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On weak determinacy of infinite binary games

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概要

In [5] and [6], we have investigated the logical strength of the determinacy of
infinite games in the Baire space up to $\Delta^0_3$. In this paper we consider infinite games
in the Cantor space. Let $\text{Det}^*$ (resp. $\text{Det}$) stand for the determinacy of infinite games
in the Cantor space (resp. the Baire space). In Section 2, we show that
$\Delta^0_1\text{-Det}^*$, $\Sigma^0_2\text{-Det}^*$ and $\text{WKL}_0$ are pairwise equivalent over $\text{RCA}_0$. In Section 3, we
show that $\text{RCA}_0 + (\Sigma^0_1 \land \Pi^0_1)\text{-Det}^*$ is equivalent to $\text{ACA}_0$. Then, we deduce that
$\text{RCA}_0 + \Delta^0_2\text{-Det}^*$ is equivalent to $\text{ATR}_0$. In the last section, we show some more
 equivalences among stronger assertions without details, which will be thoroughly
treated elsewhere.

1 Preliminaries

In this section, we recall some basic definitions and facts about second order
arithmetic. The language $\mathcal{L}_2$ of second order arithmetic is a two-sorted language
with number variables $x, y, z, \ldots$ and unary function variables $f, g, h, \ldots$, consisting
of constant symbols $0, 1, +, -, =, <$. We also use set variables $X, Y, Z, \ldots$, intending
to range over the set of $\{0, 1\}$-valued functions, that is, characteristic functions of
 sets.

The formulae can be classified as follows:

- $\varphi$ is bounded ($\Pi^0_2$) if it is built up from atomic formulae by using propositional
connectives and bounded number quantifiers $(\forall x < t), (\exists x < t)$, where $t$ does
not contain $x$. 
• $\varphi$ is $\Pi_0^1$ if it does not contain any function quantifier. $\Pi_0^1$ formulae are called *arithmetical* formulae.

• $\neg \varphi$ is $\Sigma_i^n$ if $\varphi$ is a $\Pi_i^n$ formula ($i \in \{0, 1\}, n \in \omega$).

• $\forall x_1 \cdots \forall x_k \varphi$ is $\Pi_0^{n+1}$ if $\varphi$ is a $\Sigma_0^n$ formula ($n \in \omega$).

• $\forall f_1 \cdots \forall f_k \varphi$ is $\Pi_1^{n+1}$ if $\varphi$ is a $\Sigma_1^n$ formula ($n \in \omega$).

Using above classification, we can define schemata of comprehension and induction as follows.

**Definition 1.1** Assume $n \in \omega$ and $i \in \{0, 1\}$. The scheme of $\Pi_i^n$ *comprehension*, denoted $\Pi_i^n$-CA, consists of all the formulae of the form

$$\exists X \forall x (x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ belongs to $\Pi_i^n$ and $X$ does not occur freely in $\varphi(x)$. The scheme of $\Delta_i^n$-comprehension, denoted $\Delta_i^n$-CA, consists of all the formulae of the form

$$\forall n (\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is $\Sigma_i^n$, $\psi(n)$ is $\Pi_i^n$, and $X$ is not free in $\varphi(n)$. The scheme of $\Sigma_i^n$ induction, denote $\Sigma_i^n$-IND, consists of all axioms of the form

$$(\varphi(0) \land \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is $\Sigma_i^n$.

Now we define a basic subsystem of second order arithmetic, called RCA$_0$.

**Definition 1.2** RCA$_0$ is the formal system in the language of $\mathcal{L}_2$ which consists of discrete order semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemata of $\Delta_0^i$ comprehension and $\Sigma_0^i$ induction.

The following is a formal version of the *normal form theorem* for $\Sigma_0^1$ relations.

**Theorem 1.3** (normal form theorem) Let $\varphi(f)$ be a $\Sigma_0^1$ formula. Then we can find a $\Pi_0^1$ formula $R(s)$ such that RCA$_0$ proves

$$\forall f(\varphi(f) \leftrightarrow \exists m R(f[m]))$$

where $f[m]$ is the code for the finite initial segment of $f$ with length $m$. Note that $\varphi(f)$ may contain free variables other than $f$. 
Proof. See also Simpson [7, Theorem II.2.7]. □

We loosely say that a formula is $\Sigma^i_n$ (resp. $\Pi^i_n$) if it is equivalent over a base theory (such as $\text{RCA}_0$) to a $\psi \in \Sigma^i_n$ (resp. $\Pi^i_n$).

