<table>
<thead>
<tr>
<th>Title</th>
<th>Forking in Generic Structures (Study of definability in nonstandard models of arithmetic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>IKEDA, Koichiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録, 2006, 1469: 86-91</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/48089">http://hdl.handle.net/2433/48089</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Forking in Generic Structures

池田 宏一郎 (Koichiro IKEDA) *
法政大学経営学部
(Faculty of Business Administration, Hosei University)

Abstract

Baldwin の問題とは，superstable であるが ω-stable でない generic 構造が存在するか，という問題である ([1]). この問題の部分的結果として，generic 構造が saturated のときは，理論が superstable ならば必ず ω-stable となることがわかった（定理 10）.

1 Preliminaries

Many papers [2,3,4,8] have laid out the basics of generic structures. So we do not explain all of those details here.

Let $L$ be a countable relational language and $K^*$ a class of the finite $L$-structures. Then $\delta : K^* \rightarrow \mathbb{R}^+$ is said to be a predimension, if (i) for all $AB \in K^*$, $\delta(A/B) \leq \delta(A/A \cap B)$; (ii) if $A \cong B \in K^*$, then $\delta(A) = \delta(B)$; (iii) $\delta(\emptyset) = 0$; (iv) If $A \subset A', B \subset B', C \subset C'$ and $A', B', C'$ are pairwise disjoint, then $\delta(B/A) - \delta(B/AC) \leq \delta(B'/A') - \delta(B'/A'C')$, where $\delta(A/B)$ denotes $\delta(AB) - \delta(B)$.

Let $A \subset B \in K^*$. Suppose that $A$ is finite. Then $A$ is said to be closed in $B$ (in symbol, $A \leq B$), if $\delta(X/A) \geq 0$ for any finite $X \subset B - A$. In general, $A$ is said to be closed in $B$, if $A \cap X \leq B \cap X$. We define the closure of $A$ in $B$ by $\text{cl}_B(A) = \bigcap \{C : A \subset C \leq B, |C| < \omega\}$. We define a dimension of $A$ in $B$ by $d_B(A) = \delta(\text{cl}_B(A))$.

Let $K$ be a subclass of $K^*$ that is closed under substructures, and $M$ a saturated $K$-generic structure.

$K$ is said to have finite closures, if there are no chains $A_0 \subset A_1 \subset \cdots$ of elements of $K$ with $\delta_\alpha(A_{i+1}) < \delta_\alpha(A_i)$ for each $i < \omega$. If $K$ has finite closures, then we can see that there exists a unique $K$-generic structure $M$, and moreover that any finite set of $M$ has finite closures. On the other hand, it can be seen that if a $K$-generic structure $M$ is saturated then $K$ has finite closures. We summarize our situation.

*Research partially supported by Grants-in-Aid for Scientific Research (no.16540123), Ministry of Education, Science and Culture.
Assumption \( K = (K, \leq) \) is derived from a predimension \( \delta \) such that \( K \) is closed under substructures. \( M \) is a saturated \( K \)-generic structure.

2 Smallness of algebraic types

Definition Let \( AB \) be finite \( L \)-structure. Then
\[(i) \text{ A pair } (B, A) \text{ is said to be } K\text{-normal, if } A \leq AB \in K \text{ and } A \cap B = \emptyset.\]
\[(ii) \text{ A } K\text{-normal pair } (B, A) \text{ is said to be minimal, if } \delta(C/A) > \delta(B/A) \text{ for any non-empty proper subset } C \text{ of } B.\]
\[(iii) \text{ A } K\text{-normal pair } (B, A) \text{ is said to be weakly small, if whenever } A \subset C, B \subset D \text{ and } (D, C) \text{ is } K\text{-normal, then } \delta(D/C) \geq \delta(B/C).\]
\[(iv) \text{ A } K\text{-normal pair } (B, A) \text{ is said to be pseudo-small, if whenever } A \subset C \text{ and } (B, C) \text{ is } K\text{-normal, then } \delta(B/C) \geq \delta(B/A).\]
\[(v) \text{ A } K\text{-normal pair } (B, A) \text{ is said to be small, if whenever } A \subset C, B \subset D \text{ and } (D, C) \text{ is } K\text{-normal, then } \delta(D/C) \geq \delta(B/A).\]

Note 1 A \( K\)-normal pair \((B, A)\) is small if and only if it is weakly small and pseudo-small.

