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On standard L-function for generic cusp forms on $U(2,1)$

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Introduction

The main object of our concern is ramified factors of zeta integrals. By definition, zeta integrals "interpolates" automorphic L-functions to deduce their some analytic properties, say meromorphic continuation. But this is just the first step of study of L-functions. Actually special values, entireness or poles of L-functions encode very deep and fascinating arithmetic information. It is quite hard to investigate these properties. However it has been believed since the beginning of the history of automorphic representations, that through the study of zeta integrals one can reach the depth of arithmetic nature. Besides Jacquet-Langlands theory of the standard L-function for $GL(2)$, we should cite a great work [Rama] of Ramakrishnan, which says that the Garrett integral coincides the triple L-function including the ramified factors with the help of Ikeda's archimedean calculus [Ike]. Takloo-Bighash investigate the Novodvorsky integral to determine local factors of the spinor L-function for the generic representations of $GSp(4)$ [Tak].

In this note we would like to treat the Gelbart Piatetski-Shapiro integral, which are recalled in §1, and report some results on ramified factors of the standard L-function for $U(3)$. In §2, we calculate the "GCD" of p-adic components of the zeta integral for the generic representations of $U(3)$. In §3, we redo the calculation of [K-O] to give a cleaner form of the "GCD" of archimedean component.

Contents

1 Gelbart Piatetski-Shapiro zeta integral

Note that we can obtain the same result without any loss of generality, even if we formulate the problem over an arbitrary totally real algebraic number field. So we take $\mathbb{Q}$ for our ground field.

<Group structure>

Let E be an imaginary quadratic extension of $\mathbb{Q}$ and denote the non-trivial element of its Galois group by $\bar{\cdot}$. Put

$$G := \{g \in GL(3,E) \mid {}^t\bar{g}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} g = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\}.$$

This defines a quasi-split unitary group of three variables over $\mathbb{Q}$. Let

$$G = NTK$$

be the Iwasawa decomposition of G. Then each subgroups are expressed as

$$N = \{\begin{pmatrix} 1 & b & z \\ & 1 & -\bar{b} \\ & & 1 \end{pmatrix} \in G \mid b, z \in E,\ z + \overline{z} = -|b|_E^2\},$$

$$T = \{\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \bar{\alpha}^{-1} \end{pmatrix} \in G \mid \alpha \in E^\times, \beta \in E^{(1)}\}$$

and

$$K = G \cap M_3(\mathcal{O}_E).$$

A Borel subgroup of G is given by

$$B = N \rtimes T.$$

We need a subgroup

$$H := \operatorname{Img}\Big(\iota : U(1,1) \ni \begin{pmatrix} \star & \star \\ \star & \star \end{pmatrix} \mapsto \begin{pmatrix} \star & & \star \\ & 1 & \\ \star & & \star \end{pmatrix} \in G\Big)$$

as the Euler subgroup for a Rankin-Selberg integral. The Iwasawa decomposition of H is

$$H = Z_N A K_H$$

with

$$Z_N = \{\begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} \in G \mid z \in \mathbb{R}\},$$

$$A = \{\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \in G \mid a \in \mathbb{Q}^\times\}$$

and

$$K_H = K \cap H.$$

<The standard L-function>
For a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}) = U(3)_{\mathbb{A}}$ and a Hecke character ξ of E, the standard L-function is defined by a local way as an Euler product

$$L(s; \pi \otimes \xi) := \prod_v L_v(s; \pi_v \otimes \xi_v).$$

For the unramified principal series $\pi_p \cong \mathrm{Ind}_{B_p}^{G_p}(\chi)$, the unramified factor is given by

$$L_p(s; \pi_p \otimes \xi_p) := L_{E,p}(s; \xi_p) L_p(2s; \xi_p \chi) L_p(2s; \xi_p / \chi).$$

<Zeta integral>
For a generic cusp form φ belonging to a generic π, Gelbart and Piatetski-Shapiro introduced the following zeta integral

$$\mathcal{Z}(s; \varphi, \xi) := \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \varphi|_H(h) E^{\xi,H}(s; h) \mathrm{d}h.$$

