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On a class of rigid Coxeter groups

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The purpose of this note is to introduce some results of recent papers [4] and [5] about rigid Coxeter groups.

A Coxeter group is a group $W$ having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \rangle,$$

where $S$ is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(i) $m(s,t) = m(t,s)$ for any $s,t \in S$,

(ii) $m(s,s) = 1$ for any $s \in S$, and

(iii) $m(s,t) \geq 2$ for any $s,t \in S$ such that $s \neq t$.

The pair $(W,S)$ is called a Coxeter system. For a Coxeter group $W$, a generating set $S'$ of $W$ is called a Coxeter generating set for $W$ if $(W,S')$ is a Coxeter system. Let $(W,S)$ be a Coxeter system. For a subset $T \subset S$, $W_T$ is defined as the subgroup of $W$ generated by $T$, and called a parabolic subgroup. A subset $T \subset S$ is called a spherical subset of $S$, if the parabolic subgroup $W_T$ is finite.

Let $(W,S)$ and $(W',S')$ be Coxeter systems. Two Coxeter systems $(W,S)$ and $(W',S')$ are said to be isomorphic, if there exists a bijection $\psi : S \to S'$ such that

$$m(s,t) = m'(\psi(s), \psi(t))$$

for every $s,t \in S$, where $m(s,t)$ and $m'(s',t')$ are the orders of $st$ in $W$ and $s't'$ in $W'$, respectively.
A diagram is an undirected graph $\Gamma$ without loops or multiple edges with a map $\text{Edges}(\Gamma) \rightarrow \{2, 3, 4, \ldots\}$ which assigns an integer greater than 1 to each of its edges. Since such diagrams are used to define Coxeter systems, they are called Coxeter diagrams.

In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. Indeed some counter-examples are known.

**Example** ([1, p.38 Exercise 8], [2]). It is known that for an odd number $k \geq 3$, the Coxeter groups defined by the diagrams in Figure 1 are isomorphic and $D_{2k}$.

![Figure 1](image1.png)

**Figure 1.** Two distinct Coxeter diagrams for $D_{2k}$

**Example** ([2]). It is known that the Coxeter groups defined by the diagrams in Figure 2 are isomorphic by the diagram twisting ([2, Definition 4.4]).

![Figure 2](image2.png)

**Figure 2.** Coxeter diagrams for isomorphic Coxeter groups

Here there exists the following natural problem.

**Problem** ([2], [3]). When does a Coxeter group determine its Coxeter system up to isomorphism?
A Coxeter group $W$ is said to be *rigid*, if the Coxeter group $W$ determines its Coxeter system up to isomorphism (i.e., for each Coxeter generating sets $S$ and $S'$ for $W$ the Coxeter systems $(W, S)$ and $(W, S')$ are isomorphic).

A Coxeter system $(W, S)$ is said to be *even*, if $m(s, t)$ is even for all $s \neq t$ in $S$. Also a Coxeter system $(W, S)$ is said to be *strong even*, if $m(s, t) \in \{2\} \cup 4\mathbb{N}$ for all $s \neq t$ in $S$.

The following theorem was proved by Radcliffe in [6].

**Theorem 1** ([6]). If $(W, S)$ is a strong even Coxeter system, then the Coxeter group $W$ is rigid.

In [4], we first proved the following theorem which give a new class of rigid Coxeter groups.

**Theorem 2.** Let $(W, S)$ be a Coxeter system. Suppose that

0. for each $s, t \in S$ such that $m(s, t)$ is even, $m(s, t) = 2$,
1. for each $s \neq t \in S$ such that $m(s, t)$ is odd, $\{s, t\}$ is a maximal spherical subset of $S$,
2. there does not exist a three-points subset $\{s, t, u\} \subset S$ such that $m(s, t)$ and $m(t, u)$ are odd, and
3. for each $s \neq t \in S$ such that $m(s, t)$ is odd, the number of maximal spherical subsets of $S$ intersecting with $\{s, t\}$ is at most two.

Then the Coxeter group $W$ is rigid.

**Example.** The Coxeter groups defined by the diagrams in Figure 3 are rigid by Theorem 2.

![Figure 3](image-url)
In [5], we also proved the following theorem which is an extension of Theorem 1 and Theorem 2.

**Theorem 3.** Let \((W, S)\) be a Coxeter system. Suppose that

1. for each \(s, t \in S\) such that \(m(s, t)\) is even, \(m(s, t) \in \{2\} \cup 4\mathbb{N}\),
2. for each \(s \neq t \in S\) such that \(m(s, t)\) is odd, \(\{s, t\}\) is a maximal spherical subset of \(S\),
3. there does not exist a three-points subset \(\{s, t, u\} \subset S\) such that \(m(s, t)\) and \(m(t, u)\) are odd, and
4. for each \(s \neq t \in S\) such that \(m(s, t)\) is odd, the number of maximal spherical subsets of \(S\) intersecting with \(\{s, t\}\) is at most two.

Then the Coxeter group \(W\) is rigid.

**References**


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