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<td>Author(s)</td>
<td>KAMEKO, MASAKI; YAGITA, NOBUAKI</td>
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BROWN-PETerson COHOMOlgy OF $BPu(p)$

MASAKI KAMEKO (Tomo Masaki 瀧山国際大学地域学科)
NOBUAKI YAGITA (柳田孝穂 茨城大学教育学部)

Let $p$ be a fixed odd prime and denote by $BP^{*}(X)$ (resp. $P(m)^{*}(X)$) the Brown-Peterson cohomology of a space $X$ with the coefficient ring $BP^{*} = \mathbb{Z}_{(p)}[v_{1}, v_{2}, \ldots]$ (resp. $P(m)^{*} = \mathbb{Z}/p[v_{m}, v_{m+1}, \ldots]$) where $\deg v_{k} = -2p^{k} + 2$. We denote by $PU(n)$ the projective unitary group which is the quotient of the unitary group $U(n)$ by its center $S^{1}$. Recall that the cohomologies of $PU(p)$ and exceptional Lie groups $F_{4}, E_{6}, E_{7}, E_{8}$ have odd torsion elements. In this paper, we compute the Brown-Peterson cohomologies of classifying spaces $BG$ of these Lie groups $G$ as $BP^{*}$-modules using the Adams spectral sequence. Let us write $H^{*}(X; \mathbb{Z}/p)$ by simply $H^{*}(X)$ and let $A$ be the mod $p$ Steenrod algebra.

Our main result is as follows:

**Theorem 0.1.** Let $(G, p)$ be one of cases $(G = PU(p), p)$ for an arbitrary odd prime $p$ and $G = F_{4}, E_{6}, E_{7}, E_{8}$ for $p = 3$, and $G = E_{8}$ for $p = 5$. Then the $E_{2}$-terms of the Adams spectral sequences abutting to $BP^{*}(BG)$ and $P(m)^{*}(BG)$ for $m \geq 1$

$$\text{Ext}_A^{i,j}(H^{*}(BP), H^{*}(BG)), \quad \text{Ext}_A^{i,j}(H^{*}(P(m)), H^{*}(BG))$$

have no odd degree elements.

As an immediate consequence is as follows:

**Corollary 0.2.** For $(G, p)$ in Theorem 1.1, the Adams spectral sequence abutting to $BP^{*}(BG)$ and $P(m)^{*}(BG)$ in the previous theorem collapse at the $E_{2}$-level. In particular $BP^{*}(BG) = P(m)^{*}(BG)$ for $m \geq 1$.

Recall $K(m)^{*}(X) \cong K(m)^{*} \otimes_{P(m)^{*}} P(m)^{*}(X)$ is the Morava K-theory. From above theorem and corollary, we see $K(m)^{*}(BP^{*}(BG)) = 0$. Then we have the following corollary ([Ko-Ya],[Ra-Wi-Ya])

**Corollary 0.3.** For $(G, p)$ in Theorem 1.1, the following holds:

1. $BP^{*}(BG)$ is $BP^{*}$-flat for $BP^{*}(BP)$-modules, i.e.,

$BP^{*}(BG) \cong BP^{*}(BG) \otimes_{BP^{*}} BP^{*}(X)$ for all finite complexes $X$

2. $K(n)^{*}(BG) \cong K(n)^{*} \otimes_{BP^{*}} BP^{*}(BG)$

3. $P(n)^{*}(BG) \cong P(n)^{*} \otimes_{BP^{*}} BP^{*}(BG)$

We give the $BP^{*}$-module structure of $BP^{*}(BP^{*}(BG))$ more explicitly, in this talk.

