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On weak determinacy of infinite binary games

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概要

In [5] and [6], we have investigated the logical strength of the determinacy of infinite games in the Baire space up to $\Delta^0_3$. In this paper we consider infinite games in the Cantor space. Let Det* (resp. Det) stand for the determinacy of infinite games in the Cantor space (resp. the Baire space). In Section 2, we show that $\Delta^0_1$-Det*, $\Sigma^0_2$-Det* and WKL$^0$ are pairwise equivalent over RCA$^0$. In Section 3, we show that RCA$^0 + (\Sigma^0_1 \land V^0_1)$-Det* is equivalent to ACA$^0$. Then, we deduce that RCA$^0 + \Delta^0_2$-Det* is equivalent to ATR$^0$. In the last section, we show some more equivalences among stronger assertions without details, which will be thoroughly treated elsewhere.

1 Preliminaries

In this section, we recall some basic definitions and facts about second order arithmetic. The language $\mathcal{L}_2$ of second order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and unary function variables $f, g, h, \ldots$, consisting of constant symbols $0, 1, +, -, =, <$. We also use set variables $X, Y, Z, \ldots$, intending to range over the set of $\{0, 1\}$-valued functions, that is, characteristic functions of sets.

The formulae can be classified as follows:

- $\phi$ is bounded ($\Pi^0_3$) if it is built up from atomic formulae by using propositional connectives and bounded number quantifiers ($\forall x < t), (\exists x < t)$, where $t$ does not contain $x$.
\( \varphi \) is \( \Pi_0^1 \) if it does not contain any function quantifier. \( \Pi_0^1 \) formulae are called arithmetical formulae.

\( \neg \varphi \) is \( \Sigma_n^i \) if \( \varphi \) is a \( \Pi_n^i \) formula \( (i \in \{0, 1\}, n \in \omega) \).

\( \forall x_1 \ldots \forall x_k \varphi \) is \( \Pi_0^0 \) if \( \varphi \) is a \( \Sigma_0^0 \) formula \( (n \in \omega) \).

\( \forall f_1 \ldots \forall f_k \varphi \) is \( \Pi_1^1 \) if \( \varphi \) is a \( \Sigma_1^1 \) formula \( (n \in \omega) \).

Using above classification, we can define schemata of comprehension and induction as follows.

**Definition 1.1** Assume \( n \in \omega \) and \( i \in \{0, 1\} \). The scheme of \( \Pi_n^i \) **comprehension**, denoted \( \Pi_n^i \text{-CA} \), consists of all the formulae of the form

\[ \exists X \forall x (x \in X \leftrightarrow \varphi(x)) \]

where \( \varphi(x) \) belongs to \( \Pi_n^i \) and \( X \) does not occur freely in \( \varphi(x) \). The scheme of \( \Delta_n^i \) **comprehension**, denoted \( \Delta_n^i \text{-CA} \), consists of all the formulae of the form

\[ \forall n (\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)) \]

where \( \varphi(n) \) is \( \Sigma_n^i \), \( \psi(n) \) is \( \Pi_n^i \), and \( X \) is not free in \( \varphi(n) \). The scheme of \( \Sigma_n^i \) **induction**, denote \( \Sigma_n^i \text{-IND} \), consists of all axioms of the form

\[ (\varphi(0) \land \forall n (\varphi(n) \rightarrow \varphi(n + 1))) \rightarrow \forall n \varphi(n) \]

where \( \varphi(n) \) is \( \Sigma_n^i \).

Now we define a basic subsystem of second order arithmetic, called RCA_0.

**Definition 1.2** RCA_0 is the formal system in the language of \( L_2 \) which consists of discrete order semi-ring axioms for \( (N, +, \cdot, 0, 1, <) \) plus the schemata of \( \Delta_0^1 \) comprehension and \( \Sigma_0^1 \) induction.

The following is a formal version of the normal form theorem for \( \Sigma_0^1 \) relations.

