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Note on blocks of $p$-solvable groups with same Brauer category

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Let $p$ be a prime and let $O$ be a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$. Let $G$ be finite group and $b$ be a block of $G$ with maximal $(G,b)$-subpair $(P,e_P)$ where $b$ is a block idempotent of $OG$. For any subgroup $Q$ of $P$, let $(Q,e_Q)$ be a unique $(G,b)$-subpair contained in $(P,e_P)$. Following Kessar, Linckelmann and Robinson [4], we denote by $\mathcal{F}_{(P,e_P)}(G,b)$ the category whose objects are subgroups of $P$ and for $Q$, $R \leq P$, whose set of morphisms from $Q$ to $R$ are the set of group homomorphisms $\varphi : Q \to R$ such that there exists $x \in G$ such that $\varphi(Q,e_Q) \subseteq (R,e_R)$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$. We call $\mathcal{F}_{(P,e_P)}(G,b)$ the Brauer category of $b$. Let $B_G(b)$ be the Brauer category of $b$ in the sense of Thévenaz [10], § 47. The categories $\mathcal{F}_{(P,e_P)}(G,b)$ and $B_G(b)$ are equivalent. Let $R$ be a normal subgroup of $P$ such that $N_G(P) \subseteq N_G(R)$ and $c$ be the Brauer correspondent of $b$ in $N_G(R)$, that is, $c$ is a unique block of $N_G(R)$ such that $Br_P(c) = Br_P(b)$ where $Br_P$ is the Brauer homomorphism from $(OG)^P$ onto $kC_G(P)$. Set $N = N_G(R)$. The notations $R$, $c$ and $N$ are fixed. Thus $b = c^G$ and $(P,e_P)$ is a maximal $(N,c)$-subpair. The arguments in the proof of Theorem in Kessar-Linckelmann [5] imply the following.

**Theorem 1** Assume that $G$ is $p$-solvable. With the above notations, suppose that $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e_P)}(N,c)$. Then there is an indecomposable $OGb$-$ONc$-bimodule $M$ which satisfies the following.

(i) $M$ and its $O$-dual $M^*$ induce a Morita equivalence between $OGb$ and $ONc$.

(ii) As an $O(G \times N)$-module $M$ has a vertex $\Delta P$ and an endo-permutation $O(\Delta P)$-module as a source where $\Delta P = \{(u,u) \mid u \in P\}$.

Let $H^*_r(P,e_P)(G,b)$ be the cohomology ring of $b$ in the sense of Linckelmann [6], [7], that is, $H^*_r(P,e_P)(G,b)$ is the subring of $H^*_r(P,k)$ consisting of $\zeta \in H^*_r(P,k)$ satisfying $\text{res}_Q \zeta = g \text{res}_Q \zeta$ for all $Q \leq P$ and, for all $g \in N_G(Q,e_Q)$. We prove the following.

**Theorem 2** Assume that $G$ is $p$-solvable. With the above notations, if $H^*_r(P,e_P)(G,b) = H^*_r(P,e_P)(N,c)$, then $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e_P)}(N,c)$.
We prove Theorem 1 using the following.

**Lemma 1** (Harris-Linckelmann [3], Lemma 4.2) Assume that $G$ is $p$-solvable. For any $p$-subgroup $Q$ of $G$, we have $O_{p'}(N_{G}(Q)) = O_{p'}(G) \cap N_{G}(Q) = O_{p'}(G) \cap C_{G}(Q) = O_{p'}(C_{G}(Q))$.

**Proposition 1** (Harris-Linckelmann [2], Proposition 3.1 (iii)) Let $G$ be a $p$-solvable group and $b$ be a block of $G$ such that $b$ covers a $G$-invariant block of $O_{p'}(G)$. Then $b$ is of principal type. That is, for any $p$-subgroup $Q$ of $G$, $Br_{Q}(b)$ is a block of $kC_{G}(Q)$.

**Proposition 2** (Fong[1]; Puig[9]) Let $G$ be a $p$-solvable group and $b$ be a block of $G$ with defect group $P$. Then the following holds.

