APPLICATIONS OF CR GEOMETRY TO REPRESENTATIONS OF 
SU(p, q)

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Conformal geometry on pseudo-Riemannian manifolds can be applied to the representation theory of the group SO(p, q) (cf. [3] [11] [12] [13] [14] and references therein). Kostant used the conformal invariance of the vanishing of scalar curvature on 6 dimensional manifolds to explore the minimal representation of SO(4, 4) in [14]. Recently, T. Kobayashi and B. Orsted [11] [12] [13] gave a geometric and intrinsic model of the minimal irreducible unitary representation $\varpi^{p,q}$ of SO(p, q) on $S^{p-1} \times S^{q-1}$ and on various pseudo-Riemannian manifolds which are conformally equivalent, by using the Yamabe operator. They also gave branching formulae and unitarization of various models. Here we use CR geometry to realize representations of SU(p, q).

1. Preliminaries on CR Geometry

Let $M$ be a real $(2n + 1)$-dimensional orientable $C^\infty$ manifold. A CR structure on $M$ is a $n$-dimensional complex subbundle $T_{1,0}M$ of the complexified tangent bundle $CTM$ satisfying $T_{1,0}M \cap T_{0,1}M = \{0\}$, where $T_{0,1}M = \overline{T_{1,0}M}$, and the integrability condition: $[Z_1, Z_2] \in C^\infty(M, T_{1,0}M)$ whenever $Z_1, Z_2 \in C^\infty(M, T_{1,0}M)$. $T_{1,0}M$ is usually called the complex tangential space. Set

\[ H = \text{Re}\{T_{1,0}M \oplus T_{0,1}M\}, \]

the 2n-dimensional real horizontal subbundle of $TM$. $H$ carries a complex structure $J : H \rightarrow H$ satisfying $J^2 = -\text{id}_H$ and $T_{1,0}M = \ker(J - i \cdot \text{id}_{CH})$, $T_{0,1}M = \ker(J + i \cdot \text{id}_{CH})$. When $M$ is the boundary of a domain in a complex manifold $W$, it has an induced CR structure from the complex structure of $W$ defined by

\[ T_{1,0}M = CTM \cap T_{1,0}W, \]

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if \( \dim(T_{1,0}M)_{x} = \text{const.} \) for each \( x \in M \), where \( T_{1,0}W \) is the holomorphic tangential space of complex manifold \( W \).

A mapping \( f : (M_1,T_{1,0}M_1) \rightarrow (M_2,T_{1,0}M_2) \) is called a Cauchy-Riemann mapping (or CR mapping) if

\[
f_{*}T_{1,0}M_1 \subset T_{1,0}M_2,
\]

where \( f_{*} \) is the tangential mapping of \( f \). If \( f \) is invertible, \( f \) and \( f^{-1} \) are both CR mappings, \( f \) is called a CR diffeomorphism.

Let \( \theta \) be a 1-form on \( M \) such that

\[
\ker \theta = H.
\]

We require \( \theta \) to be a contact form, i.e. \( \theta \wedge (d\theta)^n \) is non-vanishing on \( M \). Such \( \theta \) is called a pseudohermitian structure on \( (M,T_{1,0}M) \). We call the triple \( (M,T_{1,0}M,\theta) \) a pseudohermitian manifold. \( \theta \) plays the role of metric \( g \) in pseudo-Riemannian geometry.

We say \( \tilde{\theta} \) is conformal to \( \theta \) if

\[
\tilde{\theta} = \phi^2 \theta
\]

for some non-vanishing smooth function \( \phi \) on \( M \). A CR mapping between two pseudohermitian manifolds, \( f : (M_1,T_{1,0}M_1,\theta_1) \rightarrow (M_2,T_{1,0}M_2,\theta_2) \), is called conformal if \( f^{*}\theta_2 = \phi^2 \theta_1 \) for some non-vanishing smooth function \( \phi \) on \( M_1 \).

We can define a Hermitian form on \( T_{1,0}M \) associated to a pseudohermitian structure \( \theta \) by

\[
L_{\theta}(V,\overline{W}) = -id\theta(V \wedge \overline{W}),
\]

which is called the Levi form of \( \theta \).