The next theorem asserts that the universe of functions is closed under the least number operator, i.e., minimization.

**Theorem 1.4 (minimization)** The following is provable in $\text{RCA}_0$. Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for all $(n_1, \ldots, n_k) \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $f(m, n_1, \ldots, n_k) = 1$. Then there exists $g : \mathbb{N}^k \to \mathbb{N}$ such that $g(n_1, \ldots, n_k)$ is the least $m$ such that $f(m, n_1, \ldots, n_k) = 1$.

**Proof.** See Simpson [7, Theorem II.3.5]. □

## 2 \text{WKL}_0 and $\Sigma_1^0$-Det*

Let $X$ be either $\{0, 1\}$ or $\mathbb{N}$ and let $\varphi$ be a formula with a distinct variable $f$ ranging over $X^\mathbb{N}$. A two-person game $G_\varphi$ (or simply $\varphi$) over $X^\mathbb{N}$ is defined as follows:

player I and player II alternately choose elements from $X$ (starting with I) to form an infinite sequence $f \in X^\mathbb{N}$ and I (resp. II) wins iff $\varphi(f)$ (resp. $\neg\varphi(f)$). A strategy of player I (resp. II) is a map $\sigma : X^{\text{even}} = \{s \in X^{<\mathbb{N}} \mid s \text{ has even length}\} \to X$ (resp. $X^{\text{odd}} \to X$). We say that $\varphi$ is determinate if one of the players has a winning strategy, that is, a strategy $\sigma$ such that the player is guaranteed to win every play $f$ in which he played $f(n) = \sigma([f(n)])$ whenever it was his turn to play.

Given a class of formulae $C$, $C$-determinacy is the axiom scheme which states that any game in $C$ is determinate. We use $C$-$\text{Det}^*$ (resp $C$-$\text{Det}$) to denote $C$-determinacy in the Cantor space (resp. the Baire space).

A set $T$ of finite sequences is a tree if it is closed under initial segment, i.e., $t \in T$ and $s \subseteq t$ implies $s \in T$. A function $f$ is a path of $T$ if each initial segment of $f$ is a sequence of $T$.

**Definition 2.1** $\text{WKL}_0$ is a subsystem of second order arithmetic whose axioms are those of $\text{RCA}_0$ plus weak König's lemma which states that every infinite binary tree $T \subseteq \mathbb{2}^{\lt \mathbb{N}}$ has an infinite path.

Next, we prove the equivalences among $\text{WKL}_0$, $\Sigma_1^0$-$\text{Det}^*$ and $\Delta_1^0$-$\text{Det}^*$.

**Theorem 2.2** $\text{RCA}_0 \vdash \Delta_1^0$-$\text{Det}^* \rightarrow \text{WKL}_0$. 
Proof. By way of contradiction, we assume $\text{RCA}_0 + \Delta^0_1\text{-Det}^*$ and deny weak König's lemma. Let $T$ be an infinite binary tree in which there is no infinite path, i.e., there is no $f$ such that $\forall n f[n] \in T$. We consider the following game:

- Player I plays a sequence $t$ of $2^{<\mathbb{N}}$.
- Player II then answers by playing 0 or 1.
- Player I plays a new sequence $u$ of $2^{<\mathbb{N}}$.
- Player II then plays a sequence $v$ of $2^{<\mathbb{N}}$.

The winning conditions of the game are given as follows: II wins if one of the following cases holds.

- $t \ast (i) \ast u \not\in T$.
- $t \ast (1 - i) \ast v \in T$ if $|u| \leq |v|$.