Here $E^{\xi,H}$ is an Eisenstein series on $H(\mathbb{A})$ corresponding to the principal series $\mathrm{Ind}_{(B\cap H)_{\mathbb{A}}}^{H_{\mathbb{A}}}(\xi)$. By the Langlands theory of Eisenstein series the integral is continued to the whole s-plane.
<Unfolding and local integrals>
Assume the generic cusp form is localizable; $\varphi = \otimes_v \varphi_v$. By using the multiplicity one result on Whittaker models and an unfolding procedure, the Rankin-Selberg integral decomposes into a product of local integrals:

$$\mathcal{Z}(s; \varphi, \xi) = \prod_v \mathcal{Z}_v(s; W, \Phi_\xi^{(s)}),$$

with

$$\mathcal{Z}_v(s; W, \Phi_\xi^{(s)}) := \int_{Z_{N,v} \backslash H_v} W_{\varphi_v}|_{H_v}(h_v) \Phi_\xi^{(s)}(h_v) \mathrm{d}h_v.$$

Here $Z_{N,v}$ is the center of the maximal nilpotent subgroup N_v of G_v, W_{φ_v} is a Whittaker vector corresponding to $\varphi_v \in \pi_v$ and $\Phi_\xi^{(s)}$ is a special section of the principal series $\mathrm{Ind}_{(B\cap H)_p}^{H_p}(\xi|\cdot|^s)$ of H induced up from its Borel subgroup $\iota(\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right))$. Note that this integral vanishes unless φ is a generic cusp form.

Over the places where everything is unramified, Gelbart and Piatetski-Shapiro showed the coincidence of local factors of L-function and zeta integral by using the Casselman-Shalika formula.

Proposition 1 **([Ge-PS] §4)** *For the unramified (i.e. K_p-spherical)π_p's,*

$$\mathcal{Z}_p(s; W, \Phi_\xi^{(s)}) = L_p(s; \pi_p \otimes \xi_p).$$

□

Next step of investigation is to analyze ramified factors. The p-adic case was treated by Baruch in his thesis [Ba]. We record two of his results.

Proposition 2 (**[Ba]**) *For any non-archimedean π_p's, the followings hold.*
i) The family of zeta integrals for the Jacquet sections

$$\{\mathcal{Z}_p(s; W, \Phi^{(s)}_{\xi,\phi}) \mid K_p - \text{finite } W \in Wh_\psi(\pi_p),\ \phi \in \mathcal{S}(\mathbb{Q}_p^2)\}$$

admits the "GCD". Note the $\Phi^{(s)}_{\xi,\phi}$'s is dence in the representation space of princepal series.
ii) There is a rational function $\gamma_p(s; \pi_p, \xi_p; \psi_\mathbb{Q}, \chi_E)$ in q^{-s} such that the local functional equation

$$\mathcal{Z}_p(1-s; W, \Phi^{(s)}_{\xi^{-1},\widehat{\phi}}) = \gamma_p(s; \pi_p, \xi_p; \psi_\mathbb{Q}, \chi_E) \cdot \mathcal{Z}_p(s; W, \Phi^{(s)}_{\xi,\phi})$$

holds. Here $\widehat{\phi}$ is the Fourier transform of ϕ. □

Note that Baruch get the local functional equation by showing the generic multiplicity one for the space of invarient bi-linear forms ;

$$\dim_\mathbb{C} \mathrm{Bil}\left(\pi_p|_H,\ \mathrm{Ind}^{H_p}_{(B\cap H)_p}(\xi|\cdot|^s)\right) \leq 1$$

for almost all of $s \in \mathbb{C}$. And the explicit form of gamma factor has not been known so far.

The remaining problems are following.
Problems
1) Calculate $\mathcal{Z}_p(s; W, \Phi^{(s)}_\xi)$ for "bad" finite places $v = p$ to determine the ramified L-factor $L_p(s; \pi_p \otimes \xi_p)$ as the "GCD".
2) Calculate archimedean $\mathcal{Z}_\infty(s; W, \Phi^{(s)}_\xi)$ and study the "GCD".

2 p-adic factors

There are several ways to analyze ramified L-factors. Here we follow the method of Takloo-Bighash [Tak], where he calculated the Novodvorsky integral for the generic π_p's of $GSp(4)$ and determined ramified L-factors. The strategy, which is divided into three steps, is simple but relys on Shalika's ingenious fundamental work.

Step 1 : Germ expansion of Whittaker functions.
By nature of Whittaker functions, we only need their A_p-radial part, which can be expanded as

$$W_{\varphi_p}(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}) = \sum_{c \in \mathcal{S}_{\pi_p}} \phi_c(a) \cdot c(a),$$

by using finite functions c on $\mathbb{Q}_p^\times$. Here coefficient functions ϕ_c are Schwartz functions and the index set $\mathcal{S}_{\pi_p}$ is a finite set determiner by π_p. The size of set can be bounded by Shalika's argument on distribution. In our case, $|\mathcal{S}_{\pi_p}| \leq 2$.