If the Levi form has \( k \) positive eigenvalues and \( n-k \) negative eigenvalues, \( (M,T_{1,0}M,\theta) \) is said to be strictly \( k \)-pseudoconvex. The inner product \( L_{\theta}(\cdot,\cdot) \) determines a dual form \( L_{\theta}^{*}(\cdot,\cdot) \) on \( H^{*} \). \( L_{\theta}^{*}(\cdot,\cdot) \) can be naturally extended to \( T^{*}M \).

In [19], Webster showed that there exists a natural connection on the bundle \( T_{1,0}M \) adapted to a pseudohermitian structure \( \theta \). For a pseudohermitian structure \( \theta \) on a strictly
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$k$-pseudoconvex CR manifold $(M,T_{1,0}M,\theta)$, there is a unique vector field $T$, which is transversal to $H$, defined by

\begin{align}
\theta(T) = 1, \quad d\theta(T \wedge \cdot) = 0.
\end{align}

Let $\theta^\alpha$ be an admissible coframe, i.e. $(1,0)$-forms $\theta^\alpha$ form a basis for $T_{1,0}^*$ such that $\theta^\alpha(T) = 0$ for all $\alpha = 1, \ldots, n$. The integrability condition implies

\begin{align}
d\theta = ig_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^\overline{\beta}
\end{align}

for some Hermitian matrix of functions $(g_{\alpha\overline{\beta}})$, which is nondegenerate and has $k$ positive eigenvalues and $n - k$ negative eigenvalues if $(M,T_{1,0}M,\theta)$ is strictly $k$-pseudoconvex. Webster showed that there are uniquely determined 1-forms $\omega^\alpha_\beta$ and $\tau^\alpha$ on $M$ satisfying

\begin{align}
\begin{cases}
d\theta^\beta = \theta^\alpha \wedge \omega^\alpha_\beta + \theta \wedge \tau^\beta \\
\omega^\alpha_\overline{\beta} + \omega^\overline{\beta}_\alpha = dg_{\alpha\overline{\beta}} \\
\tau_\alpha \wedge \theta^\alpha = 0,
\end{cases}
\end{align}

where we use $(g_{\alpha\overline{\beta}})$ to raise and lower indices, e.g. $\omega^\alpha_\overline{\beta} = \omega^\alpha_\gamma g_{\gamma\overline{\beta}}$. Let

\begin{align}
\Omega^\alpha_\beta = d\omega^\alpha_\beta - \omega^\alpha_\gamma \wedge \omega^\gamma_\beta.
\end{align}

Webster showed that $\Omega^\alpha_\beta$ could be written as

\begin{align}
\Omega^\alpha_\beta = R^\alpha_\rho^\beta_\sigma \theta^\rho \wedge \theta^\overline{\sigma} + W^\alpha_\rho^\sigma \theta^\rho \wedge \theta - W^\sigma_\rho^\alpha \theta^\overline{\sigma} \wedge \theta + i\theta^\rho \wedge \tau^\rho - i\tau^\rho \wedge \theta^\rho
\end{align}

The Webster-Ricci tensor of $(M,T_{1,0}M,\theta)$ has components $R^\alpha_\rho_\beta_\sigma = R^\sigma_\rho_\alpha_\beta$. The Webster scalar curvature is

\begin{align}
R_\theta = g^\alpha_\beta R^\alpha_\beta.
\end{align}

The CR Yamabe problem is to find a contact form $\tilde{\theta} = u^2 \theta, u > 0$, which is conformal to the given contact form $\theta$, such that $R_\tilde{\theta} \equiv \text{constant}$. This problem is considered by Lee and Jerison [9] for strictly pseudoconvex CR manifolds and completely solved recently by N. Gamara and R. Yacoub [6] [7].

A pseudohermitian manifold $(M,T_{1,0}M,\theta)$ has a natural volume form

\begin{align}
\psi_\theta = (-1)^{n-k} \theta \wedge (d\theta)^n,
\end{align}
which is nowhere vanishing because $M$ is strictly $k$-pseudoconvex. It induces an $L^2$ inner product on functions

\[(1.14) \quad \langle u, v \rangle_{\theta} = \int_{M} u \overline{v} \psi_{\theta}, \]

and an $L^2$ inner product on sections of $H^*$,

\[(1.15) \quad \langle \omega, \eta \rangle_{\theta} = \int_{M} L_{\theta}^{*}(\omega, \eta) \psi_{\theta}. \]

