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ON CRITICAL VALUES OF ADJOINT \( L \)-FUNCTIONS FOR GSp(4)

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1. INTRODUCTION

Let \( f \in S_k(SL(2, \mathbb{Z})) \) be a normalized Hecke eigenform and \( \pi = \otimes_v \pi_v \) the irreducible cuspidal automorphic representation of \( GL(2, \mathbb{A}_\mathbb{Q}) \) determined by \( f \). Then the result of Rankin [12] says that
\[
L(1, \pi, \text{Ad}) = C_\infty \langle f, f \rangle,
\]
where \( \text{Ad} : GL(2, \mathbb{C}) \to GL(3, \mathbb{C}) \) is the adjoint representation, \( C_\infty = 2^k \) is a constant which depends only on \( \pi_\infty \), and
\[
\langle f, f \rangle = \int_{SL(2, \mathbb{Z}) \backslash \mathbb{G}} |f(\tau)|^2 \text{Im}(\tau)^{k-2} d\tau
\]
is the Petersson norm of \( f \). This formula was generalized to the case of \( GL(n) \) by Jacquet, Piatetski-Shapiro, and Shalika [6]. In this note, we give an analogue for GSp(4).

2. DELIGNE' S CONJECTURE [3]

We first give some speculation about the transcendental part of critical values of adjoint \( L \)-functions for GSp(4). Let \( f_{\text{hol}} \) be a Siegel cusp form of degree 2 and of weight \( k \) with respect to \( \text{Sp}(4, \mathbb{Z}) \). We assume that \( f_{\text{hol}} \) is a Hecke eigenform and is not a Saito-Kurokawa lift. Let \( \pi_{\text{hol}} \) be the irreducible cuspidal automorphic representation of \( \text{GSp}(4, \mathbb{A}_\mathbb{Q}) \) determined by \( f_{\text{hol}} \). By Arthur's conjecture [1], there would exist an irreducible generic cuspidal automorphic representation \( \pi_{\text{gen}} \) of \( \text{GSp}(4, \mathbb{A}_\mathbb{Q}) \) such that \( \Pi = \{ \pi_{\text{hol}}, \pi_{\text{gen}} \} \) is an \( L \)-packet. Namely,
\[
L(s, \pi_{\text{hol}}, r) = L(s, \pi_{\text{gen}}, r)
\]
for any finite dimensional representation \( r \) of \( \text{GSp}(4, \mathbb{C}) \). Let \( M \) be the hypothetical motive attached to the spinor \( L \)-function of \( f_{\text{hol}} \). Then \( M \) would be of rank 4 and of pure weight \( 2k - 3 \). Moreover, the Hodge decomposition
\[
H_{\text{DR}}(M) \otimes \mathbb{C} \cong H^{2k-3,0} \oplus H^{k-1,k-2} \oplus H^{k-2,k-1} \oplus H^{0,2k-3}
\]
would have a basis
\[
\{ f_{\text{hol}}, f_{\text{gen}}, \overline{f_{\text{gen}}}, \overline{f_{\text{hol}}} \}.
\]
Here \( f_{\text{gen}} \) is an element of \( \pi_{\text{gen}} \). By Yoshida's formula [13, (4.15)], we have
\[
c^+(\text{Sym}^2(M)) = (2\pi \sqrt{-1})^{12-6k} c^+(M)c^-(M)(f_{\text{hol}}, f_{\text{hol}}),
\]
where \( c^+(\text{Sym}^2(M)) \) is Deligne's period of \( \text{Sym}^2(M) \), etc. Moreover, the relative trace formula of Furusawa and Shalika [4] suggests that the equality
\[
\frac{|B_D(1)|^2}{\langle f_{\text{hol}}, f_{\text{hol}} \rangle} = L\left(\frac{1}{2}, \Pi \right) L\left(\frac{1}{2}, \Pi \otimes \chi_D \right) \frac{|W(1)|^2}{\langle f_{\text{gen}}, f_{\text{gen}} \rangle}
\]
should hold up to an elementary constant. Here \( D < 0 \) is a fundamental discriminant, \( \chi_D \) is the Dirichlet character associated to \( \mathbb{Q} \left( \sqrt{D} \right) / \mathbb{Q} \), \( B_D \) is the \( D \)-th Bessel function of \( f_{\text{hol}} \), and \( W \) is the Whittaker function of \( f_{\text{gen}} \). This leads to speculation that
\[
c^+(\text{Sym}^2(M)) = \langle f_{\text{gen}}, f_{\text{gen}} \rangle.
\]