**Theorem 1.3** (normal form theorem) Let \( \varphi(f) \) be a \( \Sigma_0^1 \) formula. Then we can find a \( \Pi_0^0 \) formula \( R(s) \) such that RCA_0 proves

\[ \forall f (\varphi(f) \leftrightarrow \exists m. R(f[m])) \]

where \( f[m] \) is the code for the finite initial segment of \( f \) with length \( m \). Note that \( \varphi(f) \) may contain free variables other than \( f \).
Proof. See also Simpson [7, Theorem II.2.7]. □

We loosely say that a formula is $\Sigma^i_n$ (resp. $\Pi^i_n$) if it is equivalent over a base theory (such as RCA$_0$) to a $\psi \in \Sigma^i_n$ (resp. $\Pi^i_n$).

The next theorem asserts that the universe of functions is closed under the least number operator, i.e., minimization.

**Theorem 1.4 (minimization)** The following is provable in RCA$_0$. Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for all $(n_1, \ldots, n_k) \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $f(m, n_1, \ldots, n_k) = 1$. Then there exists $g : \mathbb{N}^k \to \mathbb{N}$ such that $g(n_1, \ldots, n_k)$ is the least $m$ such that $f(m, n_1, \ldots, n_k) = 1$.

**Proof.** See Simpson [7, Theorem II.3.5]. □

# 2 WKL$_0$ and $\Sigma^0_1$-Det$^*$

Let $X$ be either $\{0, 1\}$ or $\mathbb{N}$ and let $\varphi$ be a formula with a distinct variable $f$ ranging over $X^\mathbb{N}$. A two-person game $G_{\varphi}$ (or simply $\varphi$) over $X^\mathbb{N}$ is defined as follows: player I and player II alternately choose elements from $X$ (starting with I) to form an infinite sequence $f \in X^\mathbb{N}$ and I (resp. II) wins iff $\varphi(f)$ (resp. $\neg \varphi(f)$). A strategy of player I (resp. II) is a map $\sigma : X^{\text{even}} = \{s \in X^{<\mathbb{N}} \mid s \text{ has even length}\} \to X$ (resp. $X^{\text{odd}} \to X$). We say that $\varphi$ is determinate if one of the players has a winning strategy, that is, a strategy $\sigma$ such that the player is guaranteed to win every play $f$ in which he played $f(n) = \sigma([f(n)])$ whenever it was his turn to play.

Given a class of formulas $C$, $C$-determinacy is the axiom scheme which states that any game in $C$ is determinate. We use $C$-Det$^*$ (resp $C$-Det) to denote $C$-determinacy in the Cantor space (resp. the Baire space).

A set $T$ of finite sequences is a tree if it is closed under initial segment, i.e., $t \in T$ and $s \subseteq t$ implies $s \in T$. A function $f$ is a path of $T$ if each initial segment of $f$ is a sequence of $T$.

**Definition 2.1** WKL$_0$ is a subsystem of second order arithmetic whose axioms are those of RCA$_0$ plus weak König's lemma which states that every infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has an infinite path.

Next, we prove the equivalences among WKL$_0$, $\Sigma^0_1$-Det$^*$ and $\Delta^0_1$-Det$^*$.

**Theorem 2.2** RCA$_0$ $\vdash$ $\Delta^0_1$-Det$^*$ $\rightarrow$ WKL$_0$. 

Proof. By way of contradiction, we assume RCA₀ + Δ₀¹-Det* and deny weak König’s lemma. Let T be an infinite binary tree in which there is no infinite path, i.e., there is no f such that ∀nf[n] ∈ T. We consider the following game:

- Player I plays a sequence t of 2<ℕ.
- Player II then answers by playing 0 or 1.
- Player I plays a new sequence u of 2<ℕ.
- Player II then plays a sequence v of 2<ℕ.

The winning conditions of the game are given as follows: II wins if one of the following cases holds.

- t * (i) * u ∉ T.
- t * (1 - i) * v ∈ T ∧ |u| ≤ |v|.

We shall remark that the game always terminates in finite moves, because T has no infinite path. This ensures that the game is Δ₀¹. On the other hand, we can show that player I has no winning strategy by considering two cases, in one of which player II chooses i = 0 and in the other he chooses i = 1 after player I plays t. I can not win in both of them. Therefore, by Δ₀¹-Det* player II has a winning strategy τ. Using τ, we define f : ℕ → {0, 1} as follows:

- f(0) = 1 − τ(⟨⟩),
- f(n + 1) = 1 − τ(f[n]),

By Σ₀¹-induction, we can easily see that f[n] ∈ T for all n, which contradicts with our assumption that T has no infinite path. Thus, Δ₀¹-Det* → WKL. This completes the proof of the theorem. □

Now, we turn to prove the reversal.