We shall remark that the game always terminates in finite moves, because $T$ has no infinite path. This ensures that the game is $\Delta^0_1$. On the other hand, we can show that player I has no winning strategy by considering two cases, in one of which player II chooses $i = 0$ and in the other he chooses $i = 1$ after player I plays $t$. I cannot win in both of them. Therefore, by $\Delta^0_1\text{-Det}^*$ player II has a winning strategy $\tau$. Using $\tau$, we define $f : \mathbb{N} \rightarrow \{0, 1\}$ as follows:

- $f(0) = 1 - \tau(())$,
- $f(n + 1) = 1 - \tau(f[n])$,

By $\Sigma^0_1$-induction, we can easily see that $f[n] \in T$ for all $n$, which contradicts with our assumption that $T$ has no infinite path. Thus, $\Delta^0_1\text{-Det}^* \rightarrow \text{WKL}$. This completes the proof of the theorem. $\square$

Now, we turn to prove the reversal.

Theorem 2.3 $\text{WKL}_0 \vdash \Sigma^0_1\text{-Det}^*$.

Proof. Let $\varphi(f)$ be a $\Sigma^0_1$-formula with $f \in 2^{\mathbb{N}}$. Then, by the normal form theorem, $\varphi(f)$ can be written as $\exists n R(f[n])$, where $R$ is $\Pi^0_3$. We define recursive maps $g$ and $g_n$ from $2^{<\mathbb{N}} \rightarrow \{0, 1\}$ for each $n \in \mathbb{N}$ as follows:

\[
g(s) = \begin{cases} 
1 & \text{if } \exists t \subseteq s \ R(t) \\
0 & \text{if } \forall t \subseteq s \neg R(t) 
\end{cases}
\]

\[
g_n(s) = \begin{cases} 
g(s) & \text{if } |s| \geq n \\
\text{max}\{g_n(s \ast (0)), g_n(s \ast (1))\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\
\text{min}\{g_n(s \ast (0)), g_n(s \ast (1))\} & \text{if } |s| < n \text{ and } |s| \text{ is odd} 
\end{cases}
\]
Intuitively, for \( n \in \mathbb{N} \), \( g_n(\langle \rangle) = 1 \) means "player I can win the game by stage \( n \)," and \( g_n(\langle \rangle) = 0 \) means "player I cannot win by stage \( n \)."

Claim. The following assertions hold.

1. If \( g_n(\langle \rangle) = 1 \) for some \( n \), then I has a winning strategy.
2. If \( g_n(\langle \rangle) = 0 \) for every \( n \), then II has a winning strategy.

For (1), fix \( n \) such that \( g_n(\langle \rangle) = 1 \). Define \( \sigma : 2^{\text{even}} \to \{0, 1\} \) by

\[
\sigma(s) = \begin{cases} 
0 & \text{if } g_n((s \ast \langle 0 \rangle)) = 1 \\
1 & \text{otherwise.} 
\end{cases}
\]

We can verify that \( \sigma \) is a winning strategy for player I, which completes the proof of the first assertion of the claim.

For (2), suppose that for any \( n \), \( g_n(\langle \rangle) = 0 \) and show that player II has a winning strategy. The idea of the proof is as follows. Consider an infinite binary tree which consists of the moves at which player II can prevent player I from winning the game. A path through such a tree will serve a winning strategy for II. To realize this idea, we will need some coding arguments to construct the tree.

To begin with, fix a lexicographical enumeration \( e \) of non-empty sequences of \( 2^{\text{even}} \). For instance, \( e((0)) = 0 \), \( e((1)) = 1 \), \( e((0, 0)) = 2 \), and so on. Using \( e \), we can regard any \( s \in 2^{\text{even}} \) as a partial strategy (i.e., a finite segment of the strategy) for player II (cf. [1]). We define \( T_s \) to be the tree consisting of all partial plays in which player II follows \( s \). More precisely, \( T_s \) is defined as follows:

\[
t \in T_s \iff \forall k(2k + 1 < |t| \rightarrow t(2k + 1) = s(e(\langle t(0), \ldots, t(2k) \rangle))).
\]

Finally we define \( T \), a set of all moves which avoid the winning of player I, as follows:

\[
s \in T \iff \forall t \in T_s \, g_{h(s)}(t) = 0,
\]

where \( h : 2^{\text{even}} \to \mathbb{N} \) is defined by \( h(s) = \max\{|t| : t \in T_s\} \). Clearly \( T \) is recursive, therefore it exists in \( \text{RCA}_0 \). On the other hand, the assumption \( \forall n \, g_n(\langle \rangle) = 0 \) implies that \( T \) is infinite. Thus, \( T \) has a infinite path \( f \) by weak König's lemma.

