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Kyoto University
AN EXPLICIT DIMENSION FORMULA FOR THE SPACES OF VECTOR VALUED SIEGEL CUSP FORMS OF DEGREE TWO

Satoshi Wakatsuki (若槻 聡)

1. INTRODUCTION

In this paper, we give an explicit dimension formula for the spaces of vector valued Siegel cusp forms of degree two with respect to the principal congruence subgroups of $Sp(2; \mathbb{Z})$, and certain arithmetic subgroups of non-split $\mathbb{Q}$-forms of $Sp(2; \mathbb{R})$. As for the principal congruence subgroups of $Sp(2; \mathbb{Z})$, Tsushima already gave the dimension formula by the Riemann-Roch theorem in [17], but we give an alternative proof by the Selberg trace formula and the theory of prehomogeneous vector spaces. As for the case of non-split $\mathbb{Q}$-forms, our result is new.

Our calculation is a generalization of the calculations of Morita [13], Shintani [14] and Arakawa [1]. In this short note, we explain only the points for the generalizations (we wrote the detail proof in [18]). In order to generalize their methods, first we must show the convergence of some infinite series. Then we need to calculate explicitly an integral of a certain function, which is related to the Fourier transformation of the trace of the irreducible rational representations. The integral is well-known in the scalar valued case, but the integral is unknown and nontrivial in the vector value case.

One of our motivation is as follows. Ibukiyama gave a conjecture for the Shimura correspondence between vector valued Siegel cusp forms of degree two of integral weight and half integral weight (cf. [10]). There, it is essential to take vector valued forms. In order to prove this conjecture, we must show the equality of the traces of Hecke operators. As the first step, we treat the traces of the trivial actions, which are the dimensions of the spaces.

The plan of this paper is as follows. In Section 2, we state our main results. In Section 3, we review the Godement formula. In Section 4, we explain the calculations of the vanishing part. In Section 4, we explain the calculations of the non-vanishing part. In Appendix A, for the reader’s convenience, we copy the dimension formula for $Sp(2; \mathbb{Z})$ which was given by Tsushima [17]. In Appendix B, we give the dimension formula for the full modular groups of non-split $\mathbb{Q}$-forms, which was obtained by our recent calculation.

2. MAIN RESULTS

We define the spaces of Siegel cusp forms of degree two. Let $\rho_{k,j} : GL(2; \mathbb{C}) \to GL(j + 1; \mathbb{C})$ be the irreducible rational representation of the signature $(j + k, k)$.

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Theorem 2.1. If $k \geq 5$ and $N \geq 3$, then
\[
\dim_{\mathbb{C}} S_{k,j}(\Gamma(N)) = \begin{cases} 
\frac{\Gamma(1)}{\Gamma(N)} & \text{if } k \geq 5, j = 0 \\
\frac{1}{2} \dim_{\mathbb{C}} S_{k,j}(\Gamma(N)) & \text{if } k < 5, j = 0 \\
\frac{1}{2} \dim_{\mathbb{C}} S_{k,j}(\Gamma(N)) & \text{if } k \geq 5, j > 0 \\
\frac{1}{2} \dim_{\mathbb{C}} S_{k,j}(\Gamma(N)) & \text{if } k < 5, j > 0
\end{cases}
\]

where $\Gamma(1) = \prod_{p \text{ prime}, p|N}(1-p^{-2})(1-p^{-4})$. 

We shall give the other main result. Let $B$ be an indefinite division quaternion algebra over $\mathbb{Q}$, $\mathfrak{O}$ a maximal order of $B$, $a \mapsto \overline{a}$ ($a \in B$) the canonical involution of $B$. Put
\[
G_{\mathbb{Q}} = \left\{ \left( \begin{array}{cc} a & b \\
0 & c \\
\end{array} \right) \in M(2; \mathbb{B}) ; \left( \begin{array}{cc} a & b \\
0 & c \\
\end{array} \right) \left( \begin{array}{cc} 0 & 1 \\
1 & 0 \\
\end{array} \right) \left( \begin{array}{cc} \overline{a} & \overline{c} \\
\overline{b} & b \\
\end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\
1 & 0 \\
\end{array} \right) \right\},
\]
\[
\Gamma^{*}(N) = \left\{ \left( \begin{array}{cc} a & b \\
c & d \\
\end{array} \right) \in G_{\mathbb{Q}} ; a-1, b, c, d \in N\mathfrak{O} \right\}.
\]

The following result is new.