Theorem 2.3 WKL₀ ⏐ Π₀¹-Det*.

Proof. Let ϕ(f) be a Σ₀¹-formula with f ∈ 2ℕ. Then, by the normal form theorem, ϕ(f) can be written as ∃nR(f[n]), where R is Π₀³. We define recursive maps g and gₙ from 2<ℕ to {0, 1} for each n ∈ ℕ as follows:

\[
g(s) = \begin{cases} 
1 & \text{if } \exists t \subseteq s \ R(t) \\
0 & \text{if } \forall t \subseteq s \neg R(t)
\end{cases}
\]

\[
g_n(s) = \begin{cases} 
g(s) & \text{if } |s| \geq n \\
\max\{g_n(s * (0)), g_n(s * (1))\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\
\min\{g_n(s * (0)), g_n(s * (1))\} & \text{if } |s| < n \text{ and } |s| \text{ is odd}
\end{cases}
\]
Intuitively, for $n \in \mathbb{N}$, $g_n(\langle \rangle) = 1$ means “player I can win the game by stage $n$,” and $g_n(\langle \rangle) = 0$ means “player I cannot win by stage $n$.”

Claim. The following assertions hold.

1. If $g_n(\langle \rangle) = 1$ for some $n$, then I has a winning strategy.
2. If $g_n(\langle \rangle) = 0$ for every $n$, then II has a winning strategy.

For (1), fix $n$ such that $g_n(\langle \rangle) = 1$. Define $\sigma : 2^{\text{even}} \rightarrow \{0,1\}$ by

$$\sigma(s) = \begin{cases} 0 & \text{if } g_n(s * \langle 0 \rangle) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

We can verify that $\sigma$ is a winning strategy for player I, which completes the proof of the first assertion of the claim.

For (2), suppose that for any $n$, $g_n(\langle \rangle) = 0$ and show that player II has a winning strategy. The idea of the proof is as follows. Consider an infinite binary tree which consists of the moves at which player II can prevent player I from winning the game. A path through such a tree will serve a winning strategy for II. To realize this idea, we will need some coding arguments to construct the tree.

To begin with, fix a lexicographical enumeration $e$ of non-empty sequences of $2^{<\mathbb{N}}$. For instance, $e(\langle 0 \rangle) = 0$, $e(\langle 1 \rangle) = 1$, $e(\langle 0,0 \rangle) = 2$, and so on. Using $e$, we can regard any $s \in 2^{<\mathbb{N}}$ as a partial strategy (i.e., a finite segment of the strategy) for player II (cf. [1]). We define $T_s$ to be the tree consisting of all partial plays in which player II follows $s$. More precisely, $T_s$ is defined as follows:

$$t \in T_s \iff \forall k(2k+1 < |t| \rightarrow t(2k+1) = s(e(\langle t(0), \ldots, t(2k) \rangle))).$$

Finally we define $T$, a set of all moves which avoid the winning of player I, as follows:

$$s \in T \iff \forall t \in T_s g_{h(s)}(t) = 0,$$

where $h : 2^{<\mathbb{N}} \rightarrow \mathbb{N}$ is defined by $h(s) = \max\{|t| : t \in T_s\}$. Clearly $T$ is recursive, therefore it exists in $\text{RCA}_0$. On the other hand, the assumption $\forall n g_n(\langle \rangle) = 0$ implies that $T$ is infinite. Thus, $T$ has an infinite path $f$ by weak König’s lemma.

Now, we define $\tau : 2^{\text{odd}} \rightarrow \mathbb{N}$ as:

$$\tau(s) = f(e(s(0),...,s(|s|-2))),$$

and then we can verify that $\tau$ is a winning strategy for player II, which completes the proof. $\square$
3 ATR$_0$ and $\Delta^0_2$-Det$^*$

In this section we aim to show that RCA$_0 + \Delta^0_2$-Det$^*$ and ATR$_0$ are equivalent. We first give the definitions of ACA$_0$ and ATR$_0$.