For $u \in C^{\infty}(M)$, we define a section $d_{b}u$ of $H^*$ by

\[(1.16) \quad d_{b}u = pr \circ du, \]

where $pr : T^{*}M \rightarrow H^*$ is the restriction map. We can define the SubLaplacian $\Box_{\theta}$ associated to a strictly $k$-pseudoconvex contact form $\theta$ by

\[(1.17) \quad \langle \Box_{\theta} u, v \rangle_{\theta} = \frac{1}{2} \langle d_{b}u, d_{b}v \rangle_{\theta}. \]

Since evidently, $|\theta|_{\theta} = 0$, $L_{\theta}^{*}(\cdot, \cdot)$ is degenerate on $T^{*}M$ and so the operator $\Box_{\theta}$ is a degenerate ultrahyperbolic operator.

**Proposition 1.1.** (Proposition 4.10 in [15]) If $u \in C_{0}^{\infty}$, then,

\[(1.18) \quad \Box_{\theta} u = -u_{\alpha} - \alpha u_{\overline{\alpha}}. \]

Define a product on $\mathbb{C}^{n+2}$ by

\[(1.19) \quad (\zeta, \xi)_{p,q} = \sum_{j=0}^{n+1} \epsilon_{j} \zeta_{J} \overline{\xi}_{j}, \]

where $n + 2 = p + q$, and

\[(1.20) \quad \epsilon_{j} = \begin{cases} 1, & \text{for } j = 0, 1, \ldots, p - 1, \\ -1, & \text{for } j = p, \ldots, p + q - 1. \end{cases} \]

We denote $(\zeta, \zeta)_{p,q}$ by $|\zeta|_{p,q}^2$ for $\zeta \in \mathbb{C}^{n+2}$. Similarly, we define a product on $\mathbb{C}^{n}$ by

\[(1.21) \quad (z, w)_{p-1,q-1} = \sum_{\alpha=1}^{n} \epsilon_{\alpha} z_{\alpha} \overline{w}_{\alpha}. \]

We also denote $(z, z)_{p-1,q-1}$ by $|z|_{p-1,q-1}^2$ for $z \in \mathbb{C}^{n}$. 
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The simplest CR manifold is the Heisenberg group \( \mathbb{H}^{p-1, q-1} \), whose underlying manifold is \( \mathbb{C}^{p+q-2} \times \mathbb{R} \), with coordinates \((z, t)\). Its multiplication is given by

\[(1.22) \quad (z, t) \cdot (z', t') = (z + z', t + t' + 2 \text{Im}(z, z')_{p-1, q-1}) .\]

The vector fields

\[(1.23) \quad Z_\alpha = \frac{\partial}{\partial z_\alpha} + i \epsilon_\alpha \overline{z}_\alpha \frac{\partial}{\partial t} ,\]

\(\alpha = 1, \cdots, n\), are left invariant vector fields on \( \mathbb{H}^{p-1, q-1} \). The standard CR structure on the Heisenberg group \( \mathbb{H}^{p-1, q-1} \) is given by the subbundle

\[(1.24) \quad T_{1,0} \mathbb{H}^{p-1, q-1} = \text{span}_\mathbb{C} \{Z_1, \cdots, Z_n\} .\]

Let

\[(1.25) \quad \theta_{\mathbb{H}^{p-1, q-1}} = dt + \sum_{\alpha=1}^{n} i \epsilon_\alpha (z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha)\]

be the standard contact form on \( \mathbb{H}^{p-1, q-1} \).

\[(1.26) \quad \Box_{\mathbb{H}^{p-1, q-1}} = -\frac{1}{2} \sum_{\alpha=1}^{p-1} (Z_\alpha \overline{Z}_\alpha + \overline{Z}_\alpha Z_\alpha) + \frac{1}{2} \sum_{\alpha=p}^{p+q-2} (Z_\alpha \overline{Z}_\alpha + \overline{Z}_\alpha Z_\alpha) .\]

Let us consider a real hypersurface \( Q'_{p,q} \) in \( \mathbb{C}^{n+1} \) defined by equation

\[(1.27) \quad \text{Im} z_0 = |z|_{p-1, q-1}^2 , \quad z \in \mathbb{C}^n , \quad z_0 \in \mathbb{C} ,\]

which is the boundary of the Siegel upper half space

\[(1.28) \quad S = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^n ; \text{Im} \ z_0 > |z|_{p-1, q-1}^2 \} .\]