Theorem 2.2. If $k \geq 5$ and $N \geq 3$, then
\[
\dim_{\mathbb{C}} S_{k,j}(\Gamma^{*}(N)) = \Gamma^{*}(1) : \Gamma^{*}(N)]
\]
\[

\times \left\{ 2^{-8}3^{-5}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \prod_{p \text{ prime}, p|D(B)} (p-1)(p^2+1)
\right.
\]
\[
+2^{-4}3^{-1}(j+1)5^{-3} \prod_{p \text{ prime}, p|D(B)} (p-1) \right\},
\]

where $D(B)$ is the product of prime numbers which ramify in $B$ over $\mathbb{Q}$, $p$ is prime, and $\Gamma^{*}(1) : \Gamma^{*}(N)] = N^{10} \prod_{p \text{ prime}, p|D(B)} (1-p^{-2})(1-p^{-4}) \times \prod_{p \text{ prime}, p|D(B)} (1-p^{-2})(1+p^{-1}).$

As for the scalar valued case, these dimension formulas were already known. The dimension formula of the scalar valued case for $\Gamma(N)$ ($N \geq 3, j = 0$) was calculated by Christian [3], Morita [13] and Yamazaki [19] independently. The dimension
formula of the scalar valued case for $\Gamma^*(N) \ (N \geq 3, \ j = 0)$ was calculated by Arakawa [1] and Yamaguchi independently. Christian, Morita and Arakawa used the Selberg trace formula. Yamazaki and Yamaguchi used the Riemann-Roch theorem.

Numerical examples.
(i) $\dim_{\mathbb{C}} S_{k,j}(\Gamma(3))$.

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(*) Our theorem is not valid for $k = 4$. As for $j = 0, k = 4$, Yamazaki calculated it by the Riemann-Roch theorem in [19]. We formally put $k = 4$ in the formula of our theorem. We expect that the dimension of $S_{k,j}(\Gamma(3))$ is given by putting $k = 4$ in the formula (cf. [7] and [8]). We also expect it for other arithmetic subgroups. For $j = 0, k = 1, 2, 3$, Gunji proved $\dim_{\mathbb{C}} S_{k,0}(\Gamma(3)) = 0$ in [5].

(ii) $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(3)), \ D(B) = 2 \times 3$.

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3. GODEMENT FORMULA

In this section, we explain the Godement formula and the calculations of dimension formulas. We set

$$H_{\gamma}^{k,j}(Z) = \tr \left[ \rho_{k,j}(CZ + D)^{-1} \rho_{k,j} \left( \frac{\gamma \cdot Z - \overline{Z}}{2\sqrt{-1}} \right)^{-1} \rho_{k,j}(Y) \right],$$

$$Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}, \ X = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix}, \ Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix},$$

d$Z = \det(Y)^{-3}dXdY$, $dX = dx_1 dx_{12} dx_2$, $dY = dy_1 dy_{12} dy_2,$

for $Z = X + \sqrt{-1}Y \in \mathfrak{h}_2$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2; \mathbb{R})$. Godement gave the following formula (cf. [5, Expose 10, Théorème 8]).

**Theorem 3.1** (Godement). If $k \geq 5$, then

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma) = \frac{c_{k,j}}{{\#}(Z(\Gamma))} \int_{\Gamma \backslash \mathfrak{h}_2} \sum_{\gamma \in \Gamma} H_{\gamma}^{k,j}(Z)dZ,$$$$

where $c_{k,j} = 2^{-6}\pi^{-3}(k-2)(j+k-1)(j+2k-3)$, $Z(\Gamma)$ is the center of $\Gamma$, $\#(Z(\Gamma))$ is the order of $Z(\Gamma)$. \)
We shall remark on the constant $c_{k,j}$. In [5], the constant $c_{k,j}$ was calculated for the only scalar valued case ($j = 0$), not for the vector valued case. In [12], we easily see that the constant $c_{k,j}$ is equal to $(\text{formal degree})/(j + 1) \times \text{constant}$, where the constant is independent of the signature $(k + j, k)$. Furthermore, from [6], we have an explicit form of the formal degree. Hence we get explicitly the constant $c_{k,j}$.

We give the corollary of Theorem 3.1. We can easily show the following corollary from the equality $H^{k,j}_{\eta^{-1}\gamma}(Z) = H^{k,j}_{\gamma}(g \cdot Z)$ ($g \in \text{Sp}(2; \mathbb{R})$) and the normality of $\Gamma^{(*)}(N)$ in $\Gamma^{(*)}(1)$.

**Corollary 3.2.** If $k \geq 5$ and $N \geq 3$, then

$$\dim \mathcal{S}_{k,j}(\Gamma^{(*)}(N)) = 2^{-1}c_{k,j}[\Gamma^{(*)}(1) : \Gamma^{(*)}(N)] \int_{F^{(*)}} \sum_{\gamma \in \Gamma^{(*)}(N)} \frac{1}{\#(Z(\Gamma))} \int_{\Gamma \backslash \mathfrak{H}_{2}} \sum_{\gamma \in S} H^{k,j}_{\gamma}(Z) dZ,$$

where the notation $\Gamma^{(*)}(N)$ means that $\Gamma(N)$ or $\Gamma^{*}(N)$, and $F^{(*)}$ is the fundamental domain of $\Gamma^{(*)}(1)$ in $\mathcal{S}_{2}$.

