数理解析研究所講究録 1469 巻 2006 年 92-95

On P-closure operator in quasi-minimal structures

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Abstract

Itai, Tsuboi and Wakai investigated the geometric properties of qusai-minimal structures by using the countable closure [1]. I tried to define another closure operator in such structures.

1 Quasi-minimal structure and the countable closure

We recall some definitions.

Definition 1 An uncountable structure M is called *quasi-minimal* if every definable subset of M with parameters is at most countable or co-countable.
Let M be an uncountable structure and $A \subset M$. The *n-th countable closure* $\mathrm{ccl}_n^M(A)$ of A is inductively defined as follows :
$\mathrm{ccl}_0^M(A) = A$ and
$\mathrm{ccl}_{n+1}^M(A) = \bigcup\{\phi^M : \phi(x) \in L(\mathrm{ccl}_n^M(A)),\ \phi^M \text{ is countable}\}$
We put $\mathrm{ccl}^M(A) = \bigcup_{n\in\omega} \mathrm{ccl}_n^M$ (*the countable closure of* A). We omit the superscript M if it is clear from the context.

And we recall the notion of pregeometry.

Definition 2 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say (X, cl) is a *pregeometry*.
(I) $A \subset B \Longrightarrow A \subset \mathrm{cl}(A) \subset \mathrm{cl}(B)$,
(II) (Finite character) $b \in \mathrm{cl}(A) \Longrightarrow b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subset A$,
(III) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(IV) (Exchange axiom) $b \in \mathrm{cl}(A \cup \{c\}) - \mathrm{cl}(A) \Longrightarrow c \in \mathrm{cl}(A \cup \{b\})$.

The countable closure is a closure operator.

Fact 3 [1] *Let M be a quasi-minimal structure. Then it is clear that (M, ccl) satisfies the first three properties* (I) *through* (III).

The exchange axiom (IV) does not hold in general in (M, ccl). Itai et al. showed some conditions for M such that (M, ccl) satisfies the exchange axiom.
The notion of quasi-minimal structures is a generalization of minimal structures. Thus the countable closure is the canonical closure operator for quasi-minimal structures. However, I considered that the countable closure is divided by some P-closures.

2 Internality, foreignness and P-closure

First we recall some definitions from [5].

Definition 4 A family P of partial types is *A-invariant* if it is invariant under A-automorphisms. (where A is a subset of a sufficiently large saturated model as usual.)
Let P be an A-invarint family of partial types.
A partial type q over A is *P-internal* if for every realization a of q, there is $B \mathop{\smile\!\!\!\!\!\!|}_A a$, types $\bar{p}$ from P based on B, and realizations $\bar{c}$ of $\bar{p}$, such that $a \in \mathrm{dcl}(B\bar{c})$.
A patial type q is *P-analysable* if for any $a \models q$, there are $(a_i : i < \alpha) \in \mathrm{dcl}(A, a)$ such that $\mathrm{tp}(a_i/A, a_j : j < i)$ is P-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A, a_i : i < \alpha)$.
A complete type $q \in S(A)$ is *foreign* to P if for all $a \models q$, $B \mathop{\smile\!\!\!\!\!\!|}_A a$, and realizations $\bar{c}$ of extensions of types in P over B, we always have $a \mathop{\smile\!\!\!\!\!\!|}_{AB} \bar{c}$.
And let P be an $\emptyset$-invariant family of types.
A partial type q is *co-foreign* to P if every type in P is foreign to q.
The *P-closure* $\mathrm{cl}_P(A)$ of a set A is the collection of all element a such that $\mathrm{tp}(a/A)$ is P-analysable and co-foreign to P.

Remark 5 The P-analysable assumption could be modified or even omitted, resulting in a larger P-closure.

Fact 6 *P-closure is a closure operator, i.e. it satisfies the axioms* (I) *and* (III) *in* Definition 2.

3 P-closures for quasi-minimal structures

We recall another notion from [1].

Definition 7 Let M be quasi-minimal. Then a type $p(x)$ defined by
$p(x) = \{\psi(x) \in L(M) \ : \ |\psi^M| \geqq \omega_1\}$
is a complete type in $S(M)$. The type $p(x)$ is called the *main type* of M.

We define P-closures in some particular situation to make good in arguments. They may be defined under more general assumption. From now on in this section, M denotes an ω-saturated quasi-minimal structure. And its theory $Th(M)$ is ω-stable. We recall the next theorem from [1],

Theorem 8 *Let M be a quasi-minimal structure. And $Th(M)$ is ω-stable. Then M can be elementarily embedded to an ω-saturated quasi-minimal stucture M'.*

We restricts the argument in a quasi-minimal structure M. So in the next definitions, any tuple is taken from M and any set denotes a *finite* subset of M. Moreover we assume that $p(x) \in S(M)$ is the main type of M and $p(x)$ is strongly based on $\emptyset$.

Definition 9 The set P of types is defined by $P = \{p\lceil A \ : \ A \subset M\}$.
A complete type $q \in S(A)$ is *P-internal* (over A) if for any $a \models q$, there are $B \supset A$ with $B \perp_A a$, and $\bar{c} \models \bar{r}$ over B with $\bar{r}\lceil A \in P$ such that $a \in$dcl$(B\bar{c})$.
P-analysability is defined in the same way as the originai definition from the above P-internality.
A complete type $q \in S(A)$ is *co-foreign* to P if for any $b \models r \in P$ over B, any $C \supset A$ with $b \perp_B C$, and any realization a' of an extension of q over C, $b \perp_{BC} a'$.
The *P-closure* cl'$_P(B)$ of a (*any*) set $B \subset M$ is the collection of all element a such that tp(a/A) is P-analysable and co-foreign to P for some finite $A \subset B$.

Remark 10 If there were no homogeneity of M, P does not have the invariance property.
The above P-internality is defined for partial types with respect to P in general. However, although a partial type π is P-internal over A, it is not always P-internal over $B \supset A$.

We can prove the next fact.

Fact 11 *Under the same assumptions as the former definition, the P-closure* cl'$_P$ *is a closure operator.* $(M, \text{cl'}_P)$ *satisfies the axioms* (I) *through* (III) *in* Definition 2.
And acl$(A) \subset$ cl'$_P(A) \subset$ ccl(A) *for* $A \subset M$.

Next we must proceed to the exchange axiom. I will investigate the conditions under which cl'_P has the exchange axiom. But I doubt if it is significant.

It is known that the notions of internality and foreignness are available for the characterization of stable groups. In particular, there are interpretation theorems of automorphism groups. I will consider whether such situation arises in quasi-minimal structures.

References

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