several new constructions, Ling, Colbourn, Grannell and Griggs [24] extended substantially the spectrum of 4-sparse triple systems:

Theorem 2.4 (Ling, Colbourn, Grannell and Griggs [24]) *Suppose that* $v \equiv 1, 3 \pmod 6$ *and* $v \not\equiv 13, 31, 67 \pmod{72}$. *Then there exists a* 4*-sparse* STS(v) *provided that* $v \notin 7, 13$.

To complete the remaining orders stated in the theorem avobe, Grannell, Griggs and Whitehead [18] developed a construction employing auxiliary designs.

An STS$(u, -m)$ is a triple $(U, M, \mathcal{B})$, where U is a set of points having cardinality u, $M \subseteq U$ has cardinality m, and $\mathcal{B}$ is a collection of triples of points with the property that every pair of points $\{\alpha, \beta\}$, with $\alpha \in U$, $\beta \in U \setminus M$ appears in precisely one triple from $\mathcal{B}$, and no pairs $\{\alpha, \beta\}$ with $\alpha, \beta \in M$ appears in any triple from $\mathcal{B}$. An STS$(u, -m)$ is said to be *m-bipartite* if the points of $U \setminus M$ can be partitioned into two classes A and B, each of cardinality n, in such a way that no triple of the design are labelled (M, A, A) or (M, B, B). While a quadrilateral-free STS(v), that is, an anti-Pasch STS(v) is referred to as a QFSTS(v), an m-bipartite STS$(u, -m)$ containing no Pasch configuration is denoted briefly by BQFSTS$(u, -m)$.

Theorem 2.5 (Grannell, Griggs and Whitehead [18]) *Suppose that there exist a* 4*-sparse* STS$(2n + m)$ *and a* BQFSTS$(2n + m, -m)$, *where* $n = 3$ *or* $n \geq 5$. *Suppose also that there exists a* 4*-sparse* STS(u). *Then there exists a* 4*-sparse* STS$(n(u-1) + m)$.

For example, to cover the class $v \equiv 31 \pmod{72}$, write $v = n(u-1)+m$ and consider $u = 6t+3$, $n = 12$, $m = 7$. Then, by constructing BQFSTS$(31,-7)$ directly and applying Theorem 2.5, we can construct a 4-sparse STS of the class.

Grannell, Griggs and Whitehead [18] constructed BQFSTS, which finally complete the remaining classes that were left open in Theorem 2.4, and established the Brouwer's conjecture.

Theorem 2.6 (Grannell, Griggs and Whitehead) [18] *There exists a* 4*-sparse* STS(v) *if and only if* $v \equiv 1,3 \pmod 6$ *and* $v \neq 7, 13$.

This implies that Erdős' conjecture is true for $r = 4$ and $v_0(4) = 13$. In the rest of this section, we briefly mention the existence of 4-sparse STSs with additonal properties.

An STS $(V, \mathcal{B})$ is said to be *resolvable* if there exists a partition $P = \{P_1, P_2, \ldots, P_r\}$ of $\mathcal{B}$ such that each part P_i (called parallel class) is a partition of V. A resolvable STS is also refered to as a *Kirkman triple system* and is denoted briefly KTS. A KTS is known to exist for all $v \equiv 3 \pmod 6$.

Chee, Colbourn and Ling [5] showed that 4-sparse KTSs are useful for the disk storage system called *Redundant Ararrys of Independent Disks (RAID)* and constructed such triple systems.

Theorem 2.7 (Chee, Colbourn and Ling) [5] *For all* $v \equiv 9 \pmod{18}$, *there exists a* 4*-sparse* KTS(v).

Johnson and Weller [21] point outed the usefulness of 4- and 5-sparse KTSs in *low-density parity-check (LDPC) codes*. Construction methods for 4- and 5-sparse STSs with simple automorphisms, espesially cyclic automorphisms, are also important in LDPC codes. Such an STS shall be considered in section 4.

3 anti-mitre and 5-sparse systems

In this section, we consider the existence of anti-mitre and 5-sparse Steiner triple systems. The first results on anti-mitre STSs were obtained by Colbourn, Mendelsohn, Rosa and Širáň [6]. They gave a recursive construction called "doubling construction" and a generalization of the Bose construction for anti-mitre systems.

Theorem 3.1 (Colbourn, Mendelsohn, Rosa and Širáň) [6] *If there exists an anti-mitre* STS(v) *then there exists an anti-mitre* STS$(2v+1)$.

Theorem 3.2 (Colbourn, Mendelsohn, Rosa and Širáň) [6] *There exists an anti-mitre* $\mathrm{STS}(v)$ *for* $v \equiv 3,9 \pmod{18}$ *and* $v \neq 9$.