For a subset $S$ of $\Gamma$, we put

$$I(S) = \frac{c_{k,j}}{\#(Z(\Gamma))} \int_{\Gamma \backslash \mathfrak{H}_{2}} \sum_{\gamma \in S} H^{k,j}_{\gamma}(Z) dZ.$$

We call this value $I(S)$ the contribution of $S$ to the dimension formula. We put

$$\Pi_{r} = \left\{ \gamma \in \Gamma(N) ; \gamma \text{ is } \Gamma(1)\text{-conjugate to } \begin{pmatrix} I_{2} & u \\ 0 & I_{2} \end{pmatrix} \right\},$$

$$\Pi_{0}^{*} = \{ I_{2} \},$$

$$\Pi_{2}^{*} = \left\{ \gamma \in \Gamma^{*}(N) ; \gamma \text{ is } \Gamma^{*}(1)\text{-conjugate to } \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \gamma \neq 0, \text{tr}(u) = 0 \right\}.$$
Lemma 4.1. Let \( \gamma \in \Gamma \) and \( Z \in \mathcal{H}_2(\mu) \). Then there exists a constant \( C \), which depends only on \( j \) and \( \mu \), such that
\[
|H_{k}^{h}(Z)| < C_{j,\mu} \times |H_{k}^{j}(Z)|.
\]
The constant \( C_{j,\mu} \) is independent of \( \gamma \) and \( Z \).

By Lemma 4.1, we can reduce the problems of absolute convergences of vector valued case to those of the scalar valued case. Therefore we can generalize Morita’s method [13] to the vector valued case, and calculate the vanishing part. Let \( \gamma \in \Gamma^{(*)}(N) \) \((N \geq 3)\) and \( \gamma \not\in \Pi^{(*)} \). From the results of [1], [13] and the above lemma, we can express the contribution of the \( \Gamma^{(*)}(1) \)-conjugacy classes of \( \gamma \) as
\[
\lim_{s \to 0} \int_{F_{\gamma,s}^{(*)}} I_{\gamma}^{k,j}(Z) dZ,
\]
where \( F_{\gamma,s}^{(*)} \) is the certain domain satisfying \( \lim_{s \to 0} F_{\gamma,s}^{(*)} = F_{\gamma}^{(*)} \) (see, [1],[13]) and \( F_{\gamma}^{(*)} \) is the fundamental domain of the centralizer of \( \gamma \). Furthermore we see that the integral
\[
\int_{F_{\gamma,s}^{(*)}} H_{\gamma}^{k,j}(Z) dZ = \sum_{m \in \mathbb{Z} \leq m \leq j+k} \int_{D} \left\{ \int_{-\infty}^{\infty} (f_{1}(P)p + f_{2}(P))^{-m} f_{1,m}(P)p^{l} dp \right\} dP,
\]
where \( F_{\gamma,s}^{(*)} \cong (-\infty, \infty) \times D_{s} \), \( dZ = dpdP \), \( f_{1}, f_{2}, f_{1,m} \) are polynomials of \( P \), and \( f_{1}(P)p + f_{2}(P) \neq 0 \) \((\gamma(p, P) \in (-\infty, \infty) \times D_{s}) \) (cf. [1], [3], [13]). From the partial integration and \( \int_{-\infty}^{\infty} (ap+b)^{-m} dp = \left[ a^{-1}(-m+1)^{-1}(ap+b)^{-m+1} \right]_{-\infty}^{\infty} = 0 \), we see that the contribution is zero.

Theorem 4.2. Let \( \gamma \in \Gamma^{(*)}(N) \) \((N \geq 3)\) and \( \gamma \not\in \Pi^{(*)} \). The contribution of \( \Gamma^{(*)}(1) \)-conjugacy classes of \( \gamma \) to the dimension formula is zero.

5. Non-vanishing Part

In this section, we calculate explicitly the contributions of \( \Pi_{r} \) and \( \Pi^{*}_{r} \):

5.1. Contribution of \( \Pi_{0} \) and \( \Pi^{*}_{0} \). From Corollary 3.2 and \( H_{I_{4}}^{j}(Z) = j+1 \), we get
\[
I(\Pi_{0}) = 2^{-1}c_{k,j} |\Gamma^{(*)}(1):\Gamma^{(*)}(N)|(j + 1) \int_{F^{(*)}} dZ.
\]
The volume of the fundamental domain for \( \Gamma(1) \) (resp. \( \Gamma^{*}(1) \)) was given explicitly by Siegel [16] (resp. Arakawa [1]). So we get the contributions
\[
I(\Pi_{0}) = [\Gamma(1):\Gamma(N)] \times 2^{-8} 3^{-3} 5^{-1} (j + 1)(j + k - 1)(j + 2k - 3),
\]
\[
I(\Pi^{*}_{0}) = [\Gamma^{*}(1):\Gamma^{*}(N)] \times 2^{-8} 3^{-3} 5^{-1} (j + 1)(j + k - 1)(j + 2k - 3)
\]
\[
\times \prod_{p \mid D(B)} (p - 1)(p^2 + 1).
\]
5.2. Fourier transformation. We assume that $r$ is equal to 1 or 2. We put $V_r = \{x \in M(r; \mathbb{R}); \ ^t x = x\}$, $\Omega_r = \{x \in V_r; \ x > 0\}$. For $x \in V_r$, we put 

\[ f_r^*(x) = \text{tr} \left[ \rho_{k,j} \begin{pmatrix} 1 - \sqrt{-1}x & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} \quad \text{(r = 1),} \quad \text{tr} \left[ \rho_{k,j} (I_2 - \sqrt{-1}x) \right]^{-1} \quad \text{(r = 2)}.
\]

