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Weakly o-minimal structures

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1 Introduction

Let $(M, <)$ be a dense linear ordering without endpoints and $A$ a subset of $M$. The set $A$ is said to be convex if for all $a, b \in A$ and $c \in M$ with $a < c < b$ we have $c \in A$. A structure $(M, <, \ldots)$ equipped with a dense linear ordering $<$ without endpoints is said to be o-minimal (weakly o-minimal) if every definable\(^1\) subset of $M$ is a finite union of intervals (convex sets) in $(M, <)$, respectively. A theory $T$ is said to be weakly o-minimal if every model of $T$ is weakly o-minimal.

It is well-known that the monotonicity theorem of [3] fails in a weakly o-minimal structure. However Arfiev [1] showed that the "weaker" version of the monotonicity theorem of [3] holds in any weakly o-minimal structure. In this paper we survey Arfiev's results.

2 Preliminaries

Let $M$ be a weakly o-minimal structure. For each $A, B \subseteq M$ we write $A < B$ if $a < b$ whenever $a \in A$ and $b \in B$. An ordered pair $(C, D)$ of non-empty definable subsets in $M$ is called a definable cut if $C < D$, $C \cup D = M$ and $D$ has no lowest elements. The set of all definable cuts in $M$ will be denoted by $\overline{M}$. Moreover we define a linear order on $\overline{M}$ by $(C_1, D_1) < (C_2, D_2)$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M, <)$

\(^1\)Throughout this paper 'definable' means 'definable with parameters'.
as a substructure of \((\overline{M}, <)\) by identifying an element \(a \in M\) with the definable cut \(\langle -\infty, a \rangle, (a, \infty)\).

Let \(A\) be a definable subset of \(M^n\). Then a function \(f : A \to \overline{M}\) is said to be definable if the set \(\{\langle \overline{x}, y \rangle \in A \times M : f(\overline{x}) > y\}\) is definable.

**Remark 1** Let \(A\) be a definable subset of \(M^n\). Suppose that \(f\) is a function from \(A\) into \(\overline{M}\). Then the following conditions are equivalent:

1. \(f\) is definable;
2. there exists a formula \(\varphi(\overline{x}, y)\) with parameters such that \(f(\overline{a}) = \sup_\varphi(\overline{a}, M)\) whenever \(\overline{a} \in A\).

**Definition 2** Let \(f : A \to \overline{M}\) be a function, where \(A\) is a subset of \(M\). Then \(f\) is said to be tidy if one of the following holds:

1. for each \(a \in A\) there exists an open interval \(I \subseteq A\) with \(a \in I\) such that \(f \upharpoonright I\) is strictly increasing, in which case \(f\) is said to be locally increasing on \(A\);
2. for each \(a \in A\) there exists an open interval \(I \subseteq A\) with \(a \in I\) such that \(f \upharpoonright I\) is strictly decreasing, in which case \(f\) is said to be locally decreasing on \(A\);
3. for each \(a \in A\) there exists an open interval \(I \subseteq A\) with \(a \in I\) such that \(f \upharpoonright I\) is constant, in which case \(f\) is said to be locally constant on \(A\).

**Definition 3** Let \(f : A \to \overline{M}\) be a function, where \(A\) is a subset of \(M\). Then \(f\) is said to be have the local minimum throughout \(A\) if for each \(a \in A\) there exist \(b_0, b_1 \in A\) with \(b_0 < a < b_1\) such that for each \(c \in (b_0, b_1) \setminus \{a\}\) we have \(f(a) < f(c)\). Similarly, we define that \(f\) has the local maximum throughout \(A\).

**Definition 4** A weakly \(\omega\)-minimal structure \(M\) is said to be have monotonicity if for each definable function \(f : A \subseteq M \to \overline{M}\) there exists \(n \in \mathbb{N}\) and a partition of \(A\) into definable sets \(X, I_0, \ldots, I_n\) such that \(X\) is finite, \(I_0, \ldots, I_n\) are open convex sets and for each \(i \leq n\) the function \(f \upharpoonright I_i\) is tidy.
Arfiev showed the following.