They also showed that each Netto system is anti-mitre and conjectured as follows:

Conjecture 3.3 (Colbourn, Mendelsohn, Rosa and Širáň) [6] *There exists an anti-mitre* $\mathrm{STS}(v)$ *if and only if* $v \equiv 1,3 \pmod 6$ *and* $v \neq 9$.

Ling [23] and the author [12, 13] presented further results on the existence of an anti-mitre STS, and eventually Wolfe [31] settled the conjecture.

Theorem 3.4 (Wolfe) [31] *There exists an anti-mitre* $\mathrm{STS}(v)$ *if and only if* $v \equiv 1,3 \pmod 6$ *and* $v \neq 9$.

Also, much progress had been made on 5-sparse STSs. Let G be an abelian group. An STS $(V,\mathcal{B})$ is said to be *transitive* on G if $V = G$ and for every $\alpha \in G$ and $\{a,b,c\} \in \mathcal{B}$, $\{a+\alpha, b+\alpha, c+\alpha\} \in \mathcal{B}$. If G is the cyclic group, the STS is said to be *cyclic*.

For small orders v, Colbourn, Mendelsohn, Rosa and Širáň [6] examined cyclic $\mathrm{STS}(v)$, checking whether each system is anti-Pasch, anti-mitre or both.

Theorem 3.5 (Colbourn, Mendelsohn, Rosa and Širáň) [6] *For* $19 \leq v \leq 97$ *and* $v \equiv 1,3 \pmod 6$, *there is a cyclic 5-sparse* $\mathrm{STS}(v)$ *except possibly when* $v \in \{21,25,27,31\}$. *In these cases, there is a cyclic* $\mathrm{STS}(v)$ *which is anti-Pasch but not anti-mitre, and a cyclic* $\mathrm{STS}(v)$ *which is anti-mitre but not anti-Pasch.*

Ling [23] gave a recursive constructions for 5-sparse STSs.

Theorem 3.6 (Ling) [23] *If there exists a transitive 5-sparse* $\mathrm{STS}(v)$, $v \equiv 1 \pmod 6$ *and a 5-sparse* $\mathrm{STS}(w)$, *then there exists a 5-sparse* $\mathrm{STS}(vw)$.

The author [13] generalized the BQSTS construction, that is, Theorem 2.5 and showed that there exists a 5-sparse $\mathrm{STS}(v)$ for all $v \equiv 1,19 \pmod{54}$ except for $v = 109$. Recently, Wolfe [32] constructed a 5-sparse STS for all $v \equiv 3 \pmod 6$ and $v \geq 21$. He also proved that there exists a 5-sparse STS for, in some sense, almost all admissible orders.

Let S and T be two subsets of $\mathbf{Z}^+ = \{1,2,3,\ldots\}$. Define the arithmetic density of S as compared to T as:

$$d(S;T) = \lim_{n\to\infty} \frac{|\{x \in S \cap T : x \leq n\}|}{|\{x \in T : x \leq n\}|}.$$

Theorem 3.7 (Wolfe) [32] *The arithmetic density of the spectrum of 5-sparse Steiner triple systems as compared to the set of all admissible orders is* 1.

4 Higher sparseness and automorphisms

In this section, we list very recent results on the existence of an STS with higher sparseness. As far as the author knows, these are all knowledge on r-sparse STSs for $r \geq 6$ at the time of writing this article.

In the previous two sections, we saw that the Erdős r-sparse conjecture is true for $r = 4$ and that a 5-sparse STS exists for almost all admissible orders. However, little is known about the existence of an STS with higher sparseness. In fact, no example of r-sparse systems is realized for $r \geq 7$ (and $v > 3$), and no affirmative answer to the r-sparse conjecture is known in this range. In what follows, we ignore the two trivial systems, that is, STS(1) and STS(3), unless they play a significant role.

Our primary focus in this section is on relations between group actions on an STS and its sparseness. An *automorphism* of an STS(v) $= (V, \mathcal{B})$ is a permutation on V that maps each block in $\mathcal{B}$ to a block of $\mathcal{B}$, and the *full automorphism group* is the group of all automorphisms of the STS. A *flag* of an STS $(V, \mathcal{B})$ is a pair (x, B) with $x \in V$ and $B \in \mathcal{B}$.

An STS is said to be *point-transitive* if its full automorphism group contains a subgroup which acts transitively on the point set. Similarly, we say that an STS is *block-transitive*, *flag-transitive*, *2-transitive*, or *2-homogeneous* if its full automorphism group contains a subgroup which acts transitively on the blocks, flags, ordered pairs of points, or unordered pairs of points, respectively.