For $x \in \Omega_1$, we set 

\[ f_1(x) = \sum_{i=0}^{j} (2\pi)^{k+i+1} \Gamma(k + j)^{-1} x^{k+j+1} \exp(-2\pi x), \]

where $\Gamma(s)$ is the Gamma function. For $x \notin \Omega_1$, we set $f_1(x) = 0$. The spherical polynomial $\Phi_m(x)$ for $m = (m_1, m_2) \in \mathbb{Z}_{\geq 0}$ ($m_1 \geq m_2$) is defined by $\Phi_m(x) = \int_{SO(2, \mathbb{R})} \Delta_m (^t gxg) dg$, where $\Delta_m(x) = x_1^{m_1-m_2} \det(x)^{m_2}$ and $dg$ is the Haar measure on $SO(2, \mathbb{R})$ normalized by $\int_{SO(2, \mathbb{R})} dg = 1$. Since $\text{tr}(\rho_{k,j}(x))$ is invariant for the action $x \mapsto ^t gxg$ ($g \in SO(2; \mathbb{R})$), we see that $\text{tr}(\rho_{k,j}(x)) = \sum_{m_1 + m_2 = 2k+j, m_2 \geq k} a_m \Phi_m(x)$ ($a_m \in \mathbb{R}$) (cf. [4]). For $x \in \Omega_2$, we set 

\[ f_2(x) = \sum_{m_1 + m_2 = 2k+j, m_2 \geq k} (2\pi)^{(1/2) + m_1 + m_2} a_m \Phi_m(x) \det(x)^{-3/2} \exp(-2\pi \text{tr}(x)). \]

For $x \notin \Omega_2$, we set $f_2(x) = 0$. We denote by $dx$ the Lebesgue measure on $V_r$. As for the scalar valued case ($j = 0$), the following lemma is due to Shintani [14] and Siegel [15].

Lemma 5.1. (i) If $-1 < \text{Re}(s) < k - r$, then the integral $\int_{V_r} f_r^*(x)|\det(x)|^s dx$ is absolutely convergent.

(ii) If $k > (r-1)/2$, then we get $\int_{V_r} f_r(x) \exp(2\pi i \text{tr}(xy)) dx = f_r^*(y)$. This integral is absolutely convergent.

We need the following lemma to calculate the contributions. From [5, Exposé 6, Théorème 6], we easily get the following lemma.

Lemma 5.2. Suppose $k > 2$. Then we have

\[ f_2(x) = \left\{ \begin{array}{cl} 2^{-5+2k+j+1} \text{tr}(\rho_{k,j}(x) H_{k,j}^{-1}) \det(x)^{-3/2} \exp(-2\pi \text{tr}(x)) & (x \in \Omega_2) \\ 0 & (x \notin \Omega_2) \end{array} \right. \]

where $H_{k,j} = \int_{\Omega_2} \rho_{k,j}(x) \exp(-\pi \text{tr}(x)) \det(x)^{-3/2} dx$.

5.3. Zeta integrals. We define the zeta integral $Z(P_r, L_r, s)$ by

\[ Z(P_r, L_r, s) = \int_{G_+/D} \det(g)^{2s} \sum_{x \in L_r - L_r \cap \{x \in V_r; \det(x) = 0\}} P_r( ^t gxg) dg \]

where $P_r$ is a function on $V_r$, $G_+ = \{g \in GL(r; \mathbb{R}); \det(g) > 0\}$, $D = SL(r; \mathbb{Z})$ or $\mathfrak{D}^\times$ (the unit group with norm 1 of $\mathfrak{D}$), $L_r$ is a $D$-invariant lattice of $V_r$, $dg$ is the Haar measure on $G_+$, defined by $\det(g)^{-r} \prod_{1 \leq i \leq r} \det(g_{ij})$.

For the case $D = SL(r; \mathbb{Z})$, we set $L_r = \{x \in M(r; \mathbb{Z}); \ ^t x = x\}$ or its dual lattice. For the case $D = \mathfrak{D}^\times$, we set $L_2 = \{x \in \mathfrak{D}; \det(x) = 0\}$ or its dual lattice. In
[18], we have proved the convergence, the functional equation and the meromorphic continuous of the zeta integral. The following proposition is a generalization of the results of [14] and [1] of the scalar valued case.

**Proposition 5.3.** (i) The integral $Z(f_r, L_r, s)$ is absolutely convergent if $\Re(s) > (r + 1)/2$ and $\Re(k + s) > r$. The integral $Z(f_r, L_r, s)$ is a meromorphic function of $s$ on $\mathbb{C}$.

(ii) The case $D = SL(r; \mathbb{Z})$. If $\Re(s) > (r - 1)/2$ and

\[
\begin{cases}
  k > 1, \quad \Re(s) < k & \text{for } r = 1 \\
  k > 4, \quad 2\Re(s) < k & \text{for } r = 2
\end{cases}
\]

then the integral $Z(f^*_r, L^*_r, s)$ is absolutely convergent. The integral $Z(f^*_r, L^*_r, s)$ is a meromorphic function of $s$ on $\mathbb{C}$.