Now, we define \( \tau : 2^{\text{odd}} \to \mathbb{N} \) as:

\[
\tau(s) = f(e((s(0)) \ldots e(|s| - 2)))).
\]

and then we can verify that \( \tau \) is a winning strategy for player II, which completes the proof. \( \square \)
3 ATR$_0$ and $\Delta^0_2$-Det$^*$

In this section we aim to show that RCA$_0 + \Delta^0_2$-Det$^*$ and ATR$_0$ are equivalent. We first give the definitions of ACA$_0$ and ATR$_0$.

**Definition 3.1** The system ACA$_0$ consists of the discrete order semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemes of $\Sigma^0_1$ induction and arithmetical comprehension.

Since comprehension axioms admit free variables, $\Pi^0_1$ comprehension is already as strong as arithmetical comprehension.

**Lemma 3.2** The following are pairwise equivalent over RCA$_0$.

1. arithmetical comprehension;
2. $\Pi^0_1$ comprehension.

**Proof.** See Simpson [7, Lemma III.1.3]. □

**Definition 3.3** ATR$_0$ consists of RCA$_0$ augmented by the following axiom, called *arithmetical transfinite recursion*: For any set $X \subset \mathbb{N}$ and for any well-ordering relation $\prec$, there exists a set $H \subset \mathbb{N}$ such that

- if $b$ is the $\prec$-least element, then $(H)_b = X$,
- if $b$ is the immediate successor of $a$ w.r.t. $\prec$, then $\forall n \in (H)_b \leftrightarrow \psi(n, (H)_a))$,
- if $b$ is a limit, then $\forall a \forall n ((n, a) \in (H)_b \rightarrow (a \prec b \land n \in (H)_a))$,

where $\psi$ is a $\Pi^0_1$-formula and $(H)_a = \{x : (x, a) \in H\}$, where $(x, b)$ denotes the code of the pair $(x, a)$.

ATR$_0$ is obviously stronger than ACA$_0$, but it is contained in $\Pi^1_1$-CA$_0$.

**Lemma 3.4** The following are pairwise equivalent over RCA$_0$:

$\Delta^0_1$-Det, $\Sigma^0_1$-Det and ATR$_0$.

**Proof.** See [7] or [8].

The class $\Sigma^0_1 \land \Pi^0_1$ is defined as follows. $\varphi$ is $\Sigma^0_1 \land \Pi^0_1$ if and only if $\varphi$ is of the form $\psi_0 \land \neg \psi_1$, where $\psi_0$ and $\psi_1$ are $\Sigma^0_1$. The following theorems characterize $(\Sigma^0_1 \land \Pi^0_1)$ determinacy in the Cantor space.

**Theorem 3.5** ACA$_0$ proves $(\Sigma^0_1 \land \Pi^0_1)$-Det$^*$.

**Proof.** Let $\varphi$ be of the form $\exists n R_0(f[n]) \land \forall n R_1(f[n])$. We define the functions $g, g_n, g', g'_n$ from $2^{<\mathbb{N}}$ to $\{0, 1\}$ as follows:
$g(s) = \begin{cases} 
1 & \text{if } \exists t \subseteq s R_0(t) \\
0 & \text{if } \forall t \subseteq s \neg R_0(t) 
\end{cases}$

$g_n(s) =$

$\begin{cases} 
g(s) & \text{if } |s| \geq n \\
\max\{g_n(s \cdot \langle 0 \rangle), g_n(s \cdot \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\
\min\{g_n(s \cdot \langle 0 \rangle), g_n(s \cdot \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd} 
\end{cases}$

$g'(s) = \begin{cases} 
1 & \text{if } \forall t \subseteq s R_1(t) \\
0 & \text{if } \exists t \subseteq s \neg R_1(t) 
\end{cases}$

$g'_n(s) =$

$\begin{cases} 
g'(s) & \text{if } |s| \geq n \\
\max\{g'_n(s \cdot \langle 0 \rangle), g'_n(s \cdot \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\
\min\{g'_n(s \cdot \langle 0 \rangle), g'_n(s \cdot \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd} 
\end{cases}$

Following a similar argument of the one used in the proof of Theorem 2.2, we can prove

Claim: if there exists an such that $g_n(\langle \rangle) \cdot g'_n(\langle \rangle) = 1$ for all $m > n$ then I has a winning strategy, otherwise player II has a winning strategy.