3. Result

We now give a precise description of our result. Let
\[
\text{GSp}(4) = \left\{ g \in \text{GL}(4) \mid g \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, g = \nu(g) \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \nu(g) \in \text{G}_m \right\}
\]
be the symplectic similitude group in four variables. Let \( \pi = \otimes_v \pi_v \) be an irreducible generic cuspidal automorphic representation of \( \text{GSp}(4, \mathbb{A}_{\mathbb{Q}}) \) with trivial central character. We assume that
- \( \pi_p \) is unramified for all primes \( p \),
- \( \pi_{\infty}|_{\text{Sp}(4, \mathbb{R})} = D_{(\lambda_1, \lambda_2)} \oplus D_{(-\lambda_2, -\lambda_1)} \) with \( 1 - \lambda_1 \leq \lambda_2 \leq 0 \).

Here \( D_{(\lambda_1, \lambda_2)} \) is the (limit of) discrete series representation of \( \text{Sp}(4, \mathbb{R}) \) with Blattner parameter \( (\lambda_1, \lambda_2) \). By [2], \( \pi \) has a functorial lift \( \Pi \) to \( \text{GL}(4, \mathbb{A}_{\mathbb{Q}}) \). We assume that \( \Pi \) is cuspidal.

We consider a non-zero element \( f = \otimes_v f_v \in \pi \) satisfying the following conditions:
- \( f_p \) is \( \text{GSp}(4, \mathbb{Z}_p) \)-invariant for all primes \( p \),
- \( f_{\infty} \) is the lowest weight vector of the minimal \( \text{U}(2) \)-type of \( D_{(-\lambda_2, -\lambda_1)} \).

Note that \( f \) is unique up to scalars. We may normalize \( f \) so that \( W(1) = 1 \), where \( W \) is the Whittaker function of \( f \). Let
\[
\langle f, f \rangle = \int_{\text{A}_Q^\times \text{GSp}(4, \mathbb{Q}) \backslash \text{GSp}(4, \mathbb{A}_Q)} |f(g)|^2 \, dg
\]
be the Petersson norm of \( f \), where \( dg \) is the Tamagawa measure on \( \text{GSp}(4, \mathbb{A}_Q) \).

Our main result is as follows.

**Theorem 3.1** ([5]). There exists a constant \( C_\infty \in \mathbb{C}^\times \) which depends only on \( \pi_{\infty} \) such that
\[
L(1, \pi, \text{Ad}) = C_\infty \langle f, f \rangle.
\]
Here \( \text{Ad} : \text{GSp}(4, \mathbb{C}) \to \text{GL}(10, \mathbb{C}) \) is the adjoint representation.
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4. Proof

We use the following three ingredients:

- the integral representation of $L(s, \pi, \mathrm{St})$,
- the integral representation of $L(s, \pi \times \pi') = \zeta(s)L(s, \pi, \mathrm{St})L(s, \pi, \mathrm{Ad})$,
- the Siegel-Weil formula.

Let $H = \mathrm{GSp}(8)$ and $G = \{(g_1, g_2) \in \mathrm{GSp}(4) \times \mathrm{GSp}(4) | \nu(g_1) = \nu(g_2)\}$.

We identify $G$ with its image under the embedding $G \mapsto H.$

For an automorphic form $\varphi$ on $H(\mathbb{A}_\mathbb{Q})$, let

$$
\langle \varphi|_{G}, \overline{f} \otimes f \rangle = \int_{Z_H(\mathbb{A}_\mathbb{Q})G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})} \varphi((g_1, g_2)) f(g_1) \overline{f(g_2)} \, dg_1 \, dg_2.
$$

Let $P = \{ (_{0}^{a} \, v^{-1}a^{-1} ) \in H | a \in \mathrm{GL}(4), v \in \mathbb{G}_m \}$ be the Siegel parabolic subgroup of $H$. Let $F = \otimes_v F_v$ be a holomorphic section of $\text{Ind}^{H(\mathbb{A}_\mathbb{Q})}_{P(\mathbb{A}_\mathbb{Q})}(\delta_P^{s/5})$, where $\delta_P$ is the modulus character of $P(\mathbb{A}_\mathbb{Q})$. Let $E(s, F)$ be the Siegel Eisenstein series attached to $F$.