Some classical constructions for STSs involving regular actions of $GF(q)$ on the point set generate 4- and 5-sparse STSs (see Colbourn and Rosa [7]). The direct product construction for 5-sparse triple systems developed by Ling [23], that is, Theorem 3.6 employed an abelian group which acts regularly on the point set. Forbes, Grannell and Griggs [11] discovered a construction method for block-transitive STSs and found twenty-nine examples of 6-sparse STSs in the residue class 7 modulo 12, with orders ranging from 139 to 4447. They also developed a recursive construction similar to Theorem 3.6 for block-transitive 6-sparse STSs and constructed infinitely many examples of such STSs. No 6-sparse STS other than these block-transitive systems is known and these have the highest sparseness at the time of writing.

Frequently, actions of a finite group on a triple system have helped us discover an r-sparse STS and develop a construction method. In fact, by checking for r-sparseness the block-transitive STSs arising from one of known constructions, Forbes, Grannell and Griggs [11] found the first examples of 6-sparse STSs. By limiting the search to point-transitive STS(v) over cyclic groups, that is, cyclic

STS(v), Colbourn, Mendelsohn, Rosa and Širáň [6] found a 5-sparse STS(v) for nearly all admissible $v < 100$. Most of the known recursive constructions of r-sparse STSs for $r \geq 5$ employs transitive actions of automorphism groups.

However, the author showed that such an STS can not have high sparseness. While the Erdős r-sparse conjecture says that for any $r \geq 4$ an r-sparse STS(v) exists for all sufficiently large admissible v, every point-transitive STS over an abelian group is at most 12-sparse.

Theorem 4.1 (Fujiwara) [15] *For every $r \geq 13$, there exists no point-transitive* STS *over an abelian group.*

A point-transitive STS $(V, \mathcal{B})$ over a group G has a *short orbit* if there exist a block $B \in \mathcal{B}$ and an element $x \in G$ such that $B^x = B$ and $x \neq 1$, the identity element. $(V, \mathcal{B})$ has a $\mathbf{Z}_3$-*orbit* if $\mathcal{B}$ contains a block having the form $\{a, a^x, a^{x^2}\}$, where $x^3 = 1$. $\mathbf{Z}_3$-orbit privent an STS from being high-sparse.

Theorem 4.2 (Fujiwara) [15] *Assume that there exists a point-transitive r-sparse* STS *over an abelian group G. Further, if the* STS *has a $\mathbf{Z}_3$-orbit, then $r \leq 9$.*

A cyclic STS(v) can be considered as a point-transitive STS whose full automorphism group contains a cyclic group of order v as a subgroup acting regularly on the point set. A cyclic STS(v) exists for all admissible v except for 9. Theorem 3.5 provides many examples of cyclic 5-sparse STSs. The author [14] developed some general recursive constructions for cyclic 4- and 5-sparse STSs, and constructed such an STS for infinitely many orders.

Theorem 4.3 (Fujiwara) [14] *There exists a cyclic 4-sparse* STS(v) *for* $v \equiv 3 \pmod 6$ *satisfying one of the condition (i)* $(v, 27) \neq 9$, *(ii)* $v \equiv 0 \pmod 7$, *or (iii)* $v \equiv 0 \pmod 5$.

Theorem 4.4 (Fujiwara) [14] *If there exist a cyclic 5-sparse* STS(v) *and a cyclic 5-sparse* STS(w), *where* $v, w \equiv 1 \pmod 6$, *then there exists a cyclic 5-sparse* STS(vw).

Theorem 4.5 (Fujiwara) [14] *If there exist a cyclic 5-sparse* STS(v), $v \equiv 1 \pmod 6$) *and a cyclic 5-sparse* STS(w), *where v and w are relatively prime, then there exists a cyclic 5-sparse* STS(vw).

However, by Theorems 4.1 and 4.2, we have:

Corollary 4.6 (Fujiwara) [15] *For every $r \geq 13$, there exists no cyclic r-sparse* $\mathrm{STS}(v)$. *In particular, when $v \equiv 3 \pmod 6$, no cyclic r-sparse* $\mathrm{STS}(v)$ *exists for every $r \geq 10$.*

The classification of STSs admitting other types of transitive actions and Theorem 4.1 gives further nonexistence results on an STS with higer sparseness. The details shall be presented in a future paper so we only mention the consequence.

Corollary 4.7 (Fujiwara) [15] *For every $r \geq 5$, there exists no 2-transitive r-sparse* STS.

Corollary 4.8 (Fujiwara) [15] *For every $r \geq 6$, there exists no 2-homogeneous r-sparse* STS.