Step 2 : Shahidi theory of intertwiners.
Let $I_\sigma(\nu)$ denote the principal series $\mathrm{Ind}^{G_p}_{B_p}\sigma|\cdot|^\nu_E$ of G_p. Here σ is given by

$$\begin{array}{cccc} \sigma : & T & \to & \mathbb{C}^\times \\ & \mathrm{diag}(\alpha, \beta, \bar{\alpha}^{-1}) & \mapsto & \chi(\alpha)\chi'(\alpha\beta\bar{\alpha}^{-1}), \end{array}$$

where χ and χ' are quasi-character and character of $E_p^\times$ respectively. And $\nu \in \mathbb{C}$ is a complex parameter. Consider the Whittaker vector corresponding to a section $f \in I_\sigma(\nu)$. Then the A_p-radial part can be expressed as

$$W_f(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}) = \lambda_\sigma(f)\cdot\chi(a) + C(\sigma,w)\Big(\lambda_{\sigma^w}A(\nu;\sigma,w)\Big)(f)\cdot\chi^{-1}(a).$$

Here $\lambda_\sigma : I_\sigma(\nu) \to \mathbb{C}$ is the Whittaker functional for $I_\sigma(\nu)$ and

$$A(\nu;\sigma,w) \ : \ I_\sigma(\nu) \to I_{\sigma^w}(-\nu)$$

is the intertwiner. Similarly $\lambda_{\sigma^w} : I_{\sigma^w}(-\nu) \to \mathbb{C}$ is the Whittaker functional for $I_{\sigma^w}(-\nu)$. The action of Weyl element w is given by $\sigma^w := (\bar{\chi}^{-1}, \chi')$. The proportional factor $C(\sigma, w)$ is Shahidi's local coefficients.

Note that if the section f sits in π_p then f is our φ_p. So we need to know which π_p is kernel/image of the intertwiner. We appeal to

Step 3 : Classification result of π_p's.

A general result of Shahidi says that the reducing points of the standard moddules induced up from *maximal* parabolic subgroup can be 0, ± 1 or $\pm 1/2$. In our case, all of these occur.

Proposition 3 ([Ke]) *The standard module $I_\sigma(\nu)$ is irreducible except the following three cases.*

1) When $\nu = \pm 1$, the constituent of the standard module is given by the exact sequence

$$0 \to St_{\chi'} \to I_\sigma(+1) \to \chi'\cdot\det \to 0.$$

Here $St_{\chi'}$ is the twisted Steinberg representation and is the kernel of $A(+1;\sigma,w)$. For $\nu = -1$, submodule and quotient in the above exact sequence are substituted.

2) When $\nu = \pm 1/2$ and $\chi|_{\mathbb{Q}_p^\times} = \mathrm{sgn}$,

$$0 \to \pi^2 \to I_\sigma(+1/2) \to \pi^{nt} \to 0$$

is an exact sequence. Here π^2 is a square integrable representation and is the kernel of $A(+1/2;\sigma,w)$. The quotient π^{nt} is a non-tempered unitary representation. For $\nu = -1/2$, submodule and quotient are substituted.

3) When $\nu = \pm 0$ and $\chi|_{\mathbb{Q}_p^\times} = \mathbf{1}_{\mathbb{Q}_p^\times}$, $\chi \neq \mathbf{1}_{E_p^\times}$, the standard module decomposes into a direct sum

$$I_\sigma(0) = \pi^{deg} \oplus \pi^{nd},$$

where π^{deg}, π^{nd} stands for irreducible degenerate, irreducible non-degenerate representations respectively. A general result of Shahidi says that non-degenerate representations can not be annihirated by intertwiners. So the kernel of $A(0;\sigma,w)$ is π^{deg}. □

Now we define the local factor $L_p(s;\pi_p \otimes \xi_p)$ as the "GCD" of Baruch's family. Then the above argument gives the following result for special representations and the fact that $Wh_\psi(\pi_p)$ coincides the full $S(\mathbb{Q}_p^\times)$ for super-cuspidals.