The Cayley transformation \( C \) is defined by

\[(1.29) \quad w_0 = \frac{z_0 - i}{z_0 + i} , \quad w_\alpha = \frac{2z_\alpha}{z_0 + i} ,\]

which transforms the hypersurface \( Q'_{p,q} \) into the hyperquadric \( Q_{p,q} \),

\[(1.30) \quad Q_{p,q} = \{w = (w_0, w') ; w_0 \in \mathbb{C} , w' \in \mathbb{C}^n , |w_0|^2 + |w'|_{p-1, q-1}^2 = 1 \} .\]

Now introduce homogeneous coordinates \( \zeta_j , j = 0, \cdots, n+1 \). By equations

\[(1.31) \quad z_j = \frac{\zeta_j}{\zeta_{n+1}} , \quad j = 0, \cdots, n+1 ,\]
\begin{align}
\overline{Q}_{p,q} = \{\zeta = (\zeta_0, \cdots, \zeta_{n+1}) \in \mathbb{C}P^{n+1}; |\zeta|_{p,q}^2 = 0\}.
\end{align}

Projective hyperquadric $\overline{Q}_{p,q}$ is the compactification of $Q_{p,q}$ in $\mathbb{C}P^{n+1}$. The hypersurface $Q'_{p,q}$ and the projective hyperquadric $\overline{Q}_{p,q}$ have induced CR structures by (1.2) from complex manifolds $\mathbb{C}^{n+1}$ and $\mathbb{C}P^{n+1}$, respectively.

$SU(p, q)$ is the group of unimodular transformations preserving the Hermitian form (1.19). Its center $K$ consists of $n+2$ transformations. Then $SU(p, q)/K$ acts on $\overline{Q}_{p,q}$ effectively and $PU(p, q) = SU(p, q)/K$. It is well known that $Aut_{CR}\overline{Q}_{p,q} = PU(p, q)$ [4].

Pseudo-Riemannian geometry

\begin{align}
\begin{array}{ll}
\text{A metric } g & \text{A contact form } \theta \\
\text{conformal } \tilde{g} = \phi^2 g & \tilde{\theta} = \phi^2 \theta \\
\text{pseudo-Riemannian connection} & \text{Webster connection} \\
\text{the Laplacian } \square_g & \text{the SubLaplacians } \square_{\theta} \\
SO(p, q) & SU(p, q) \\
\text{the flat model } \mathbb{R}^{p-1,q-1} & \mathbb{H}^{p-1,q-1} \\
S^{p-1} \times S^{q-1} & \text{the projective hyperquadric } \overline{Q}_{p,q} \\
\text{the Yamabe operator} & \text{the CR Yamabe operator} \\
\end{array}
\end{align}

2. Representations realized as conformal CR diffeomorphisms

Let $Q = \dim M + 1 = 2n + 2$, the homogeneous dimension of $M$. The following transformation formula is due to Lee.

**Proposition 2.1.** Let $(M, T_0 M, \theta)$ be a pseudohermitian manifold with $\dim M = 2n + 1$. The Webster scalar curvature $R_{\tilde{\theta}}$ associated with the pseudohermitian structure $\tilde{\theta} = u^{\frac{4}{Q-2}} \theta$ satisfies

\begin{align}
b_n \square_{\tilde{\theta}} u + R_{\tilde{\theta}} u = R_{\tilde{\theta}} u^{\frac{Q+2}{Q-2}},
\end{align}

where $b_n = 2 + \frac{2}{n}$.
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The following is a transformation formula for the SubLaplacians under a conformal CR transformation.

**Proposition 2.2.** Let $(M_1, T_{1,0}M_1)$ and $(M_2, T_{1,0}M_2)$ be two CR manifolds with strictly k-pseudoconvex pseudohermitian structure $\theta_1$ and $\theta_2$, respectively. Suppose $\Phi : (M_1, T_{1,0}M_1) \rightarrow (M_2, T_{1,0}M_2)$ is a CR diffeomorphism with $\Phi^*\theta_2 = u^{\frac{4}{Q-2}}\theta_1$ for some positive smooth function $u$ on $M_1$. Then

$$\square_{\theta_1}(u \cdot \Phi^* f) - u^{\frac{Q+2}{Q-2}}\Phi^*(\coprod_{\theta_2} f) = \square_{\theta_1}u \cdot \Phi^* f,$$

for any smooth real function $f$ on $M_2$.