**Theorem 4.1** (Piatetski-shapiro and Rallis [11]). We have

$$
\langle E(s, F)|_{G}, \overline{f} \otimes f \rangle = \langle f, f \rangle d_P^S(s)^{-1} L^S(s + \frac{1}{2}, \pi \times \pi') \prod_{v \in S} Z_v(s, \phi_v, F_v).
$$

Here $d_P^S(s) = \zeta^S(s + \frac{5}{2}) \zeta^S(2s + 1) \zeta^S(2s + 3)$, $\phi_v$ is the matrix coefficient of $\pi_v$ associated to $f_v$ such that $\phi_v(1) = 1$, and $Z_v(s, \phi_v, F_v)$ is the local zeta integral.

Let $Q = \{ (_{0}^{a} \, v^{-1}a^{-1} ) \in H | a \in \mathrm{GL}(3), v \in \mathbb{G}_m \}$ be a maximal parabolic subgroup of $H$. Let $\mathcal{F} = \otimes_v \mathcal{F}_v$ be a holomorphic section of $\text{Ind}^{H(\mathbb{A}_\mathbb{Q})}_{Q(\mathbb{A}_\mathbb{Q})}(\delta_Q^{s/6})$, where $\delta_Q$ is the modulus character of $Q(\mathbb{A}_\mathbb{Q})$. Let $\mathcal{E}(s, \mathcal{F})$ be the Eisenstein series attached to $\mathcal{F}$.

**Theorem 4.2** (Jiang [7]). We have

$$
\langle \mathcal{E}(s, \mathcal{F})|_{G}, \overline{f} \otimes f \rangle = d_Q^S(s)^{-1} L^S(s + \frac{1}{2}, \pi \times \pi') \prod_{v \in S} Z_v(s, W_v, \mathcal{F}_v).
$$
Here $d_Q^S(s) = \zeta^S(s+1)\zeta^S(s+2)\zeta^S(s+3)\zeta^S(2s+2)$, $W_{\psi}$ is the Whittaker function of $\pi_{\psi}$ associated to $f_{\psi}$ such that $W_{\psi}(1) = 1$, and $Z_v(s, W_{\psi}, F_{\psi})$ is the local zeta integral.

To compare these two integral representations, we use the Siegel-Weil formula. Recall the analytic behavior of the Eisenstein series:

- $E(s, F)$ has at most a simple pole at $s = \frac{1}{2}$ (Kudla and Rallis [10]),
- $\mathcal{E}(s, F)$ has at most a double pole at $s = 1$ (Jiang [7]).

On the other hand, since $\Pi$ is cuspidal,

- $L^S\left(s + \frac{1}{2}, \pi, \text{St}\right)$ is holomorphic and non-zero at $s = \frac{1}{2}$,
- $L^S\left(s + \frac{1}{2}, \pi \times \pi^\vee\right)$ has a simple pole at $s = 1$.

Hence the first terms in the Laurent expansions of the Eisenstein series do not contribute to special values. This means that we must compare the second terms.

**Proposition 4.3.** There exist $F$ and $\mathcal{F}$ which are $H(\mathcal{Z})$-invariant and which satisfies the following:

- For $\varphi = \text{Res}_{s=1} E(s, F) - \zeta(4)^{-1} \text{CT}_{s=\frac{1}{2}} E(s, F)$, we have
  $$\langle \varphi|_G, \overline{f} \otimes f \rangle = 0.$$
- $Z_{\infty}(s, \phi_{\infty}, F_{\infty})$ is holomorphic and non-zero at $s = \frac{1}{2}$.
- $Z_{\infty}(s, W_{\infty}, F_{\infty})$ is holomorphic and non-zero at $s = 1$.

The proof of this proposition is based on the regularized Siegel-Weil formula of Kudla and Rallis [10], Kudla [9], and Jiang [8].

Now it is easy to check that

$$\frac{L(1, \pi, \text{Ad})}{\langle f, f \rangle} = C'_{\infty} \frac{Z_{\infty}\left(\frac{1}{2}, \phi_{\infty}, F_{\infty}\right)}{Z_{\infty}(1, W_{\infty}, F_{\infty})},$$

where

$$C'_{\infty} = 2^{-1} \zeta_{\infty}(4)^{-1} L_{\infty}(1, \pi_{\infty}, \text{Ad}) \in C^\times.$$  

Since $\phi_{\infty}$ and $W_{\infty}$ depend only on $\pi_{\infty}$, the right-hand side depends only on $\pi_{\infty}$, $F_{\infty}$, and $F_{\infty}$. However, the left-hand side is independent of $F_{\infty}$ and $F_{\infty}$. This completes the proof of Theorem 3.1.

**References**


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