Definition 3.1 The system ACA$_0$ consists of the discrete order semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemes of $\Sigma^0_1$ induction and arithmetical comprehension.

Since comprehension axioms admit free variables, $\Pi^0_1$ comprehension is already as strong as arithmetical comprehension.

Lemma 3.2 The following are pairwise equivalent over ACA$_0$.

(1) arithmetical comprehension;

(2) $\Pi^0_1$ comprehension.

Proof. See Simpson [7, Lemma III.1.3]. □

Definition 3.3 ATR$_0$ consists of RCA$_0$ augmented by the following axiom, called arithmetical transfinite recursion: For any set $X \subset \mathbb{N}$ and for any well-ordering relation $\prec$, there exists a set $H \subset \mathbb{N}$ such that

- if $b$ is the $\prec$-least element, then $(H)_b = X$,
- if $b$ is the immediate successor of $a$ w.r.t. $\prec$, then $\forall n ((H)_b \leftrightarrow \psi(n, (H)_a))$,
- if $b$ is a limit, then $\forall a \forall n ((n, a) \in (H)_b \rightarrow (a \prec b \wedge n \in (H)_a))$,

where $\psi$ is a $\Pi^0_1$-formula and $(H)_a = \{x : (x, a) \in H\}$, where $(x, b)$ denotes the code of the pair $(x, a)$.

ATR$_0$ is obviously stronger than ACA$_0$, but it is contained in $\Pi^1_1$-ACA$_0$.

Lemma 3.4 The following are pairwise equivalent over RCA$_0$:

$\Delta^0_1$-Det, $\Sigma^0_1$-Det and ATR$_0$.

Proof. See [7] or [8].

The class $\Sigma^0_1 \land \Pi^0_1$ is defined as follows. $\varphi$ is $\Sigma^0_1 \land \Pi^0_1$ if and only if $\varphi$ is of the form $\psi_0 \land \neg \psi_1$, where $\psi_0$ and $\psi_1$ are $\Sigma^0_1$. The following theorems characterize ($\Sigma^0_1 \land \Pi^0_1$) determinacy in the Cantor space.

Theorem 3.5 ACA$_0$ proves $(\Sigma^0_1 \land \Pi^0_1)$-Det$^*$. 

Proof. Let $\varphi$ be of the form $\exists n R_0(f[n]) \land \forall n R_1(f[n])$. We define the functions $g$, $g_n$, $g'$, and $g'_n$ from $2^{<\mathbb{N}}$ to $\{0, 1\}$ as follows:
\[
\begin{align*}
\bullet \ g(s) &= \begin{cases} 
1 & \text{if } \exists t \subseteq s R_0(t) \\
0 & \text{if } \forall t \subseteq s \neg R_0(t)
\end{cases} \\
\bullet \ g_n(s) &= \begin{cases} 
g(s) & \text{if } |s| \geq n \\
\max\{g_n(s \langle 0 \rangle), g_n(s \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\
\min\{g_n(s \langle 0 \rangle), g_n(s \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd}
\end{cases} \\
\bullet \ g'(s) &= \begin{cases} 
1 & \text{if } \forall t \subseteq s R_1(t) \\
0 & \text{if } \exists t \subseteq s \neg R_1(t)
\end{cases} \\
\bullet \ g_n'(s) &= \begin{cases} 
g'(s) & \text{if } |s| \geq n \\
\max\{g_n'(s \langle 0 \rangle), g_n'(s \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\
\min\{g_n'(s \langle 0 \rangle), g_n'(s \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd}
\end{cases}
\end{align*}
\]
Following a similar argument of the one used in the proof of Theorem 2.2, we can prove

**Claim:** if there exists a such that \( g_a(\langle \rangle) \cdot g'_a(\langle \rangle) = 1 \) for all \( m > n \) then I has a winning strategy, otherwise player II has a winning strategy.

This complete the proof of the theorem. \( \square \)

**Theorem 3.6** RCA_0 \vdash (\Sigma^0_1 \land \Pi^0_1)-Det^* \rightarrow ACA_0

**Proof.** Let \( \varphi(n) \) be a \( \Sigma^0_1 \)-formula. We need to construct a set \( X \) such that for any \( n \in \mathbb{N} \), \( \varphi(n) \leftrightarrow n \in X \). To construct \( X \), consider the following game: player I asks II about \( n \) by playing 0 consecutively \( n \) times and playing 1 after that (if he plays 0 for ever, he loses). II ends the game by answering 0 or 1.