**Theorem 5** ([1]) *Every weakly o-minimal structure M has monotonicity.*

In the next section we give the proof for Theorem 5.

### 3 Proof of Theorem 5

Throughout this section we assume that \((M, <, \ldots)\) is a weakly o-minimal structure and \(f\) is a definable function from definable subset \(A\) of \(M\) into \(\overline{M}\). We now define the following formulas:

\[
\varphi_0(x) \equiv \exists x_1 > x (\forall y \in (x, x_1) f(y) < f(x));
\]

\[
\varphi_1(x) \equiv \exists x_1 > x (\forall y \in (x, x_1) f(y) = f(x));
\]

\[
\varphi_2(x) \equiv \exists x_1 > x (\forall y \in (x, x_1) f(y) > f(x));
\]

\[
\psi_0(x) \equiv \exists x_0 < x (\forall y \in (x_0, x) f(y) < f(x));
\]

\[
\psi_1(x) \equiv \exists x_0 < x (\forall y \in (x_0, x) f(y) = f(x));
\]

\[
\psi_2(x) \equiv \exists x_0 < x (\forall y \in (x_0, x) f(y) > f(x));
\]

\[
\theta_{ij}(x) \equiv \psi_i(x) \land \varphi_j(x) \text{ for each } i, j \leq 2.
\]

To show Theorem 5, we first prove some lemmas needed later.

**Lemma 6** ([2]) *For each \(x \in \text{Int}(A)\), there exist \(i, j \leq 2\) such that \(\theta_{ij}(x)\) holds.*

*Proof.* Suppose that there exists some \(x \in \text{Int}(A)\) such that \(\varphi_j(x)\) does not hold for some \(j \leq 2\). Then the set \(\{y \in A : f(y) \leq f(x)\}\) cannot be written as a union of finitely many convex sets, contradicting that \(M\) is weakly o-minimal. Thus, for each \(x \in \text{Int}(A)\) there exists some \(j \leq 2\) such that \(\varphi_j(x)\) holds. Similarly, for each \(x \in \text{Int}(A)\) there exists some \(i \leq 2\) such that \(\psi_i(x)\) holds. \(\square\)

**Lemma 7** ([2]) *There exists a partition of \(A\) into finitely many points and open convex sets such that each open convex set lies in the solution set of some formula \(\theta_{ij}\).*
Proof. By Lemma 6, there exists a finite subset $X$ of $A$ such that we have $X \cup \bigcup_{i,j \leq 2} \theta_{ij} = A$. For each $i,j \leq 2$, by weak $o$-minimality of $M$, the set $\theta_{ij}$ can be written as a union of finitely many points and open convex sets. This finishes the proof. 

Lemma 8 ([2, Lemma 3.6]) Let $I$ be an open interval of $M$. Then it cannot happen that one of the formulas $\theta_{01}, \theta_{10}, \theta_{12}, \theta_{21}$ holds throughout $I$.

Proof. Suppose that the formula $\theta_{01}$ holds throughout $I$. Let $x$ be an element of $I$. Since $\varphi_1(x)$ holds, there exists some $x_1 > x$ with $x_1 \in I$ such that for each $y \in (x, x_1)$ we have $f(x) = f(y)$. Let $z$ be an element of open interval $(x, x_1)$. Since $\psi_0(z)$ holds, there exists some $w$ with $x < w < z$ such that for each $y \in (w, z)$ we have $f(y) < f(z)$, a contradiction.

The other cases are similar. 

We show the next lemma later.

Lemma 9 Let $I$ be an open interval of $M$. Suppose that $h : I \to \overline{M}$ has the local minimum or maximum throughout $I$. Then $h$ is not definable.