(ii') The case $D = \mathcal{O}^\times$. If $0 < \Re(s) < k - 1/2$, then the integral $Z(f^*_r, L^*_r, s)$ is absolutely convergent. The integral $Z(f^*_r, L^*_r, s)$ is a meromorphic function of $s$ on $\mathbb{C}$.

(iii) We have the functional equation

\[
Z(f_r(x), L_r, s) = \text{vol}(L_r)^{-1}Z(f^*_r(x), L^*_r, (r + 1)/2 - s).
\]

5.4. Contributions of $\Pi_1$, $\Pi_2$ and $\Pi^*_2$.

**Theorem 5.4.** If $k \geq 5$, then we obtain

\[
\begin{align*}
I(\Pi_1) &= \left[\Gamma(1) : \Gamma(N)\right] \times (-1)2^{-6}3^{-2}(j + 1)(j + 2k - 3)N^{-2}, \\
I(\Pi_2) &= \left[\Gamma(1) : \Gamma(N)\right] \times 2^{-5}3^{-1}(j + 1)N^{-3}, \\
I(\Pi^*_2) &= \left[\Gamma^*(1) : \Gamma^*(N)\right] \times 2^{-4}3^{-1}(j + 1)N^{-3} \prod_{p|D(B)}(p - 1).
\end{align*}
\]

**Proof.** We put $L_r = \{x \in M(r; \mathbb{Z}); \xi(x) = x\}$ in case of $\Gamma(N)$, $L_r = \{x \in \mathcal{O}; \text{tr}(x) = 0\}$ in case of $\Gamma^*(N)$. By the method of [14, Section 3, Chapter 2] and Proposition 5.3 (ii), we get

\[
I(\Pi^*_2) = c_{k,j} \times c^{(*)}(r) \times \left[\Gamma^*(1) : \Gamma^*(N)\right] \times Z(f^*_r, L^*_r, 2 - 2^{-1}(r - 1)),
\]

where $c(1) = 2 \times 3^{-1}N^{-2}2^r$, $c(2) = 2^3N^{-3} \pi^{-1}$, $c^*(2) = 2^3N^{-3} \pi^{-1}D(B)$. By the functional equation, we get

\[
Z(f^*_r, L^*_r, 2 - 2^{-1}(r - 1)) = \text{vol}(L^*_r) \times Z(f^*_r, L^*_r, r - 2),
\]

where $\text{vol}(L^*_r) = 1$ for $\Pi_1$, $2^{-1}$ for $\Pi_2$, $2^{-1}D(B)^{-1}$ for $\Pi^*_2$. Furthermore we have

\[
Z(f_r, 2N^{-1}L^*_r, r - 2) = \xi^*_r(r - 2) \times P_r.
\]

Here $2 \times \xi_1(s)$ is the Riemann zeta function, $\xi^*_2(s)$ is zeta functions of quadratic forms, $\xi_2(s)$ is defined in [14], $\xi^*_2(s)$ is defined in [1], and we set the integrals

\[
P_1 = \int_{\Omega_1} f_1(x)x^{-2}dx, \quad P_2 = \int_{\Omega_2} f_2(x)\det(x)^{-3/2}dx.
\]

We know $\xi_1(-1) = -1/24$. The special value $\xi_2(0) = 2^{-5}3^{-1} \pi$ was given in [14]. The special value $\xi^*_2(0) = 2^{-4}3^{-1} \pi \prod_{p|D(B)}(p - 1)$ was given in [1]. By simple calculation,
we get $P_1 = (2\pi)^2(j+1)(k-2)^{-1}(j+k-1)^{-1}$. By Lemma 5.2, we calculate

$$P_2 = 2^{-5+2k+j}c_{k,j}^{-1}\int_{\Omega_2} \text{tr}(\rho_{k,j}(x)H_{k,j}^{-1})\exp(-2\pi \text{tr}(x)) \det(x)^{-3} dx$$

$$= 2^{-2}c_{k,j}^{-1}\int_{\Omega_2} \text{tr}(\rho_{k,j}(x)H_{k,j}^{-1})\exp(-\pi \text{tr}(x)) \det(x)^{-3} dx$$

$$= 2^{-2}c_{k,j}^{-1}\text{tr} \left\{ \left( \int_{\Omega_2} \rho_{k,j}(x)\exp(-\pi \text{tr}(x)) \det(x)^{-3} dx \right) H_{k,j}^{-1} \right\}$$

$$= 2^{-2}c_{k,j}^{-1}\text{tr}(H_{k,j}H_{k,j}^{-1}) = 2^{-2}c_{k,j}^{-1}(j+1).$$

So we get explicitly the contributions of $\Pi_1$, $\Pi_2$ and $\Pi_2^*$. □

APPENDIX A. DIMENSION FORMULA FOR $\Gamma(1)$

The following dimension formula is due to [17]. For the scalar valued case ($j = 0$), the dimension was also calculated in [11] and [7]. Let $i$, $\rho$, $\omega$ and $\sigma$ be $\sqrt{-1}$, $e^{2\pi i/3}$, $e^{2\pi i/5}$ and $e^{\pi i/6}$ respectively. We denote $\text{tr}_{\mathbb{Q}[\alpha]/\mathbb{Q}}$ by $\text{tr}_\alpha$ for an algebraic number $\alpha$. We remark on $\text{dimc}_S k_{ij}(Sp(2;\mathbb{Z})) = 0$ if $j$ is odd.