Now define the CR Yamabe operator to be

$$\widetilde{\square}_\theta = \square_\theta + \frac{1}{b_n} R_\theta,$$

where $b_n = 2 + \frac{2}{n}$, $R_\theta$ is the Webster scalar curvature (1.12). The transformation formula for the CR Yamabe operator is a consequence of Corollary 2.1 and Proposition 2.2 as follows.

**Proposition 2.3.** Under the same assumption as in proposition 2.2, we have that

$$\widetilde{\square}_{\theta_1}(u \cdot \Phi^* f) = u^{\frac{2+3}{2+3}}\Phi^*(\widetilde{\square}_{\theta_2} f),$$

for any smooth function $f$ on $M_2$.

Suppose $(M_1, T_{1,0}M_1, \theta_1)$ and $(M_2, T_{1,0}M_2, \theta_2)$ are two pseudohermitian manifolds of homogeneous dimension $Q$. Let conformal CR mapping $\Phi : (M_1, T_{1,0}M_1, \theta_1) \rightarrow (M_2, T_{1,0}M_2, \theta_2)$ be a local diffeomorphism such that

$$\Phi^*\theta_2 = \Omega^2\theta_1,$$

for some positive function $\Omega$ on $M_1$. We can define twisted pull back

$$\Phi^* : C^\infty(M_2) \rightarrow C^\infty(M_1), \quad f \mapsto \Omega^\lambda(\Phi^* f).$$

Let $G$ be a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$. We write the action of $h \in G$ as $L_h : (M, T_{1,0}M, \theta) \rightarrow$
There exists a positive valued function $\Omega(h,x)$ for $h \in G$ and $x \in M$ such that

$$L_h^* \theta = \Omega(h,x)^2 \theta.$$  

We have the cocycle formula for $\Omega(\cdot, \cdot)$.

**Proposition 2.4.** For $h_1, h_2 \in G$ and $x \in M$, we have

$$\Omega(h_1 h_2, x) = \Omega(h_1, L_{h_2} x) \Omega(h_2, x).$$

Now for $\lambda \in \mathbb{C}$, we can define a representation $\varpi_\lambda$ of the group $G$ on $C^\infty(M)$ as follows. For $h \in G$, $f \in C^\infty(M)$ and $x \in M$,

$$(\varpi_\lambda(h^{-1}) f)(x) = \Omega(h, x)^\lambda f(L_h x).$$

Proposition 2.4 assures that $\varpi_\lambda(h_1 h_2) = \varpi_\lambda(h_1) \varpi_\lambda(h_2)$, i.e., $\varpi_\lambda$ is a representation of $G$. Thus, $\square f = 0$ if and only if $\Omega^\frac{Q-2}{2} \tilde{\Phi}^* f = 0$. In summary, we have the following theorem.

**Theorem 2.5.** Suppose $G$ is a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$ of homogeneous dimension $Q$. Then,

1. the CR Yamabe operator $\square$ is an intertwining operator from $\varpi_{\frac{Q-2}{2}}$ to $\varpi_{\frac{Q+2}{2}}$.
2. The kernel $\ker \square$ is a subrepresentation of $G$ through $\varpi_{\frac{Q-2}{2}}$.

3. The CR Yamabe operator on the hypersurface $Q_{p,q}'$

Let $\xi \mapsto [\xi]$ denote the canonical projection of $\mathbb{C}^{n+2}\backslash\{0\}$ into the complex projective space $\mathbb{C}P^{n+1}$. It is easy to see that the transformation

$$I(z_0, z_1, \cdots, z_n) = \left[ \frac{z_0 - i}{2}, z_0 + i \right],$$

maps the hypersurface $Q_{p,q}'$ defined by (1.27) into the projective hyperquadric $\overline{Q}_{p,q}$ (1.32). Define a 1-form

$$\theta = \frac{\sum_{j=0}^{p+1} i \varepsilon_j (\xi_j d\overline{\xi}_j - \overline{\xi}_j d\xi_j)}{\sum_{j=0}^{p-1} |\xi_j|^2},$$

$$I(z_0, z_1, \cdots, z_n) = \left[ \frac{z_0 - i}{2}, z_0 + i \right].$$
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on $\mathbb{C}^{n+2}\backslash\{\xi \in \mathbb{C}^{n+2}; \xi_0 = \cdots = \xi_{p-1} = 0\}$. It induces a 1-form on the projective hyperquadric $\overline{Q}_{p,q}$ in (1.32). We denote it by $\theta_{\overline{Q}_{p,q}}$. The hyperquadric $Q_{p,q}$ in (1.30) has a contact form

\begin{equation}
\theta_{Q_{p,q}} = \sum_{\alpha=0}^{n} i\epsilon_{\alpha}(z_{\alpha}d\overline{z}_{\alpha} - \overline{z}_{\alpha}dz_{\alpha}),
\end{equation}