Now, suppose that player I plays \( n \)'s and a 1 consecutively. Player II wins if one of the following cases holds.

- II answers 1 and \( \varphi(n) \).
- II answers 0 and \( \neg \varphi(n) \).

Clearly, I has no winning strategy. By \( (\Sigma^0_1 \land \Pi^0_1)-\text{Det}^* \), let \( \tau \) be a winning strategy of player II. We defined a set \( X \) by:

\[
\forall n, \varphi(n) \leftrightarrow n \in X.
\]

The set \( X \) exists by \( \Pi^0_1 \) comprehension. Moreover, we can verify that \( \forall n, \varphi(n) \leftrightarrow n \in X \), which completes the proof. \( \square \)

Let \( < \) be a recursive well-ordering on \( \mathbb{N} \). We define a recursive well-ordering \( <^* \) on \( \mathbb{N} \times \{0,1\} \) as follows:

\[
(x, i) <^* (y, j) \text{ iff } x < y \lor (x = y \land i < j).
\]
Let \( X \) be either \( \mathbb{N} \) or \( \{0,1\} \). We say that a formula \( \varphi(n,i,f) \) with distinct free variable \( f \) ranging over \( X^\mathbb{N} \) is decreasing along \( \prec^* \) if and only if
\[
\forall n \forall i \forall m \forall j \quad (((m,j) \prec^*(n,i) \land \varphi(n,i,f)) \rightarrow \varphi(m,j,f)),
\]
for all \( f \).

The following lemma will play a key role to characterize \( \Delta_2^0 \)-Det*.

**Lemma 3.7** It is provable in \( \text{RCA}_0 \) that a formula \( \psi \) is \( \Delta_2^0 \) if and only if:
\[
\psi(f) \rightarrow \exists x (\varphi(x,0,f) \land \neg \varphi(x,1,f))
\]
where \( \varphi \) is \( \Pi_1^0 \) and it is decreasing along some recursive well-ordering relation \( \prec^* \).

**Proof.** See [8] for the proof. \( \square \)

**Theorem 3.8** \( \text{ATR}_0 \) is equivalent to \( \text{RCA}_0 + \Delta_2^0 \)-Det*.

**Proof.** The proof is a modification of the proof of Theorem 6.1 in [8]. By Theorem 3.6 and Lemma 3.7, \( \Delta_2^0 \)-Det* is just a transfinite iteration of arithmetical comprehension, which is the same as \( \text{ATR}_0 \). \( \square \)

### 4 Further classes of games

In this section, we summarize our results about the determinacy of Boolean combinations of \( \Sigma_2^0 \)-games. The detailed treatment of these results will appear in our forthcoming paper.

We start by formalizing the inductive definition of a class of operators.

**Definition 4.1** Given a a class of formulas \( \mathcal{C} \), the axiom \( \mathcal{C} \)-ID asserts that for any operator \( \Gamma \in \mathcal{C} \), there exists \( W \subset \mathbb{N} \times \mathbb{N} \) such that
1. \( W \) is a pre-wellordering on its field \( F \),
2. \( \forall x \in F \quad W_x = \Gamma(W_{<x}) \cup W_{<x} \),
3. \( \Gamma(F) \subset F \).

For a class of formulas \( \mathcal{C} \), \( \Gamma \) is a monotone \( \mathcal{C} \)-operator if and only if \( \Gamma \in \mathcal{C} \) and \( \Gamma \) satisfies \( \Gamma(X) \subset \Gamma(Y) \) whenever \( X \subset Y \). The class of monotone \( \mathcal{C} \)-operators is denoted by \( \text{mon-} \mathcal{C} \). We also use \( \mathcal{C} \)-M1 to denote \([\text{mon-} \mathcal{C}]\)-ID. We refer the reader to our papers [9], [5] for more information on this formalization.

**Theorem 4.2** The following assertions hold over \( \text{RCA}_0 \).
1. \( \Sigma_2^0 \)-M1 \( \rightarrow \Sigma_2^0 \)-Det*.
\((2) \Sigma^0_2 - \text{Det}\^* \rightarrow \Sigma^0_2 - \text{ID}\).