This complete the proof of the theorem. □

**Theorem 3.6** RCA₀ ⊢ (Σ₁⁰ ∩ Π₁⁰)-Det* → ACA₀

**Proof.** Let $\varphi(n)$ be a $\Sigma₁⁰$-formula. We need to construct a set $X$ such that for any $n \in \mathbb{N}$, $\varphi(n) \iff n \in X$. To construct $X$, consider the following game: player I asks II about $n$ by playing 0 consecutively $n$ times and playing 1 after that (if he plays 0 for ever, he loses). II ends the game by answering 0 or 1.

Now, suppose that player I plays $n$ 0's and a 1 consecutively. Player II wins if one of the following cases holds.

- II answers 1 and $\varphi(n)$.
- II answers 0 and $\neg \varphi(n)$.

Clearly, I has no winning strategy. By $(\Sigma₁⁰ ∩ Π₁⁰)$-Det*, let $\tau$ be a winning strategy of player II. We defined a set $X$ by:

$$n \in X \iff \tau(0^n1) = 1.$$ 

The set $X$ exists by $\Pi₁⁰$ comprehension. Moreover, we can verify that $\forall n, \varphi(n) \iff n \in X$, which completes the proof. □

Let $<$ be a recursive well-ordering on $\mathbb{N}$. We define a recursive well-ordering $\prec^*$ on $\mathbb{N} \times \{0, 1\}$ as follows:

$$(x, i) \prec^* (y, j) \text{ iff } x < y \vee (x = y \land i < j).$$
Let $X$ be either $\mathbb{N}$ or $\{0, 1\}$. We say that a formula $\varphi(n, i, f)$ with distinct free variable $f$ ranging over $X^\mathbb{N}$ is *decreasing along $\prec^*$ if and only if*

$$\forall n \forall i \forall m \forall j ( ((m, j) \prec^* (n, i) \land \varphi(n, i, f)) \rightarrow \varphi(m, j, f)),$$

for all $f$.

The following lemma will play a key role to characterize $\Delta^0_2$-Det$^*$.

**Lemma 3.7** It is provable in RCA$_0$ that a formula $\psi$ is $\Delta^0_2$ if and only if:

$$\psi(f) \rightarrow \exists x ( (\varphi(x, 0, f) \land \neg \varphi(x, 1, f))),$$

where $\varphi$ is $\Pi^0_1$ and it is decreasing along some recursive well-ordering relation $\prec^*$.

**Proof.** See [8] for the proof. $\square$

**Theorem 3.8** ATR$_0$ is equivalent to RCA$_0$ + $\Delta^0_2$-Det$^*$.

**Proof.** The proof is a modification of the proof of Theorem 6.1 in [8]. By Theorem 3.6 and Lemma 3.7, $\Delta^0_2$-Det$^*$ is just a transfinite iteration of arithmetical comprehension, which is the same as ATR$_0$. $\square$

### 4 Further classes of games

In this section, we summarize our results about the determinacy of Boolean combinations of $\Sigma^0_2$-games. The detailed treatment of these results will appear in our forthcoming paper.

We start by formalizing the inductive definition of a class of operators.

**Definition 4.1** Given a a class of formulas $\mathcal{C}$, the axiom $\mathcal{C}$-ID asserts that for any operator $\Gamma \in \mathcal{C}$, there exists $W \subset \mathbb{N} \times \mathbb{N}$ such that

1. $W$ is a pre-wellordering on its field $F$,
2. $\forall x \in F \ W_x = \Gamma(W_{<x}) \cup W_{<x}$,
3. $\Gamma(F) \subset F$.

For a class of formulas $\mathcal{C}$, $\Gamma$ is a monotone $\mathcal{C}$-operator if and only if $\Gamma \in \mathcal{C}$ and $\Gamma$ satisfies $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y$. The class of monotone $\mathcal{C}$-operators is denoted by mon-$\mathcal{C}$. We also use $\mathcal{C}$-$\text{ML}$ to denote [mon-$\mathcal{C}$]-ID. We refer the reader to our papers [9], [5] for more information on this formalization.