(here we use variables $z_{\alpha}$ instead of $w_{\alpha}$, $\alpha = 0, \cdots, n$, in the definition of $Q_{p,q}$ in (1.30)) and the hypersurface $Q'_{p,q}$ in (1.27) has a contact form

\begin{equation}
\theta_{Q'_{p,q}} = \sum_{\alpha=1}^{n} i\epsilon_{\alpha}(z_{\alpha}d\overline{z}_{\alpha} - \overline{z}_{\alpha}dz_{\alpha}) + \frac{1}{2}(d\overline{z}_0 + dz_0).
\end{equation}

Contact forms (3.3) and (3.4) are actually

\begin{equation}
i(\overline{\partial} - \partial)r
\end{equation}

for corresponding defining functions $r$ of $Q_{p,q}$ and $Q'_{p,q}$, respectively.

**Proposition 3.1.**

\begin{equation}
I^{\ast} \theta_{\overline{Q}_{p,q}} = \frac{1}{\frac{1}{4}|z_0 - i|^2 + \sum_{j=1}^{p-1}|z_j|^2} \theta_{Q'_{p,q}},
\end{equation}
on the hypersurface $Q'_{p,q}$.

**Proposition 3.2.** Let $S_0 = \sum_{j=0}^{n+1} a_j|\xi_j|^2$ with $a_j = \epsilon_j$ or 0, but $a_0 = 1$ and $a_{n+1} = 0$. Then the function

\begin{equation}
S(z_0, z) = S_0\left(\frac{z_0 - i}{2}, z, \frac{z_0 + i}{2}\right)
\end{equation}
on hypersurface $Q'_{p,q}$ satisfies where it is positive

\begin{equation}
\Delta_{\theta_{Q'_{p,q}}} S^{-\frac{Q-2}{4}} = \frac{n+1}{2} \left(\sum_{j=1}^{n} 2a_j\epsilon_j - n\right) S^{-\frac{Q+2}{4}},
\end{equation}

where $Q = 2n + 2$.

**Corollary 3.3.** The scalar curvature of the projective hyperquadric $\overline{Q}_{p,q}$ with contact form $\theta_{\overline{Q}_{p,q}}$ is $\frac{n+1}{2}(p - q)$.
4. Representations on the projective hyperquadric $\overline{Q}_{p,q}$

**Proposition 4.1.** For $g \in SU(p, q)$ and $z \in Q_{p,q}$, we have

\[(4.1) \quad g^* \theta_{Q_{p,q}}(z) = \frac{1}{|g(z,1)_{n+1}|^2} \theta_{Q_{p,q}}(z).\]

Define the light cone to be

\[(4.2) \quad \Xi := \{ \xi \in \mathbb{C}^{n+2}; |\xi|_{p,q} = 0\} \backslash \{0\},\]

and

\[(4.3) \quad \Sigma := \left\{ \xi \in \mathbb{C}^{n+2}; \sum_{j=0}^{p-1} |\xi_j|^2 = \sum_{j=p}^{p+q-1} |\xi_j|^2 = 1 \right\} \simeq S^{2p-1} \times S^{2q-1}.\]

The multiplicative group $\mathbb{R}_+^\times$ acts on $\Xi$ as a dilation and the quotient space $\Xi/\mathbb{R}_+^\times$ is identified with $\Sigma$. By definition, $\Xi/\mathbb{C}^\times \simeq \Sigma/S^1 \approx \overline{Q}_{p,q}$. Because the action of $SU(p, q)$ on $\mathbb{C}^{n+2}$ commutes with that of $\mathbb{C}^\times$, we can define the action of $SU(p, q)$ on the quotient space $\Xi/\mathbb{C}^\times$, and also on $\overline{Q}_{p,q}$ through the above diffeomorphism. This action will be denoted by

\[(4.4) \quad L_h : \overline{Q}_{p,q} \rightarrow \overline{Q}_{p,q}, \quad \xi \mapsto L_h \xi,\]

for $h \in SU(p, q), \xi \in \overline{Q}_{p,q}$.