**Proof.** The idea of the proof is similar to the one used in [9] and [5]. We just mention that since the game is played over the Cantor space, rather than the Baire space, we can replace the \(\Sigma^1_1\)-operator in [9] and [5] by a \(\Sigma^0_2\)-operator. □

Now, we turn to investigate the strength of \(\Sigma^0_2\)-ID. The following lemma provides an alternative definition of \(\Pi^1_1\)-CA₀.

**Lemma 4.3** The following assertions hold over RCA₀.

\((1) \Pi^1_1\)-CA \(\leftrightarrow (\Sigma^0_1 \land \Pi^0_1\)-Det.

\((2) \Pi^0_1\)-MI \(\rightarrow \Pi^1_1\)-CA.

**Proof.** The proof of the assertion (1) can be found either in [8] or in [7]. The assertion (2) is a straightforward formalization of Hinman's proof [4]. □

**Theorem 4.4** \(\Pi^1_1\)-CA \(\vdash \Pi^1_1\)-MI.

**Proof.** Let \(\Gamma\) be a monotone \(\Pi^1_1\)-operator. Using the strategy of a certain \((\Sigma^0_1 \land \Pi^0_1)\)-game, we can construct \(W\) which satisfies conditions (1), (2) and (3) of Definition 4.1. This completes the proof by the assertion (1) of Lemma 4.3. □

Finally, we give the following corollary.

**Corollary 4.5** The following are equivalent over RCA₀:

\(\Sigma^0_2\)-Det\^*, \(\Pi^1_1\)-CA₀, \(\Pi^0_1\)-MI, \(\Sigma^0_2\)-ID and \(\Pi^1_1\)-MI.

**Proof.** It is straightforward from Theorems 4.2 and 4.4. □

Next, we turn to the games which can be written as Boolean combinations of \(\Sigma^0_2\) formulas. We first recall the following definitions from [6]. The class \((\Sigma^0_2)^k\) of formulas is defined as follows. For \(k = 1\), \((\Sigma^0_2)^1 = \Sigma^0_2\). For \(k > 1\), \(\psi \in (\Sigma^0_2)^k\) if it can be written as \(\psi_1 \land \psi_2\), where \(\neg \psi_1 \in (\Sigma^0_2)^{k-1}\) and \(\psi_2 \in \Sigma^0_2\). It can be shown that for any formula \(\psi\) in the class of Boolean combinations of \(\Sigma^0_2\)-formulas, there is a \(k \in \omega\) such that \(\psi \in (\Sigma^0_2)^k\), or more strictly, \(\psi\) is equivalent to a formula in \((\Sigma^0_2)^k\).

**Theorem 4.6** Assume \(0 < k < \omega\). Then, \((\Sigma^0_2)^{k+1}\)-Det\^* \(\leftrightarrow (\Sigma^0_2)^k\)-Det.

**Proof.** \((\rightarrow)\). Let \(\psi\) be a \((\Sigma^0_2)^k\)-formula and \(G_\psi\) the infinite game over \(\mathbb{N}^\mathbb{N}\) associated with \(\psi\). We explain how to turn \(G_\psi\) to a \((\Sigma^0_2)^{k+1}\)-game over \(2^\mathbb{N}\), which will be denoted \(G_\psi^*\). The idea is the following: In \(G_\psi^*\), I starts by playing \(n_0\) 0's, then plays 1. Then, II plays \(n_1\) 1's and plays 0 and so on. We need to avoid some trivial situation. For instance, player I must not play 0's consecutively for ever. He must
stop after playing finitely may 0's to give II a chance to play. This will make \( G_{\psi}^{*} \) a \( (\Sigma^{2})_{k+1} \)-game and hence determinate by our assumption. On the other hand the player who wins \( G_{\psi}^{*} \) can win \( G_{\psi} \), which completes the proof of the first direction.

The direction \((\leftarrow)\) can be proved by using the inductive definition of a combination of \( k \Sigma_{1}^{1} \)-operators, which is equivalent to \((\Sigma^{2})_{k}\)-Det by [6]. \( \square \)

**Reference Text**