Theorem A.1 (R. Tsushima). $k \geq 5$, $j > 0$ or $k \geq 4$, $j = 0$. $j$ is even.

\[
\text{dimc}_S k_{ij}(Sp(2;\mathbb{Z})) =
2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) - 2^{-5}3^{-2}(j+1) + 2^{-3}j\]

\[+(-1)^k(2^{-7}3^{-2}7(k-2)(j+k-1) - 2^{-4}3^{-1}(j+2k-3) + 2^{-5}3)\]

\[+(-1)^{j/2}(2^{-7}3^{-1}5(j+k-1) - 2^{-3} + (-1)^k(-1)^{j/2}2^{-7}(j+1)\]

\[+\text{tr}_i(i)^{k}(-i)^{j/2}(2^{-6}3^{-1}(i+j-1) - 2^{-4}(i)) + \text{tr}_i(-1)^k(i)^{j/2}2^{-5}(i+1)\]

\[+\text{tr}_i(i)^{k}(-i)^{j/2}(2^{-6}3^{-1}(k-2) - 2^{-4} + \text{tr}_i(-1)^k(i)^{j/2}2^{-5}(i+1)\]

\[+\text{tr}_i(-1)^k(i)^{j/2}3^{-3}(\rho+1) + \text{tr}_\rho(\rho)^{k}(\rho)^{j/2}2^{-2}3^{-4}(2\rho+1)(j+1)\]

\[+\text{tr}_\rho(\rho)^{k}(-\rho)^{j/2}2^{-2}3^{-2}(2\rho+1) + \text{tr}_\rho(-\rho)^{k}(\rho)^{j/2}3^{-3}\]

\[+\text{tr}_\rho(\rho)^{k}(2^{-1}3^{-4}(1-\rho)(j+2k-3) - 2^{-1}3^{-2}(1-\rho))\]

\[+\text{tr}_\rho(\rho)^{k}(2^{-3}3^{-4}(2+\rho)(j+k-1) - 2^{-2}3^{-3}(6+5\rho))\]

\[+\text{tr}_\rho(-\rho)^{k}(2^{-3}3^{-3}(2+\rho)(j+k-1) - 2^{-2}3^{-2}(2+\rho))\]

\[+\text{tr}_\rho(\rho)^{k}(2^{-3}3^{-4}(1-\rho)(k-2) + 2^{-2}3^{-3}(-5+\rho))\]

\[+\text{tr}_\rho(-\rho)^{k}(2^{-3}3^{-3}(1-\rho)(k-2) - 2^{-2}3^{-2}(1-\rho))\]

\[+\text{tr}_\omega(\omega)^{k}(\omega^4)^{j/2}5^{-2} - \text{tr}_\omega(\omega)^{k}(\omega^3)^{j/2}5^{-2}\omega^2\]

\[+\text{tr}_\sigma(\sigma)^{k}(-1)^{j/2}2^{-3}3^{-2}(\sigma^2+1) - \text{tr}_\sigma(\sigma)^{k}(\sigma^3)^{j/2}2^{-3}3^{-2}(\sigma+\sigma^3)\]
APPENDIX B. DIMENSION FORMULA FOR $\Gamma^*(1)$

In order to get the dimension formula for $\Gamma^*(1)$, we need to calculate explicitly the contributions of elliptic elements and quasi-unipotent elements. Because $\Gamma^{(a)}(N)$ ($N \geq 3$) have no such elements (cf. [13], [1] and [6]). We can get easily the contributions of elliptic elements by the results of [12], [8] and [9]. So we have only to calculate explicitly the orbital integrals of quasi-unipotent elements (we calculated it explicitly in [18]). For the scalar valued case, the orbital integrals were calculated in [7].

For the scalar valued case ($j = 0$), the following dimension formula was given by Hashimoto [8]. We generalized it to the vector valued case in [18]. We also remark on $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1)) = 0$ if $j$ is odd.

**Theorem B.1.** $k \geq 5$. $j$ is even.

\[
\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1)) = \sum_{i=1}^{12} H_i
\]

where $H_i$ is the total contribution of elements $\Gamma^*(1)$ with principal polynomial $f_i(\pm x)$.