**Theorem 0.4.** There is a $BP^{*}$-algebra isomorphism

$$0 \rightarrow BP^{*} \widehat{\otimes} M \rightarrow grBP^{*}(BP^{*}(BG)) \rightarrow BP^{*} \widehat{\otimes} \mathbb{Z}/(f_{0}, f_{1}) \rightarrow 0$$

where

1. $M \cong \mathbb{Z}_{(p)}[x_{4}, x_{6}, \ldots, x_{2p}]$ as $\mathbb{Z}_{(p)}$-modules (but not $\mathbb{Z}_{(p)}$-algebras).
2. $IN \cong \mathbb{Z}_{(p)}[x_{2p+2}, x_{2p(p-1)}] \{x_{2p+2}\}$; the principal ideal of $\mathbb{Z}[x_{2p+2}, x_{2p(p-1)}]$ generated by $x_{2p+2}$. 


3. relations $f_0, f_1$ are given with modulo $(p, v_1, v_2, \cdots)^2$

$$f_0 \equiv v_0 - v_2 x_{2p+1}^2 + \cdots, \quad f_1 \equiv v_1 - v_2 x_{2p(p-1)} + \cdots.$$ 

**Remark 0.5.** In the above theorem, suffix $i$ of $x_i$ means its degree. $BP^*(BPU(p))$ does not contain the subalgebra $BP^* \otimes \mathbb{Z}[(p)] [x_4, \ldots, x_{2p}]$, but contains a subalgebra which is isomorphic as $BP^*$-modules to the above $BP$-subalgebra.

For an algebraic group $G$ over $\mathbb{C}$, Totaro defines its Chow ring [To] and conjectures that $BP^*(BG) \otimes_{BP} \mathbb{Z}[(p)] \cong CH^*(BG)_{(p)}$. Recall that $PGL(p, \mathbb{C})$ is the algebraic group over $\mathbb{C}$ corresponding to the Lie group $PU(p)$.

**Theorem 0.6.** There is the isomorphism

$$BP^*(BPU(p)) \otimes_{BP} \mathbb{Z}[(p)] \cong CH^*(BGL(p, \mathbb{C}))_{(p)}.$$ 

Hence there is the additive isomorphism

$$CH^*(BGL(p, \mathbb{C}))_{(p)} \cong \mathbb{Z}[(p)][x_4, x_5, \ldots, x_{2p}] \oplus \mathbb{F}_p[x_{2p+2}, x_{2p(p-1)}] \{x_{2p+2}\}.$$ 

**Remark.** Recently Vistoli [Vi] also determined the additive structure of the Chow ring and integral cohomology of $BPGL(p, \mathbb{F}_p)$ by using stratified methods of Vessosoi. Moreover he shows that for $G = PGL(p, \mathbb{C})$

$$H^*(G; \mathbb{Z}) \to H^*(BT; \mathbb{Z})^{W_G(T)}$$

is epic.

Let $MGL^{2*}(X)$ be the motivic cobordism ring defined by V.Voevodsky [Vo] and $MGL^{2*}(X) = \oplus_i MGL^{2i}(X)$.

**Corollary 0.7.** $MGL^{2*}(BPGL(p, \mathbb{C}))_{(p)} \cong MU^*(BPU(p))_{(p)}$.

We prove Theorem 1.1 using the Adams spectral sequence converging to the Brown-Peterson cohomology. The $E_1$-term of the spectral sequence could be given by

$$\mathbb{F}_p[v_0, v_1, \cdots] \otimes H^*(X) \quad \text{with} \quad d_1 x = \sum_{k=0}^{\infty} v_k Q_k x$$

where $Q_k$'s are Milnor's operations. By the change-of-rings isomorphism, the $E_2$-term is

$$\text{Ext}_A(H^*(BP), H^*(X)) \cong \text{Ext}_E(\mathbb{F}_p, H^*(X))$$

where $E = \Lambda(Q_0, Q_1, \cdots)$. The $E_\infty$-term is given by $grBP^*(X)$.