Lemma 2 Let \((B, A)\) be \( K\)-minimal with \( A \leq AB \leq M \). If \( \text{tp}(B/A) \) is algebraic, then \((B, A)\) is weakly \( K\)-small.

Proof Suppose by way of contradiction that \((B, A)\) is not weakly \( K\)-small. Then there are \( C \supset A \) and \( D \supset B \) such that \((D, C)\) is \( K\)-normal and \( \delta(D/C) < \delta(B/C) \).

Claim 1: There is a set \( \{B_i\}_{i<\omega} \) of copies of \( B \) with the following conditions:
\[(i) B_i \cong_{CB_0...B_{i-1}} B \text{ for each } i < \omega; \]
\[(ii) CB_0...B_i, CB_0...B_{i-1}D \leq CB_0...B_iD \in K \text{ for each } i < \omega; \]
\[(iii) D, B_0, B_1, B_2, ... \text{ are pairwise disjoint.}\]

Proof of Claim 1: We construct \( \{B_i\}_{i<\omega} \) inductively. Suppose that \( \{B_i\}_{i\leq n} \) has been defined. By (ii), \( CB_0...B_n \leq CB_0...B_nD \in K \), and so we have \( CB_0...B_n \leq CB_0...B_nB \in K \). By the amalgamation property, we can take a copy \( B_{n+1} \) of \( B \) over \( CB_0...B_n \) such that
\[(*) \text{ } CB_0...B_nD, CB_0...B_nB_{n+1} \leq CB_0...B_nB_{n+1}D \in K.\]

Hence \( B_{n+1} \) satisfies (i) and (ii). On the other hand, \( B_0, B_1, ..., B_{n+1} \) are pairwise disjoint, since \( B_{n+1} \cong_{CB_0...B_n} B \) and \( B \subset D \). So, to see that (iii) holds it is enough to show that \( B' = B_{n+1} \cap D = \emptyset \). If \( B' = B_{n+1} \) would hold, then we have \( B_{n+1} \subset D \), and so \( CB_0...B_{n+1} \leq CD \), since \( \delta(D/C) < \delta(B/C) = \delta(B_{n+1}/C) \). This contradicts (*), and hence we have \( B' \neq B_{n+1} \). By (*) again, we have \( CB_0...B_nD \leq CB_0...B_nB_{n+1}D \), and so \( AB' \leq AB_{n+1} \). Since \( (B, A) \) is a minimal pair, we have \( B' = \emptyset \). (End of Proof of Claim 1)

Claim 2: \( AB, AB_j \leq AB_0...B_jB \in K \) for \( j \leq i < \omega \)
Proof: We prove by induction on \( i \). By (ii) of claim 1, \( AB_0...B_iB \leq AB_0...B_{i+1}B \).
By induction hypothesis, we have $AB, AB_j \leq AB_0...B_i B$ for $j \leq i$. Hence $AB, AB_j \leq AB_0...B_{i+1}B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0...B_{i+1}B$. By induction hypothesis again, we have $AB \leq AB_0...B_i B$. From (i) of claim 1, it follows that $AB_{i+1} \leq AB_0...B_{i+1}$. By (ii) of claim 1, $AB_0...B_{i+1} \leq AB_0...B_{i+1}B$. Hence we have $AB_{i+1} \leq AB_0...B_{i+1}$. (End of Proof of Claim 2)

We show that $\text{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $AB, AB_j \leq AB_0...B_i B$ for each $i, j$ with $j \leq i < \omega$. So we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (iii) of claim 1, $B_j$’s are pairwise disjoint. Hence $\text{tp}(B/A)$ is not algebraic.

**Definition** We say that $K$ is closed under $\delta$-substructures, if for any disjoint $A, B, C$ with $ABC \in K$, there is a copy $B^{*}$ of $B$ over $A$ with $\delta(B^{*}/CA) = \delta(B^{*}/A)$.