(i) There is a subgroup $H$ of $G$ and an $H$-invariant block $e$ of $O_{p'}(H)$ such that $O_{p'}(G)P \subseteq H$ and $OGb \cong \text{Ind}_{H}^{G}(OHe)$ as interior $G$-algebras.

(ii) $P$ is a Sylow $p$-subgroup of $H$ and $P$ is a defect group of $e$ as a block of $H$. Moreover, let $(P,e_{P})$ be a maximal $(H,e)$-subpair and let $e_{P} = \mathcal{T}_{C_{H}(P)}^{G}(e_{P})$. Then $(P,e_{P})$ is a maximal $(G,b)$-subpair.

Note that in the above proposition $\mathcal{F}_{(Pe_{P})}(G,b) = \mathcal{F}_{(Pe_{P})}(H,e)$ since $OGb \cong \text{Ind}_{H}^{G}(OHe)$ as interior $G$-algebras.

**Proposition 3** ([5]. Proposition 6) With the notations in the above proposition, let $R$ be a subgroup of $P$ such that $N_{G}(P) \subseteq N_{G}(R)$. Denote by $c$ the Brauer correspondent of $b$ in $N_{G}(R)$, and by $f$ the Brauer correspondent of $e$ in $N_{H}(R)$. Then $f$ is an $N_{H}(R)$-invariant block of $O_{p'}(N_{H}(R))$ and $ON_{G}(R)c \cong \text{Ind}_{N_{H}(R)}^{N_{G}(R)}(ON_{H}(R)f)$ as interior $N_{G}(R)$-algebras.

The following is shown in the proof of Theorem in [5].

**Theorem 3** (Kessar-Linckelmann) Let $G$ be a $p$-solvable group and $b$ be a block of $G$ with defect group $P$. Let $R$ be a subgroup of $P$ such that $N_{G}(P) \subseteq N_{G}(R)$ and let $c$ be the Brauer correspondent of $b$ in $N$ where we set $N = N_{G}(R)$. If $b$ covers a $G$-invariant block of $O_{p'}(G)$ and if $G = O_{p'}(G)N$, then there is an indecomposable $OGb \cdot ONc$-bimodule $M$ which satisfies the following.

(i) $M$ and its $O$-dual $M^{*}$ induce a Morita equivalence between $OGb$ and $ONc$.

(ii) As an $O(G \times N)$-module $M$ has a vertex $\Delta P$ and an endo-permutation $O(\Delta P)$-module as a source.

**Proof of Theorem 1.** We prove by induction on $|G|$. Let $H$, $e$, $e_{P}$ and $e_{P}$ be as in Proposition 2, and let $f$ be as in Proposition 3. We may assume that $e_{P}$'s in Theorem 1 and Proposition 2 are equal by replacing $H$, $e$, $e_{P}$ and $f$, by $H^{x}$, $e^{x}$, $(e_{P})^{x}$ and $f^{x}$ respectively for some $x \in N_{G}(P)$ if necessary. By Proposition 2,