To state the cohomology $H^*(BPU(p))$, we recall the Dickson algebra. Let $A_n$ be an elementary abelian $p$-group of rank $n$, and

$$H^*(BA_n) \cong \mathbb{F}_p[t_1, \ldots, t_n] \otimes \Lambda(dt_1, \ldots, dt_n) \quad \text{with} \quad \beta(dt_i) = t_i.$$ 

The Dickson algebra is

$$D_n = \mathbb{F}_p[t_1, \ldots, t_n]^{GL(n, \mathbb{F}_p)} \cong \mathbb{F}_p[c_{n,0}, \ldots, c_{n,n-1}]$$

with $|c_{n,i}| = 2(p^n - p^i)$. The invariant ring under $SL(n, \mathbb{F}_p)$ is also given

$$SD_n = \mathbb{F}_p[t_1, \ldots, t_n]^{SL(n, \mathbb{F}_p)} \cong D_n \{e_1, e_2, \ldots, e_n^{p-2}\} \quad \text{with} \quad e_n^{p-1} = c_{n,0}.$$ 

We also recall the Mui's ([Mu]) result by using $Q_i$ by [Ka-Mi]

$$grH^*(BA)^{SL_n(\mathbb{F}_p)} \cong SD_n / (e_n) \oplus SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \{u_n\}$$

where $u_n = dt_1 \cdots dt_n$ and $e_n = Q_0 \cdots Q_{n-1} u_n$. 
Theorem 0.8. There is the short exact sequence
\[ 0 \to M/p \to H^*(BPU(p)) \to N \to 0 \]
where $M/p$ is the trivial $E$-module given in Theorem 1.4 and
\[ N = SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \cong \mathbb{F}_p[x_{2p+2}, x_{2(p^2-p)}] \otimes \Lambda(Q_0, Q_1)\{u_2\} \]
identifying $x_{2p+2} = e_2$ and $x_{2(p^2-p)} = c_{2,1}$.

This theorem is proved by using the following facts. The group $G = PU(p)$ has just two conjugacy classes of maximal elementary abelian $p$-subgroups, one of which is toral and the other is non-toral $A$ of rank $p = 2$. The cohomology $H^*(BG)$ is detected by this two subgroups. The restriction image to the non-toral subgroup is $i_\ast^*(H^*PU(p)) \cong H^*(BA)_{SL(2, \mathbb{F}_p)}$. Similar (but not same) facts also hold for the exceptional Lie groups in Theorem 1.1.

Algebraic main result in this talk is as follows:

Theorem 0.9. For $m \geq 0$, define $f_0, \ldots, f_{n-1} \in P(m)^* \otimes SD_n$ by
\[ d_1 u_n = \sum_{k \geq m} v_k Q_k(u_n) = f_0 Q_0 u_n + \cdots + f_{n-1} Q_{n-1} u_n. \]
Then the sequence $f_0, \ldots, f_{n-1}$ is a regular sequence in $P(m)^* \otimes SD_n$.

With the notation in this theorem, we prove that the complex
\[ C = (P(m)^* \otimes SD_n \otimes \Lambda(Q_0, Q_1, \ldots, Q_{n-1}\{u_n\}, d_1) \]
with the differential $d_1 u_n = \sum_{i=0}^{n-1} f_i Q_i u_n$ is a Koszul complex. This means that
\[ H_i(C, d_1) = \begin{cases} P(m)^* \otimes SD_n\{e_n\}/(f_0, \ldots, f_{n-1}) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases} \]
Thus Theorem 1.1 follows from the above theorem.

Remark about the convergence of the Adams spectral sequence. By Theorem 15.6 in Boardman’s paper [Bo2], since $H^*(BP)$ is of finite type, the above Adams spectral sequence is conditionally convergent. Moreover, since we prove the above Adams spectral sequence collapses at the $E_2$-level, by the remark after Theorem 7.1 in [Bo1], the above Adams spectral sequence is strongly convergent, so that we know the Brown-Peterson cohomology up to group extension.

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Faculty of Regional science, Toyama University of Internal Studies, Toyama, Japan
Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan
kameko@tuins.ac.jp, yagita@mx.ibaraki.ac.jp