**Lemma 3** Assume that $K$ is closed under $\delta$-substructures. Let $(B, A)$ be $K$-minimal with $A \leq AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then $(B, A)$ is pseudo-$K$-small.

**Proof** Suppose by way of contradiction that $(B, A)$ is not pseudo-$K$-small. Then there is a set $\{B_i\}_{i<\omega}$ of copies of $B$ over $A$ with the following conditions:

(i) $C \subseteq CB_j \leq CB_0B_1...B_i \in K$ for each $j \leq i < \omega$

(ii) $B_0, B_1, B_2, ...$ are pairwise disjoint.

(iii) $B_i \cap C = \emptyset$ for each $i < \omega$;

(iv) $\delta(B_i/C) = \delta(B_i/A)$ for each $i < \omega$.

Proof: Suppose that $\{B_i\}_{i\leq n}$ has been defined. By our assumption, we have $C \leq CB \in K$, and by (i) we have $C \leq CB_0B_1...B_n \in K$. So we can take a copy $B^{*}$ of $B$ over $C$ such that $CB_0...B_n, CB^{*} \leq CB_0...B_nB^{*} \in K$. By (iv), $\delta(B_i/C) = \delta(B_i/A)$ for each $i \leq n$. On the other hand, we have $\delta(B^{*}/C) < \delta(B^{*}/A)$. So we have $B_i \neq B^{*}$ for all $i \leq n$. Since $(B, A)$ is $K$-minimal, $B$ and $B_i$’s are pairwise disjoint. Since $K$ is closed under $\delta$-substructures, $B_{n+1}$ with $B_{n+1} \cong AB_0B_1...B_n B^{*}, CB_0B_1...B_nB_{n+1} \in K$ and $\delta(B_{n+1}/C) = \delta(B_{n+1}/A)$. Then we can see that (i)-(iv) hold. (End of Proof of Claim)

By claim, we have $AB_j \leq AB_0...B_i \in K$ for $j \leq i < \omega$. So we can assume that $AB_j \leq AB_0...B_i \leq M$ for each $i, j$ with $j \leq i < \omega$. Thus we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, $B_j$’s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Lemma 4** If $(B, A)$ and $(C, BA)$ are $K$-small, then so is $(BC, A)$.

**Proof** Take any $K$-normal pair $(E, D)$ such that $BC \subseteq E$ and $A \subseteq D$. Then note that $(E - B, BD)$ is $K$-normal. (Proof: Take any $X \subseteq E - B$. Note
that \((XB, D)\) is \(K\)-normal since \((E, D)\) is so. Since \((B, A)\) is \(K\)-small, we have
\[
\delta(X/BD) = \delta(XB/D) - \delta(B/D) \geq \delta(XB/D) - \delta(B/A) \geq 0.
\]
Hence \((E - B, BD)\) is \(K\)-normal.) Since \((C, BA)\) is \(K\)-small, we have \(\delta(E/BD) \geq \delta(C/BA)\). On the other hand, since \((B, A)\) is \(K\)-small and \((B, D)\) is \(K\)-normal, we have \(\delta(B/D) = \delta(B/A)\). It follows that \(\delta(BC/A) = \delta(C/AB) + \delta(B/A) \leq \delta(E/BD) + \delta(B/D) = \delta(E/D)\). Hence \((BC, A)\) is \(K\)-small.

**Theorem 5** Assume that \(K\) is closed under \(\delta\)-substructures. Let \((B, A)\) be \(K\)-normal with \(AB \leq M\). If \(\text{tp}(B/A)\) is algebraic, then \((B, A)\) is \(K\)-small.

**Proof** Let \(\text{tp}(B/A)\) be algebraic. Take a sequence \(A = B_0 \leq B_0B_1 \leq \ldots \leq B_0B_1 \ldots B_n = AB\) with \((B_{i+1}, B_i)\) \(K\)-minimal for each \(i\). Since each \(\text{tp}(B_{i+1}/B_0 \ldots B_i)\) is algebraic, it is \(K\)-small by lemma 2 and 3. So, by lemma 4, 
\((B, A) = (B_0B_1 \ldots B_n, B_0)\) is \(K\)-small.