For $a \in \mathbb{C}$, denote by $S^a(\Xi)$ the space of smooth function on $\Xi$ homogeneous of degree $a$, i.e.

\[(4.5) \quad S^a(\Xi) = \{ f \in C^\infty(\Xi); f(t\xi) = t^a f(\xi), \xi \in \Xi, t \in \mathbb{R}_+^\times \}.\]

A character $\psi$ of $\mathbb{C}^\times$ has the form

\[(4.6) \quad \psi(t) = |t|^a \left( \frac{t}{|t|} \right)^m,\]

for some $a \in \mathbb{C}, m \in \mathbb{Z}$, which can be formally written as

\[(4.7) \quad \psi(t) = \psi^{\alpha,\beta}(t) = t^\alpha \overline{t}^\beta,\]

with $\alpha + \beta = a$ and $\alpha - \beta = m$. We see that a pair $(\alpha, \beta)$ can occur if and only if $\alpha - \beta$ is an integer. For such a pair, we define $S^{\alpha,\beta}(\Xi) \subset S^a(\Xi)$ to be the $\psi^{\alpha,\beta}$ eigenspace for $\mathbb{C}^\times$. 
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Then, we have a decomposition

\[(4.8) \quad S^\alpha(\Xi) = \sum_{\alpha+\beta=q, \alpha-\beta\in \mathbb{Z}} S^{\alpha,\beta}(\Xi).\]

Let \( \nu : \Xi \rightarrow \mathbb{R}_+ \) be defined by

\[(4.9) \quad \nu(\xi) = \left( \sum_{j=0}^{p-1} |\xi_j|^2 \right)^{\frac{1}{2}} = \left( \sum_{j=p}^{p+q-1} |\xi_j|^2 \right)^{\frac{1}{2}}.\]

**Proposition 4.2.** For \( g \in \text{SU}(p, q) \) and \( \xi \in \overline{Q}_{p,q} \), we have

\[(4.10) \quad g^* \theta_{Q_{p,q}}(\xi) = \frac{1}{\nu(g(\xi))} \theta_{Q_{p,q}}(\xi),\]

if we require the coordinates of \( \xi \) satisfying \( \sum_{j=0}^{p-1} |\xi_j|^2 = 1 \).

**Proposition 4.3.** \( S^{-\frac{\lambda}{2},-\frac{\lambda}{2}}(\Xi) \) is isomorphic to \( (\varpi_{\lambda}, C^\infty(\overline{Q}_{p,q})) \) as \( \text{U}(p, q) \) modules.

Define the representation \((\varpi^{p,q}, V^{p,q})\) to be \( (\varpi_{\frac{Q-2}{2}}, \ker \Delta_{Q_{p,q}}) \).

We can identify \( S^{\alpha,\beta}(\Xi) \) with degenerate principal series representations in standard notation (cf. [5]).

**Corollary 4.4.** \((\varpi^{p,q}, V^{p,q})\) is a subrepresentation of \( S^{-\frac{\xi-1}{2},-\frac{\xi-1}{2}}(\Xi) \), or equivalently, of \( C^\infty - \text{Ind}_{P_{\text{max}}}^{G}(\chi_0 \otimes C_{-1}) \)

5. Basic properties of \((\varpi^{p,q}, V^{p,q})\)

There is a natural action of \( S^1 \) on \( \Sigma \) defined by

\[(5.1) \quad \mu_\sigma : \Sigma \rightarrow \Sigma, \quad (\xi_1, \cdots, \xi_{n+1}) \mapsto (e^{i\sigma} \xi_1, \cdots, e^{i\sigma} \xi_{n+1}),\]

for \( \sigma \in [0, 2\pi) \). We can define the projection

\[(5.2) \quad \Pi : \Sigma \simeq S^{2p-1} \times S^{2q-1} \rightarrow \overline{Q}_{p,q},\]

by \( \Pi(\xi_1, \cdots, \xi_{n+1}) = [\xi_1, \cdots, \xi_{n+1}] \in \mathbb{C}P^{n+1} \). Namely, \( \Sigma \) is a \( S^1 \) fiber bundle over the projective hyperquadric \( \overline{Q}_{p,q} \). Let

\[(5.3) \quad \theta_{S^{2p-1}} = i \sum_{j=0}^{p-1} \xi_j d\xi_j - \overline{\xi}_j d\xi_j, \quad \theta_{S^{2q-1}} = i \sum_{j=p}^{n+1} \xi_j d\overline{\xi}_j - \overline{\xi}_j d\xi_j,\]
the standard contact forms on spheres $S^{p-1}$ and $S^{q-1}$, respectively. Then,

\begin{equation}
\Pi^{*}\theta_{Q_{p,q}} = \theta_{S^{2p-1}} - \theta_{S^{2q-1}}.
\end{equation}