and are as follows:

\[
H_1 = H_1^e + H_1^u, \quad H_1^u = 2^{-3}3^{-1}(j + 1) \prod_{p \mid D(B)} (p - 1),
\]

\[
H_1^e = 2^{-7}3^{-5}5^{-1}(j + 1)(k - 2)(j + k - 1)(j + 2k - 3) \times \prod_{p \mid D(B)} (p - 1)(p^2 + 1).
\]

\[
H_2 = 2^{-7}3^{-2}(-1)^k(j + k - 1)(k - 2) \prod_{p \mid D(B)} (p - 1)^2 \times \begin{cases} 7 & \text{if } 2 \notmid D(B) \\ 13 & \text{if } 2 \mid D(B) \end{cases},
\]

\[
H_3 = 2^{-5}3^{-1} \left\{ (j + k - 1) \sin(k - 2) \frac{\pi}{2} - (k - 2) \sin(j + k - 1) \frac{\pi}{2} \right\} \prod_{p \mid D(B)} (p - 1) \left( 1 - \left( \frac{-1}{p} \right) \right).
\]

\[
H_4 = 2^{-3}3^{-3} \left\{ (j + k - 1) \sin(k - 2) \frac{2\pi}{3} - (k - 2) \sin(j + k - 1) \frac{2\pi}{3} \right\}
\]

\[
\times \left( \sin \frac{2\pi}{3} \right)^{-1} \prod_{p \mid D(B)} (p - 1) \left( 1 - \left( \frac{-3}{p} \right) \right).
\]
$H_5 = 2^{-3}3^{-2}\left\{(j + k - 1)\sin(k - 2)\frac{\pi}{3} - (k - 2)\sin(j + k - 1)\frac{\pi}{3}\right\}$
\[\times \left(\sin\frac{\pi}{3}\right)^{-1} \times \prod_{p\mid D(B)} (p - 1)\left(1 - \left(-\frac{3}{p}\right)\right)\].

$H_6 = H_6^{pe} + H_6^{e}$,
$H_6^{pe} = -2^{-3}(-1)^{j/2} \prod_{p\mid D(B)} \left(1 - \left(-\frac{1}{p}\right)\right)$,
$H_6^{e} = 2^{-3}3^{-2}(j+k-1)\sin(k-2)\frac{\pi}{3} - (k-2)\sin(j+k-1)\frac{\pi}{3} \times \prod_{p\mid D(B)} \left(1 - \left(-\frac{1}{p}\right)\right)$.

$H_7 = 2^{-7}3^{-1}(-1)^{j/2+k}(j+1) \sum_{D_0\mid 2D(B) \neq D_0} \prod_{q\mid D_0} (q-1) \times \prod_{p\mid 2D(B)/D_0} \left(1 - \left(-\frac{3}{p}\right)\right) \times A$
$+ 2^{-7}3^{-1}(-1)^{j/2}(j+2k-3) \sum_{D_{\epsilon}\mid 3D(B)} \prod_{q\mid D_{\epsilon}} (q-1) \times \prod_{p\mid 3D(B)/D_{\epsilon}} \left(1 - \left(-\frac{3}{p}\right)\right) \times B$

$A (\text{resp. } B) = \begin{cases} 3 & \text{if } 2 \! \not\mid D(B), 2 \! \not\mid D^* \\ 5 & \text{if } 2 \mid D(B), 2 \mid D^*; \text{ or } 2 \mid D(B), 2 \mid D^* \\ 11 & \text{if } 2 \mid D(B), 2 \not\mid D^* \end{cases}$

and $D^* = D_0$ (resp. $D_\epsilon$) runs through the set of divisors of $2D(B)$ which are the product of odd (resp. even) number of distinct primes.

$H_7 = H_7^{pe} + H_7^{e}$,
$H_7^{pe} = -2^{-1}3^{-1}\left(\sin(j+1)\frac{2\pi}{3}\right) \left(\sin\frac{2\pi}{3}\right)^{-1} \prod_{p\mid D(B)} \left(1 - \left(-\frac{3}{p}\right)\right)$,
$H_7^{e} = 2^{-3}3^{-3}(j+1)\sin(j+2k)\frac{2\pi}{3} \left(\sin\frac{2\pi}{3}\right) \times \prod_{p\mid 3D(B)/D_{\epsilon}} \left(1 - \left(-\frac{3}{p}\right)\right) \times A$
$+ 2^{-3}3^{-3}(j+2k-3) \sin(j+1)\frac{2\pi}{3} \left(\sin\frac{2\pi}{3}\right) \times \prod_{p\mid 3D(B)/D_{\epsilon}} \left(1 - \left(-\frac{3}{p}\right)\right) \times B$

$A (\text{resp. } B) = \begin{cases} 1 & \text{if } 3 \! \not\mid D^* \\ 4 & \text{if } 3 \mid D(B), 3 \not\mid D^* \\ 16 & \text{if } 3 \mid D(B), 3 \mid D^* \end{cases}$

and $D^* = D_0$ (resp. $D_\epsilon$) runs through the set of divisors of $3D(B)$ which are the product of odd (resp. even) number of distinct primes.

We set $C(\mu, \nu) = c(\mu, \nu) + c(-\mu, \nu) + c(\mu, -\nu) + c(-\mu, -\nu)$, where

$c(\mu, \nu) = \frac{e^{\sqrt{-1}(k-2)\mu}e^{\sqrt{-1}(j+k-1)\nu} - e^{\sqrt{-1}(j+k-1)\mu}e^{\sqrt{-1}(k-2)\nu}}{1 - e^{2\sqrt{-1}\nu}}$.

$H_8 = 2^{-3}3^{-2}C\left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \prod_{p\mid D(B)} \left(1 - \left(-\frac{1}{p}\right)\right) \left(1 - \left(-\frac{3}{p}\right)\right)$.