$$\mathcal{F}_{(Pe_{P})}(G,b) = \mathcal{F}_{(Pe_{P})}(H,e).$$
By Proposition 3, \((P, e'_p)\) is a maximal \((N_H(R), f)\)-subpair and
\[
\mathcal{F}_{(P,e'_p)}(N, c) = \mathcal{F}_{(P,e'_p)}(N_H(R), f).
\]
So by the assumption we have \(\mathcal{F}_{(P,e'_p)}(H, e) = \mathcal{F}_{(P,e'_p)}(N_H(R), f)\). Since \(\mathcal{O}Gb \cong \text{Ind}^G_H(\mathcal{O}He)\) as interior \(G\)-algebras, the \(\mathcal{O}Gb-\mathcal{O}He\)-bimodule \(b\mathcal{O}Ge = \mathcal{O}Ge\) and the \(\mathcal{O}He-\mathcal{O}Gb\)-bimodule \(e\mathcal{O}G\) induce a Morita equivalence between \(\mathcal{O}Gb\) and \(\mathcal{O}He\). Similarly the \(\mathcal{O}Nc-\mathcal{O}N_H(R)f\)-bimodule \(\mathcal{O}Nf\) and the \(\mathcal{O}N_H(R)f-\mathcal{O}Nc\)-bimodule \(f\mathcal{O}N\) induce a Morita equivalence between \(\mathcal{O}Nc\) and \(\mathcal{O}N_H(R)f\). Suppose that \(H < G\). By the induction hypothesis for \(H\) and \(e\), there is an indecomposable \(\mathcal{O}He-\mathcal{O}N_H(R)f\)-bimodule \(M_0\) such that \(M_0\) and \(M'_0\) induce a Morita equivalence between \(\mathcal{O}He\) and \(\mathcal{O}N_H(R)f\) and that \(M_0\) as an \(\mathcal{O}(H \times N_H(R))\)-module has a vertex \(\Delta P\) and an endo-permutation \(\mathcal{O}(\Delta P)\)-module as a source. Set \(M = b\mathcal{O}G \otimes_{\mathcal{O}He} M_0 \otimes_{\mathcal{O}N_H(R)f} \mathcal{O}Nc \cong M_0^{G \times N}\). Then \(M\) satisfies (i) and (ii) in Theorem 1. Therefore we may assume that \(H = G\). Then \(b = e\).

Let \(Y = O_{p',p}(G)\). Then \(b\) is a \(G\)-invariant block of \(Y\) because \(Y/O_{p'}(G)\) is a \(p\)-group. Furthermore we have \(Y = O_{p'}(G)(Y \cap P)\). Set \(Q = P \cap Y\). Then \(Q\) is a defect group of \(b\) as a block of \(Y\). Now since \(G\) is constrained, \(C_Y(Q) = C_G(Q)\). Therefore we see that \((Q, e_Q)\) is a maximal \((Y, b)\)-subpair. By the Frattini argument and the assumption that \(\mathcal{F}_{(P,e_p)}(G, b) = \mathcal{F}_{(P,e_p)}(N, c)\).

\[
G = N_G(Q, e_Q)Y \subseteq N_N(Q)C_G(Q)Y \subseteq NY \subseteq NO_{p'}(G).
\]
So we have \(G = NO_{p'}(G)\). This and Theorem 3 complete the proof.

**Proof of Theorem 2.** We prove by induction on \(|G|\). Let \(H, e, e'_p\) and \(e_p\) be as in Proposition 2, and let \(f\) be as in Proposition 3. We may assume that \(e_p\)'s in Theorem 2 and Proposition 2 are equal as in the proof of Theorem 1. Since \(\mathcal{F}_{(P,e_p)}(G, b) = \mathcal{F}_{(P,e'_p)}(H, e)\) and \(\mathcal{F}_{(P,e_p)}(N, c) = \mathcal{F}_{(P,e'_p)}(N_H(R), f)\) we have
\[
\]
\[
\]
From the assumption, we have \(H^*_*(P,e'_p)(H, e) = H^*_*(P,e'_p)(N_H(R), f)\). Suppose that \(H < G\). Then by the induction hypothesis, \(\mathcal{F}_{(P,e_p)}(H, e) = \mathcal{F}_{(P,e'_p)}(N_H(R), f)\), and hence \(\mathcal{F}_{(P,e_p)}(G, b) = \mathcal{F}_{(P,e_p)}(N, c)\). Therefore we may assume that \(H = G\). Then \(b\) covers a \(G\)-invariant block of \(O_{p'}(G)\) and \(P\) is a Sylow \(p\)-subgroup of \(G\). Note that the element \(b \in O_{p'}(G)\).

From Proposition 1, \(b\) is of principal type. On the other hand, by Lemma 1, \(\text{Br}_R(b)\) is an \(N\)-invariant block idempotent of \(kO_{p'}(N)\) and \(c\) is a lifting of \(\text{Br}_R(b)\) to \(\mathcal{O}N\). So by Proposition 1, \(c\) is also of principal type. So we may assume that \(b\) is a principal block. Therefore by a theorem of Mislin [8], we obtain \(\mathcal{F}_{(P,e_p)}(G, b) = \mathcal{F}_{(P,e_p)}(N, c)\). This completes the proof.
References