## 3 Forking and dimension

In this section, we assume that \(K\) is closed under \(\delta\)-substructures.

**Lemma 6** Let \(A \subseteq B\). If \(B\) is closed, then so is \(B \cup \text{acl}(A)\).

**Proof** We can assume that \(A, B\) are finite. It is enough to show that \(BA^*\) is closed for any finite \(A^*\) with \(A \subseteq A^* \subseteq \text{acl}(A)\). Let \(A' = A^* \cap B\). Since \(\text{tp}(A^*/A')\) is algebraic and \(A' \leq A^*\) is closed, \((A^* - A', A')\) is small. Then we can see that \(BA^*\) is closed as follows. If not, then there is finite \(X \subseteq N - BA^*\) with \(\delta(X/BA^*) < 0\). Then we have \(0 \leq \delta(XA^*/B) = \delta(X/BA^*) + \delta(A^*/B) < \delta(A^*/B) \leq \delta(A^*/A')\). This contradict that \((A^* - A', A')\) is small.

**Fact([7], [8])** Let \(B, C\) be closed and \(A = B \cap C\) algebraically closed. Then the following are equivalent.
(i) \(d(B/A) = d(B/C)\);
(ii) \(B\) and \(C\) are free over \(A\), and \(BC\) is closed;
(iii) \(\text{tp}(B/C)\) does not fork over \(A\).

**Lemma 7** Assume that \(K\) is closed under \(\delta\)-substructures. Let \(B, C\) be closed and \(A = B \cap C\). If \(B, C\) are free over \(A\) and \(BC\) is closed, then \(\text{tp}(B/C)\) does not fork over \(A\).

**Proof** By lemma 6, \(\text{acl}(A), \text{Cacl}(A)\) is closed. Since \(BC\) is closed, by lemma 6 again, \(BC\text{acl}(A)\) is closed. So, by fact, to show that \(\text{tp}(B/C)\) does not fork over \(A\), it is enough to prove that \(\text{acl}(A)B, \text{acl}(A)C\) are free over \(\text{acl}(A)\). Take finite closed \(B_0 \subseteq B, C_0 \subseteq C\) such that \(B_0C_0\) is closed. Let \(A_0 = B_0 \cap C_0\). Take
finite closed $A^* \subset \text{acl}(A_0)$. Let $D = A^* - B_0C_0$ and $A' = A^* \cap B_0C_0$.

Claim 1: $\delta(B_0C_0/A^*) = \delta(B_0C_0/A')$.

Proof: Since $\text{tp}(D/A^*)$ is algebraic and $(D, A')$ is normal, $(D, A')$ is small. So we have $\delta(D/B_0C_0) = \delta(D/A')$. Then $\delta(B_0C_0/A^*) - \delta(B_0C_0/A') = \delta(D/B_0C_0) - \delta(D/A') = 0$. (End of Proof of Claim 1)

Claim 2: $\delta(B_0/C_0A^*) = \delta(B_0/A^*)$.

Proof: Since $B, C$ are free over $A$, we can see that $\delta(C_0/A'B_0) = \delta(C_0/A')$. Then $\delta(B_0/C_0A^*) = \delta(B_0C_0/A^*) - \delta(C_0/A')$.

By claim 2, $\text{Bacl}(A), \text{Cacl}(A)$ is free over $\text{acl}(A)$.

**Lemma 8** Assume that $K$ is closed under $\delta$-substructures. Let $A, B, C$ be such that $B, C$ are closed and $A = B \cap C$. Suppose that $\text{tp}(B/C)$ does not fork over $A$. Then

(i) $B, C$ are free over $A$, and moreover $\text{acl}(B), \text{acl}(C)$ are free over $\text{acl}(A)$;

(ii) $B \cup C \cup \text{acl}(A)$ is closed.

**Proof** Let $A^* = \text{acl}(A), B^* = \text{acl}(B)$ and $C^* = \text{acl}(C)$. Clearly $\text{tp}(B^*/A^*)$ does not fork over $A^*$, and so $B^* \cap C^* = A^*$.