$H_9 = 2^{-1}3^{-2}C\left(\frac{\pi}{2}, \frac{\pi}{3}\right) \prod_{p\mid D(B), p \neq 1} \left(1 - \left(-\frac{3}{p}\right)\right)^2 \times \begin{cases} 2 & \text{if } 2 \mid D(B) \\ 5 & \text{if } 2 \mid D(B) \end{cases}$.
\[ H_{10} = 2^{-1}5^{-1}C \left( \frac{2\pi}{5}, \frac{4\pi}{5} \right) \prod_{p \mid D(B)} 2 \times \prod_{p \not\mid D(-1;5)} 2 \times \begin{cases} 0 & \text{if } \bigcup_{i=1}^{n} D(i;5) \neq \emptyset \\ 1 & \text{if } \bigcup_{i=1}^{n} D(i;5) = \emptyset, \ 5 \mid D(B) \end{cases}, \]

where we set \( D(i; j) = \{ p \mid D(B); p \equiv i (\text{mod} j) \}. \)

\[ H_{11} = 2^{-3}C \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \prod_{p \mid D(B), p \not\equiv 2} 2 \times \prod_{p \not\equiv D(-1;8)} 2 \times \begin{cases} 0 & \text{if } D(1;8) \neq \emptyset \\ 1 & \text{if } D(1;8) = \emptyset \end{cases}. \]

if the otherwise \( H_{12} = 2^{-2}3^{-1} \prod_{p \mid D(B)} 2 \times \prod_{p \not\equiv D(-1;12)} 2 \times \left( c \left( \frac{5\pi}{6} \right) + c \left( \frac{\pi}{6}, \frac{-5\pi}{6} \right) \right) \times A \]

\[ + 2^{-2}3^{-1} \prod_{p \mid D(B)} 2 \times \prod_{p \not\equiv D(-1;12)} 2 \times \left( c \left( \frac{\pi}{6}, \frac{-5\pi}{6} \right) + c \left( \frac{5\pi}{6} \right) \right) \times B, \]

where

(i) if \( 2 \parallel D(B), \ 3 \parallel D(B), \)

\[ A \ (\text{resp. } B) = \begin{cases} 1/2 & \text{if } D(-1;12) \neq \emptyset \\ 0 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is even (resp. odd)} \\ 1 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is odd (resp. even).} \end{cases} \]

(ii) if \( 2 \parallel D(B), \ 3 \parallel D(B), \)

\[ A \ (\text{resp. } B) = \begin{cases} 3/4 & \text{if } D(-1;12) \neq \emptyset \\ 1/2 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is even (resp. odd)} \\ 1 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is odd (resp. even).} \end{cases} \]

(iii) if \( 2 \parallel D(B), \ 3 \parallel D(B), \)

\[ A \ (\text{resp. } B) = \begin{cases} 3/4 & \text{if } D(-1;12) \neq \emptyset \\ 1 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is even (resp. odd)} \\ 1/2 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is odd (resp. even).} \end{cases} \]

(iv) if \( 6 \parallel D(B), \)

\[ A \ (\text{resp. } B) = \begin{cases} 9/8 & \text{if } D(-1;12) \neq \emptyset \\ 5/4 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is even (resp. odd)} \\ 1 & \text{if } D(-1;12) = \emptyset, \#D(5;12) \text{ is odd (resp. even).} \end{cases} \]

\[ \left( \frac{-1}{p} \right) = \begin{cases} 0, & p = 2, \\ 1, & p \equiv 1 \ (\text{mod} 4), \\ -1, & p \equiv 3 \ (\text{mod} 4), \end{cases} \]

\[ \left( \frac{-3}{p} \right) = \begin{cases} 0, & p = 3, \\ 1, & p \equiv 1 \ (\text{mod} 3), \\ -1, & p \equiv 2 \ (\text{mod} 3). \end{cases} \]

Principal polynomials (see, [9]) : 

\[ f_{1}(x) = (x - 1)^{4}, \ f_{1}(-x), \ f_{2} = (x - 1)^{2}(x + 1)^{2}, \ f_{3}(x) = (x - 1)^{2}(x^{2} + 1), \ f_{6}(x), \]

\[ f_{4}(x) = (x - 1)^{2}(x^{2} + x + 1), \ f_{4}(-x), \ f_{6}(x) = (x - 1)^{2}(x^{2} - x + 1), \ f_{6}(-x), \ f_{6}(x) = (x^{2} + 1)^{2}, \]

\[ f_{7}(x) = (x^{2} + x + 1)^{2}, \ f_{8}(x) = (x^{2} + 1)(x^{2} + x + 1), \ f_{8}(-x), \ f_{8}(x) = (x^{2} + x + 1)(x^{2} - x + 1), \]

\[ f_{10}(x) = (x^{4} + x^{3} + x^{2} + x + 1), \ f_{10}(-x), \ f_{11}(x) = (x^{4} + 1), \ f_{12}(x) = (x^{2} - x^{2} + 1). \]
Numerical examples of \( \dimc S_{k,j}(\Gamma^*(1)) \).

(i) \( D(\mathbf{B}) = 2 \times 3 \).

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(ii) \( D(\mathbf{B}) = 2 \times 5 \).

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REFERENCES


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