タイプ変換とシャープな文字の記号

タイトル

タイプ変換

数理解析研究所講究録

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Type transformations for sharp characters

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1 Introduction

Let $G$ be a finite group and $\chi$ be a faithful character of $G$ of degree $n$. Put $L = \{\chi(g) | g \in G, g \neq 1\}$. Then we have the following

Theorem 1 (Blichfeldt[B]) $|G|$ divides the integer $\prod_{l \in L} (n - l)$.

Theorem 1 gives us the upper bound of the order of $G$. We are interested in the case $G$ attains the bound.

Definition 1 We call $(G, \chi)$ sharp of type $L$ (or $L$-sharp) if $|G| = \prod_{l \in L} (n - l)$ holds.

Problem 1 For a given $L$, determine all $L$-sharp pairs $(G, \chi)$.

Example 1 Let $G$ be a sharply $t$-transitive permutation group, which is different from $S_t$, the symmetric group of degree $t$. Let $\pi$ be the permutation character of $G$. Then $(G, \pi)$ is sharp of type $\{0, 1, \cdots, t - 1\}$.

Note that $(G, \chi)$ is sharp if and only if $(G, \chi + 1_G)$ is sharp, where $1_G$ is the trivial character of $G$. So we may assume $(\chi, 1_G) = 0$ holds, when we consider sharp characters $\chi$. We call such character normalized sharp character.

We have the following results concerning Problem 1. When $L$ contains an irrational number, $L$-sharp pairs $(G, \chi)$ are completely classified by Alvis-Nozawa[A-N]. Hence we may assume that $L \subset \mathbb{Z}$ holds. The cases $L = \{l\}, \{l, l + 1\}, \{l, l + 2\}, \{l, l + 1, l + 2\}, \{l, l + 1, l + 2, l + 3\}$ are treated in Cameron-Kiyota [C-K], Cameron-Kataoka-Kiyota [C-K-K], Nozawa [N]. We do not have any classification results for "big" $L$ in case $L \subset \mathbb{Z}$, and so we should ask the following

Problem 2 Can we reduce the classification of $L$-sharp pairs to that of $L'$-sharp pairs for some $L'$ with $|L'| < |L|$?
2 Transformations of types

Let \( L_1, L_2 \) be finite sets of complex numbers with \(|L_1| = |L_2| = m \geq 2\).

Definition 2 We write \( L_1 \sim L_2 \) if \( e_1(L_1) = e_1(L_2), e_2(L_1) = e_2(L_2), \ldots, e_{m-1}(L_1) = e_{m-1}(L_2) \) hold, where \( e_k(L_1) \) is the \( k \)-th elementary symmetric function with variables in \( L_1 \). For example, \( e_1(L_1) = \sum_{l \in L_1} l \), \( e_m(L_1) = \prod_{l \in L_1} l \).

Example 2 \( \{a, b\} \sim \{c, d\} \iff a + b = c + d \), 
\( \{a, b, c\} \sim \{d, e, f\} \iff a + b + c = d + e + f, \ ab + bc + ca = de + ef + fd \)

The following two lemmas are fundamental but easy to prove.

Lemma 1
(1) \( L_1 \sim L_2 \iff L_1 + l \sim L_2 + l \), where we denote \( L_1 + l = \{a + l \mid a \in L_1\} \).
(2) If \( L_1 \sim L_2 \), then we have 
\[ L_1 = L_2 \iff L_1 \cap L_2 \neq \emptyset \iff e_m(L_1) = e_m(L_2). \]

Lemma 2 Assume \( L \subset \mathbb{C}, \ |L| = rm (m \geq 2) \). Then the followings are equivalent.
(1) There exists a monic polynomial \( f(X) \in \mathbb{C}[X] \) of degree \( m \) with \( |f(L)| = r \).
(2) There exists a decomposition of \( L, L = L_1 \cup \cdots \cup L_r \) with \( |L_k| = m, L_1 \sim \cdots \sim L_r \).

Using the above lemmas, we can prove the following Theorem.

Theorem 2 Let \( \chi \) be a faithful character of a finite group \( G \). Set \( L = \{\chi(g) \mid g \in G, g \neq 1\} \). Suppose that there exists a decomposition of \( L, L = L_1 \cup \cdots \cup L_r \) with \( |L_k| = m \geq 2, L_1 \sim \cdots \sim L_r \). Assume further that each \( L_k \) is algebraically closed. Then there exists a monic \( f(X) \in \mathbb{Z}[X] \) which satisfies the following two conditions.

(i) \( (G, \chi) \) is sharp of type \( L \iff (G, f(\chi)) \) is sharp of type \( f(L) \).
(ii) \( f(L) = \{(-1)^{m-1}e_m(L_1), \ldots, (-1)^{m-1}e_m(L_r)\} \).

We will give some examples that shows how to apply Theorem 2.
Example 3  Let \((G, \chi)\) be normalized sharp of type \(L = \{-1, 0, 1, 2\}\). Note that \(L = \{-1, 2\} \cup \{0, 1\}, \{-1, 2\} \sim \{0, 1\}\). So \(L\) satisfies the conditions of Theorem 2. If we put \(f(X) = X^2 - X\), then \((G, f(\chi))\) is sharp of type \(\{2, 0\}\) (but not necessarily normalized). Using the classification of sharp of type \(\{l, l + 2\}\), we get \(G = S_5, A_6, M_{11}\). Thus, \(G\) is a sharply 4-transitive group except \(S_4\).

Example 4  \(L = \{-1, 0, 2, 3\} = \{-1, 3\} \cup \{0, 2\}\) satisfies the conditions of Theorem 2. Using \(f(X) = X^2 - 2X\), we can reduce the determination of \(L\)-sharp pairs to that of \(\{3, 0\}\)-sharp pairs. But unfortunately we do not have complete classification of \(\{l, l + 3\}\)-sharp pairs.

Example 5  \(L = \{-2, -1, 0, 2, 3, 4\} = \{-1, 0, 4\} \cup \{-2, 2, 3\}\) satisfies the conditions of Theorem 2. Using \(f(X) = X^3 - 3X^2 - 4X\), we can reduce the determination of \(L\)-sharp pairs to that of \(\{0, -12\}\)-sharp pairs. But again we do not have complete classification of \(\{l, l + 12\}\)-sharp pairs.

Remarks  In Theorem 2, \(f(\chi)\) is a generalized character of \(G\) and is not necessarily character. \(f(\chi)\) is not necessarily normalized, even if \(\chi\) is so.

References