Theorem 4 *The local factor $L_p(s;\pi_p\otimes\xi_p)$ of standard L-function for $U(3)$ is given as follows.*

0) When π_p is an irreducible standard module $I_\sigma(\nu)$,

$$L_{E,p}(s;\xi_p)L_p(2s+2\nu;\xi_p\chi)L_p(2s+2\nu;\xi_p/\chi).$$

1) When π_p is the twisted Steinberg representation $St_{\chi'}$,

$$L_{E,p}(s;\xi_p)L_p(2s+2;\xi_p\chi).$$

2) When π_p is the twisted Steinberg representation $St_{\chi'}$,

$$L_{E,p}(s;\xi_p)L_p(2s+1;\xi_p\chi).$$

3) When π_p is the irreducible square-integrable representation π^2,

$$L_{E,p}(s;\xi_p)L_p(2s;\xi_p\chi).$$

4) When π_p is a super-cuspidal representation,

$$L_{E,p}(s;\xi_p).$$

□

We note that Watanabe studied related subject in a broader setting [Wat], where the representations are limitted to regular and tamely ramified though, and covers much of Theorem 4. For the twisted Steinberg representation $St_{\chi'}$, the author culculated the zeta integral for Iwahori sherical vector by using a Casselman-Shalika type formula of Li [Li], which was reported in annual work shop at RIMS in 2004.

3 Archimedean factors

As is seen in the previous section, p-adic story is completely similar to the GL_2-case. No new difficulties come out for $U(3)$. However even for this small group, archimedean factor is hard to handle. In fact we can not mimic Shalika's idea over archimedean places. So we have to find other ways. Here we appeal to a very direct method. That is to compute out the zeta integral by using an explicit formula for Whittaker functions. This strategy is not so smart nor elegant, but sometimes powerful and useful. Gross and Kudla [Gr-Ku] adopted the strategy to compute the ramified factor of the Garrett integral to reduce it to some Igusa local zeta when one of three cusp form has Iwahori level structure. Over archimedean places, Moriyama [Mo], Ishii studied the Novodvorsky integral for $GSp(4)$, and the author [Is] a Shimura type integral for $U(2,1)$ discovered by Shintani [Shi].

We again devide the story into three steps. The crucial step is the first one, where we compute the A_∞-radial part of Whittaker function corresponding to the minimal/corner K_∞-type vectors in discrete series /principal series representations.

Step 1 : Explicit formula for the minimal K_∞-type Whittaker functions.

Again by nature of Whittaker functions, their A_∞-radial part is essencial. Note K_∞ is isomorphic to $U(2) \times U(1)$. So the functions are of vector-valued in the representation space of K_∞-type. Fixing a realization, we can expand the A_∞-radial part with respect to the basis. And the coefficient functions can be determined by solving the system of difference–differential equations. Koseki and Oda obtained an explicit formula for the group $SU(2,1)$. We record a $U(2,1)$-version, which has cleaner form to handle easier, by the Gel'fand-Zetlin realization. For our group $U(2,1)$, the representations admitting Whittaker model is "large" discrete series or principal series representations. We omit the principal series case.

Theorem 5 **([K-O], [I])** *Let π_∞ be a "large" discrete series representation π_Λ with Harish-Chandra parameter $(\Lambda_1, \Lambda_2, \Lambda_3)$, i.e. $\Lambda_1 > \Lambda_3 > \Lambda_2$. If the Whittaker function W for π_Λ has the minimal K_∞-type, this Whittaker vector is written as*

$$W\left(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}\right) = \sum_{\Lambda_1 \geq k \geq \Lambda_2} c_k \cdot a^{\Lambda_1 - \Lambda_2 - \frac{1}{2}} W_{0,k-\Lambda_1-\Lambda_2+\Lambda_3}(2\sqrt{b}a) \times \left(\left| \begin{matrix} \Lambda_1, \Lambda_2 \\ k \end{matrix} \right\rangle \otimes \mathbf{1}_{\Lambda_3} \right).$$

Here c_k and b are constants and $\left| \begin{smallmatrix} \Lambda_1, \Lambda_2 \\ k \end{smallmatrix} \right\rangle$ stands for the Gel'fand-Zetlin base for $U(2)$-representation. □

Step 2 : Recursion relations among arbitrary K_∞-type $\mathcal{Z}_\infty(s; W, \Phi_\xi^{(s)})$.
By the branching rule of $U(2)$-module, we can see

$$\mathcal{Z}_\infty(s; W, \Phi_\xi^{(s)}) = 0 \quad \text{unless} \quad \Lambda_1 \geq m - \Lambda_3 \geq \Lambda_2.$$