Let $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^{p})$ denote the space of harmonic polynomials of bi-degree $(\alpha, \beta)$ in $\mathbb{C}^{p}$, i.e., harmonic polynomials which are homogeneous of degree $\alpha$ in the $z_j$'s and of degree $\beta$ in the $\overline{z}_j$'s

\begin{equation}
L^{2}(S^{2p-1}) \simeq \sum_{\alpha, \beta=0}^{\infty} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^{p}).
\end{equation}

For a function $f \in L^{2}(\overline{Q}_{p,q})$, $\Pi^{*}f$ is an $L^{2}$ function in $\Sigma$ invariant under the action of $S^{1}$. Thus,

\begin{equation}
L^{2}(\overline{Q}_{p,q}) \simeq \sum_{m=0}^{\infty} \mathcal{H}^{m_{1},n_{1}}(\mathbb{C}^{p}) \boxtimes \mathcal{H}^{m_{2},n_{2}}(\mathbb{C}^{q})
\end{equation}
as Hilbert direct sum. We denote by $C^{\infty}(S^{2p-1} \times S^{2q-1})_{0}$ the space of $S^{1}$-invariant functions in $C^{\infty}(S^{2p-1} \times S^{2q-1})$. We can identify $C^{\infty}(\overline{Q}_{p,q})$ with the subspace $C^{\infty}(S^{2p-1} \times S^{2q-1})_{0}$ by the mapping $\Pi$.

**Proposition 5.1.** For $u \in C^{\infty}(S^{2p-1} \times S^{2q-1})_{0}$, we have

\begin{equation}
\nabla_{\theta_{Q_{p,q}}}(u \circ \Pi^{-1}) = \nabla_{\theta_{S^{2p-1}}}u - \nabla_{\theta_{S^{2q-1}}}u.
\end{equation}

The Yamabe operator on the projective hyperquadric $\overline{Q}_{p,q}$ is

\begin{equation}
\nabla_{\theta_{Q_{p,q}}} = \nabla_{\theta_{S^{2p-1}}} + \frac{n}{4}(p - q).
\end{equation}

**Proposition 5.2.** $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^{p})$ is an eigenspace of $\nabla_{\theta_{S^{2p-1}}}$ on $S^{2p-1}$ with eigenvalue $\frac{1}{4}(\alpha + \beta)(2p - 2 + \alpha + \beta) - \frac{1}{4}(\alpha - \beta)^{2}$.

**Theorem 5.3.** The underlying $(\mathfrak{g}, K)$-module $(\mathfrak{w}^{p,q})_{K}$ has the following $K$-type formula

\begin{equation}
(\mathfrak{w}^{p,q})_{K} \simeq \bigoplus_{m_{1}+n_{1}+p=m_{2}+n_{2}+q, \atop m_{1}+m_{2}=n_{1}+n_{2}} \mathcal{H}^{m_{1},n_{1}}(\mathbb{C}^{p}) \boxtimes \mathcal{H}^{m_{2},n_{2}}(\mathbb{C}^{q})
\end{equation}
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Remark 5.4. For other rank-1 Lie groups $\mathrm{Sp}(1)\mathrm{Sp}(n+1,1)$ and $F_{4}^{-20}$, there exist quaternionic and octonianic CR geometries. For example, we have corresponding Webster connections, corresponding conformal geometry, corresponding Yamabe operators, etc. (cf. [3]). It is interesting to study the representation theories of $\mathrm{Sp}(1)\mathrm{Sp}(n+1,1)$ (more generally, of $\mathrm{Sp}(p,q)$) and $F_{4}^{-20}$ by using corresponding conformal geometries.

REFERENCES


[10] KOBAYASHI, T., Singular unitary representations and discrete series for indefinite Stiefel manifolds $U(p,q;F)/U(p-m,q;F)$, Mem. Amer. Math. Soc., 95, no. 462, 1992, 1-106.


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