**Theorem 4.2** The following assertions hold over RCA$_0$.

1. $\Sigma^0_2$-$\text{ML} \rightarrow \Sigma^0_2$-Det$^*$. 


(2) $\Sigma^0_2$-Det$^*$ $\rightarrow$ $\Sigma^0_2$-ID.

**Proof.** The idea of the proof is similar to the one used in [9] and [5]. We just mention that since the game is played over the Cantor space, rather than the Baire space, we can replace the $\Sigma^1_1$-operator in [9] and [5] by a $\Sigma^0_2$-operator. □

Now, we turn to investigate the strength of $\Sigma^0_2$-ID. The following lemma provides an alternative definition of $\Pi^1_1$-CA$^\omega$.

**Lemma 4.3** The following assertions hold over RCA$^\omega$.

(1) $\Pi^1_1$-CA $\leftrightarrow$ $(\Sigma^0_1 \land \Pi^0_1)$-Det.

(2) $\Pi^0_1$-MI $\rightarrow$ $\Pi^1_1$-CA.

**Proof.** The proof of the assertion (1) can be found either in [8] or in [7]. The assertion (2) is a straightforward formalization of Hinman's proof [4]. □

**Theorem 4.4** $\Pi^1_1$-CA $\vdash$ $\Pi^1_1$-MI.

**Proof.** Let $\Gamma$ be a monotone $\Pi^1_1$-operator. Using the strategy of a certain $(\Sigma^0_1 \land \Pi^0_1)$-game, we can construct $W$ which satisfies conditions (1), (2) and (3) of Definition 4.1. This completes the proof by the assertion (1) of Lemma 4.3. □

Finally, we give the following corollary.

**Corollary 4.5** The following are equivalent over RCA$^\omega$:

$\Sigma^0_2$-Det$^*$, $\Pi^1_1$-CA$^\omega$, $\Pi^0_1$-MI, $\Sigma^0_2$-ID and $\Pi^1_1$-MI.

**Proof.** It is straightforward from Theorems 4.2 and 4.4. □

Next, we turn to the games which can be written as Boolean combinations of $\Sigma^0_2$-formulas. We first recall the following definitions from [6]. The class $(\Sigma^0_2)_k$ of formulas is defined as follows. For $k = 1$, $(\Sigma^0_2)_1 = \Sigma^0_2$. For $k > 1$, $\psi \in (\Sigma^0_2)_k$ if it can be written as $\psi_1 \land \psi_2$, where $\neg \psi_1 \in (\Sigma^0_2)_{k-1}$ and $\psi_2 \in \Sigma^0_2$. It can be shown that for any formula $\psi$ in the class of Boolean combinations of $\Sigma^0_2$-formulas, there is a $k \in \omega$ such that $\psi \in (\Sigma^0_2)_k$, or more strictly, $\psi$ is equivalent to a formula in $(\Sigma^0_2)_k$.

**Theorem 4.6** Assume $0 < k < \omega$. Then, $(\Sigma^0_2)_{k+1}$-Det$^* \leftrightarrow (\Sigma^0_2)_k$-Det.

**Proof.** $(\rightarrow)$. Let $\psi$ be a $(\Sigma^0_2)_k$-formula and $G^*_\psi$ the infinite game over $\mathbb{N}^\omega$ associated with $\psi$. We explain how to turn $G^*_\psi$ to a $(\Sigma^0_2)_{k+1}$-game over $2^{\mathbb{N}}$, which will be denoted $G^*_\psi$. The idea is the following: In $G^*_\psi$, I starts by playing $n_0$ 0's, then plays 1. Then, II plays $n_1$ 1's and plays 0 and so on. We need to avoid some trivial situation. For instance, player I must not play 0's consecutively for ever. He must
stop after playing finitely may 0's to give II a chance to play. This will make $G^*_{\psi}$ a $(\Sigma^0_k)_{k+1}$-game and hence determinate by our assumption. On the other hand the player who wins $G^*_{\psi}$ can win $G_{\psi}$, which completes the proof of the first direction.

The direction (→) can be proved by using the inductive definition of a combination of $k \Sigma^1_1$-operators, which is equivalent to $(\Sigma^0_k)_{k+1}$-Det by [6]. □

参考文献