Corollary 4.9 (Fujiwara) [15] *For every $r \geq 6$, there exists no flag-transitive r-sparse* STS.

Corollary 4.10 (Fujiwara) [15] *For every $r \geq 13$, there exists no block-transitive r-sparse* STS.

It is notable that the construction developed by Grannell, Griggs and Murphy [17] can generate finitely many examples of 6-sparse STSs but none of them is 7-sparse (see Forbes, Grannell and Griggs [11]).

We next consider Steiner triple systems admitting a nontrivial automorphism with fixed points.

An $\mathrm{STS}(v)$ is said to be *1-rotational* over a group G if it admits G as a subgroup of the full automorphism group and G fixes exactly one point and acts regularly on the other points. A 1-rotational automorphism is closely related to an involution.

An STS is said to be *reverse* if it admits an involutory automorphism fixing exactly one point. Any 1-rotational STS is reverse. Indeed, for every 1-rotational $\mathrm{STS}(v)$ over a group G, the order of G is $v-1$ and even. Hence, G has at least one involution.

Buratti [3] showed that there exists a 1-rotational $\mathrm{STS}(v)$ over an abelian group if and only if $v \equiv 3,9 \pmod{24}$ or $v \equiv 1,19 \pmod{72}$. He also gave partial answers for an arbitrary group. The combined work of Doyen [8], Rosa [27] and Teirlinck [28] established the fact that the spectrum for reverse STS is the set of all $v \equiv 1,3,9$ or $19 \pmod{24}$. An STS admitting an automorphism with more than one fixed point is known to exist (see Hartman and Hoffman [20]) and may also be considered. However, the fixed points must induce a smaller STS as a subsystem, and hence sparseness of the original Steiner system can not exceed that of the small sub-STS. Most interesting is the case when the induced subsystem is a trivial STS, that is, one point and no block, or three points and one block. The following theorem shows that such an STS is at most 4-sparse.

Theorem 4.11 (Fujiwara) [15] *For every $r \geq 5$, there exists no r-sparse* STS *admitting an involutory automorphism fixing exactly one or three points.*

The following is an immediate corollary of the theorem above.

Corollary 4.12 (Fujiwara) [15] *For every $r \geq 5$, there exists no reverse r-sparse* STS.

Since a 1-rotational STS is also reverse, we have:

Corollary 4.13 (Fujiwara) [15] *For every $r \geq 5$, there exists no* 1*-rotaional r-sparse* STS.

It is well known that the points and lines of $AG(n,3)$ forms the elements and triples of a 1-rotaional, and thus reverse, 4-sparse STS(3^n). In this sense, the bounds of Theorem 4.11, Corollary 4.12 and 4.13 are best possible.

Corollary 4.13 limits the sparseness of a 1-rotational STS over any finite group even if it is nonabelian. The same bound for a rotational group action fixing three points inducing the other trivial subsystem follows from the same argument. However, if groups are restricted to abelian ones, we can easily obtain much stronger theorem. In fact, sparseness is limited to the lowest.

Theorem 4.14 (Fujiwara) [15] *If the full automorphism group of an* STS *S contains an abelian subgroup which fixes more than one point and acts transitively on the other points, then S is not* 4*-sparse.*

In the remainder of this paper, we list two sporadic results on automorphisms, similar to those we have discussed.

An STS is said to be *bicyclic* if it admits a permutation on points consisting of a pair of cycles of length k and $v-k$ as an automorphism. Calahan and Gardner [4] proved that there exists a bicyclic STS(v) for $k > 1$ if and only if $v \equiv 1,3 \pmod 6$, $k \mid v$, and either $k \equiv 1 \pmod 6$ and $3k \mid v$; or $k \equiv 3 \pmod 6$ and $k \neq 9$.

Theorem 4.15 (Fujiwara) [15] *Let S be a bicyclic r-sparse* STS *and l be length of the smaller cycle of its bicyclic automorphism. Then,*

$$r \leq \begin{cases} 4 & \textit{when} \quad l = 1,3, \\ 9 & \textit{when} \quad l \equiv 3 \pmod 6, \\ 12 & \textit{when} \quad l \equiv 1 \pmod 6. \end{cases}$$

An STS is said to be 1*-transrotational* if it admits an automorphism consisting one fixed point, a transposition and a cycle of length $(v-3)$. Gardner [16] showed that a 1-transrotational STS(v) exists if and only if $v \equiv 1,7,9,15 \pmod{24}$.

Theorem 4.16 (Fujiwara) [15] *For every $r \geq 5$, there exists no* 1*-transrotational r-sparse* STS .

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