Here m is the parameter of Hecke character ; $\xi_\infty(\delta) =: |\delta|_{\mathbb{C}}^t \left(\frac{\delta}{|\delta|}\right)^m$. Moreover when the K_∞-type of Whittaker function W is $[\Lambda_1 + a, \Lambda_2 - b; \Lambda_3 - a + b]$, the $K_{H\infty}$-type of section $\Phi_\xi^{(s)}$ should be $[m - \Lambda_3 + a - b, \Lambda_3 - a + b]$, if not the zeta integral vanishes. Therefore most of the members of family $\{\mathcal{Z}_\infty(s; W, \Phi_\xi^{(s)})\}$ are zero-function. Among the non-trivial $\mathcal{Z}_\infty(s; W, \Phi_\xi^{(s)})$'s there are recusive relations, which is inherited from ones of K_∞-finite Whittaker functions deduced from differential equations.
Step 3 : Normalization of Eisenstein series part.
We go back to principal series of $H_\infty \cong U(1,1)$ and consider the intertwiner

$$A(s; \xi, w) \ : \ \mathrm{Ind}_{(B \cap H)_\infty}^{H_\infty}(\xi_\infty \otimes e^{2s} \otimes 1_N) \to \mathrm{Ind}_{(B \cap H)_\infty}^{H_\infty}(\bar{\xi}_\infty^{-1} \otimes e^{-2s} \otimes 1_N).$$

We normalize this intertwiner following Langlands, Arthur and Shahidi. Put

$$A^*(s; \xi, w) \ := \ \frac{\varepsilon(s; \xi, \psi) L_{E,\infty}(s; \xi)}{L_{E,\infty}(s+1; \xi)} A(s; \xi, w)$$

then we have local functional equation of "Eisenstein series"

$$A^*(-s; \bar{\xi}^{-1}, w) \cdot A^*(s; \xi, w) \ = \ Id.$$

Moreover we take natural sections

$$\Phi_\xi^\natural(s;*) \ := \ L_{E,\infty}^{\mathcal{T}}(s;\xi)\cdot\Phi_\xi(s;*),$$

where $L_{E,\infty}^{\mathcal{T}}(s;\xi)$ is the archimedean factor of Hecke L modified by Harish-Chandra c-function and $\Phi_\xi(s;*)$ is a standard section normalized by $\Phi_\xi(-;e)=1$. Then we have symmetrized functional equation

$$A^*(s;\xi,w)\Phi_\xi^\natural(s) \ = \ \varepsilon(s;\xi,\psi)\cdot\Phi_{\xi^{-1}}^\natural(-s).$$

Now we define the archimedean L-factor $L_\infty(s;\pi_\infty\otimes\xi_\infty)$ as the "GCD" of the family of zeta integrals for K_∞-finite Whittaker vectors and the natural sections. Then the above argument gives the following result.

Theorem 6 *The archimedean factor $L_\infty(s;\pi_\infty\otimes\xi_\infty)$ for the "large" discrete series representation $\pi_\infty=\pi_\Lambda$ with Harish-Chandra parameter $(\Lambda_1,\Lambda_2,\Lambda_3)$ and a Hecke character ξ with parameter $(t,m)\in\mathbb{C}\times\mathbb{Z}$ is given as follows.*

$$2^s\Gamma(s+t+\Lambda_1-\frac{m}{2})\Gamma(s+t-\Lambda_2+\frac{m}{2})\begin{cases}\Gamma(s+t-\Lambda_3-\Lambda_1+\frac{3m}{2}) & \textit{when } m\geq\Lambda_3 \textit{ and } m\geq\Lambda_1\\ \Gamma(s+t-\Lambda_3+\frac{m}{2}) & \textit{when } \Lambda_1\geq m\geq\Lambda_3\\ \Gamma(s+t-\Lambda_1+\frac{m}{2}) & \textit{when } \Lambda_3\geq m\geq\Lambda_1\\ \Gamma(s+t-\frac{m}{2}) & \textit{when } \Lambda_3\geq m \textit{ and } \Lambda_1\geq m\end{cases}$$

supposing $\Lambda_1+\Lambda_3\geq m\geq\Lambda_2+\Lambda_3$. □

We reported archimedean local functional equation in the occasion of the talk in Jan/21. However some serious mistake was found afterward and has not been removed so far. We would like to reconsider this probrem in near future.

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