(i) By fact, $B^*$ and $C^*$ are free over $A^*$. So we obtain that $B$ and $C$ are free over $A^*$. Let $B' = B \cap A^*, C' = C \cap A^*$. Note that $\delta(B/B'C') = \delta(B/B'C'').$

First we show that $B'$ and $C$ are free over $A$. If not, then there are finite closed $A_0 \subset A, B_0' \subset B', C_0 \subset C$ such that $B_0', C_0$ are not free over $A_0 = B_0' \cap C_0$. We can assume that $\text{tp}(B_0'/A_0)$ is algebraic. By theorem 5, $(B_0' - A_0, A_0)$ is small. So we have $\delta(B_0'/A_0) = \delta(B_0/C_0)$. This contradicts that $B_0'$ and $C_0$ are not free over $A_0$. Thus $B'$ and $C$ are free over $A$. Similarly we see that $B$ and $C'$ are free over $A$. Then $\delta(B/C) = \delta(BB'/C') = \delta(B/B'C) + \delta(B'/C) = 

\delta(B/B'C') + \delta(B'/C') = \delta(B/C') = \delta(B/A)$. Hence $B$ and $C$ are free over $A$.

(ii) By fact, we obtain that $B^*C^*$ is closed. If $BCA^*$ is not closed, then there are finite $X \subset B^*C^* - BCA^*, B_0 \subset B, C_0 \subset C, A_0^* \subset A^*$ such that $\delta(X/B_0C_0A_0^*) < 0$. By lemma, $BA^*, CA^*$ are closed, and hence we can assume that $B_0A_0^*, C_0A_0^*$ are closed. Let $X_B = X \cap B^*$ and $X_C = X \cap C^*$. Then $\delta(X_B/B_0C_0X_CA_0^*) = \delta(X_B/B_0A_0^*)$ and $\delta(X_C/B_0C_0A_0^*) = \delta(X_C/C_0A_0^*)$ since $B^*, C^*$ are free over $A^*$. Therefore we have $\delta(X/B_0C_0A_0^*) = \delta(X_B/B_0C_0X_CA_0^*) + \delta(X_C/B_0C_0A_0^*) = \delta(X_B/B_0A_0^*) + \delta(X_C/C_0A_0^*) \geq 0 + 0 = 0$. A contradiction.

**Lemma 9** Assume that $\text{Th}(M)$ is superstability. Then for any countable model $N$ and $p \in S(N)$ there is finite $A \subset N$ such that $p|A$ is stationary.

**Proof** Take a realization $\bar{b}$ of $p$. By superstability, there is finite $X \subset N$ such that $p$ does not fork over $X$. Let $B = \text{cl}(X\bar{b})$ and $A = B \cap N$. Clearly $\text{tp}(\bar{b}/N)$ does not fork over $A$. We show that $\text{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such
that $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A)$ and $\text{tp}(\bar{b}'/N)$ does not fork over $A$. Let $B' = \text{cl}(\bar{b}'A)$.

Then we have $B \cong_A B'$. Note that $B' \cap N = A$. By lemma 8, $B, N$ and $B', N$ are free over $A$ respectively, and so we have $B \cong_N B'$. By lemma 8 again, $BN, B'N$ are closed since $\text{acl}(A) \subseteq N$. It follows that $\text{tp}(BN) = \text{tp}(B'N)$ and hence $\text{tp}(b/N) = \text{tp}(b'/N)$.

By lemma 9, we have the following theorem.

**Theorem 10** Let $L$ be a countable relational language. Let $K = (K, \leq)$ be a class of finite $L$-structures that is derived from a predimension $\delta$ and that is closed under substructures. Let $M$ be a saturated $K$-generic structure. If $\text{Th}(M)$ is superstable, then it is $\omega$-stable.

**Reference**


Faculty of Business Administration
Hosei University
2-17-1, Fujimi, Chiyoda
Tokyo, 102-8160
JAPAN
ikeda@i.hosei.ac.jp