Lemma 10 Let $I$ be an open interval of $M$. Then it cannot happen that one of the formulas $\theta_{00}$ and $\theta_{22}$ holds throughout $I$.

Proof. This lemma follows from Lemma 9.

Lemma 11 Let $I$ be a non-empty open definable convex subset of $M$ such that $\theta_{02}$ holds throughout $I$. Then there exists $n \in \mathbb{N}$ and a partition of $I$ into definable sets $X, I_0, \ldots, I_n$ such that $X$ is finite, $I_0, \ldots, I_n$ are open convex sets and for each $i \leq n$ the function $f \upharpoonright I_i$ is locally increasing. Similarly, if $\theta_{20}$ holds throughout $I$, the same conclusion holds with ‘locally increasing’ replaced by ‘locally decreasing’.

Proof. Suppose that $\theta_{02}(x)$ holds throughout $I$. We define the following formulas:

\[ \chi_0(x) \equiv \forall x_1 > x [\exists y, z (x < y < z < x_1 \land f(z) \leq f(y))]; \]
\[ \chi_2(x) \equiv \forall x_0 < x [\exists y, z (x_0 < y < z < x \land f(z) \leq f(y))]. \]

Claim $\chi_0(x)$ and $\chi_2(x)$ cannot hold throughout a subinterval of $I$. 
Proof of Claim. Suppose for a contradiction that $\chi_0(x)$ holds throughout a subinterval $I_0 \subseteq I$. The argument for $\chi_2(x)$ is similar. For each $a \in I_0$, we define the following:

$$V_a := \{x \in I_0 : x < a \text{ and if } y \in [x, a), \text{then } f(y) < f(a)\}$$

$$\cup \{x \in I_0 : x > a \text{ and if } y \in (a, x], \text{then } f(y) > f(a)\} \cup \{a\};$$

$$g(a) := \inf V_a.$$

Since $\theta_{02}(x)$ holds throughout $I_0$, the set $V_a$ is an infinite definable convex set and $a$ is not a boundary point of $V_a$. Then, by Lemma 9, it suffices to show that $g$ has the local minimum throughout $I_0$. We define the following formulas:

$$\mu_0(x, a) \equiv x < a \land g(x) \leq g(a);$$

$$\mu_1(x, a) \equiv x < a \land g(x) > g(a);$$

$$\nu_0(x, a) \equiv x > a \land g(x) \leq g(a);$$

$$\nu_1(x, a) \equiv x > a \land g(x) > g(a).$$

By weak $\omega$-minimality of $M$, for each $a \in I_0$, there exist open interval $J \subseteq I_0$ and $K \subseteq I_0$ with $J < a < K$ such that $a$ is a boundary point of $J$ and $K$, either $\mu_0(x, a)$ or $\mu_1(x, a)$ holds throughout $J$, and either $\nu_0(x, a)$ or $\nu_1(x, a)$ holds throughout $K$. Then, it suffices to show that $\mu_1(x, a)$ holds throughout $J$ and $\nu_1(x, a)$ holds throughout $K$. Suppose for a contradiction that $\mu_0(x, a)$ holds throughout $J$. The argument for $\nu_0(x, a)$ is similar. Since $a$ is not a boundary point of $V_a$, there exists $b \in V_a \cap J$. Since $\chi_0(b)$ holds, there exist $c, d \in V_a \cap J$ such that $b < c < d < a$ and $f(d) \leq f(c)$. Hence, by the definition of $g$, we have $b < c \leq g(d)$. Now, since $b$ is an element of $V_a$, we have $g(a) \leq b < g(d)$, contradicting that $\mu_0(d, a)$ holds.

By the claim, the set $\{x \in I : \chi_0(x) \lor \chi_2(x)\}$ is finite. Hence, we finish the proof.

Proof of Theorem 5. By Lemma 6 through 11, the theorem follows.

Finally, we show Lemma 9.

Proof of Lemma 9. Suppose that $h : I \to \overline{M}$ has the local minimum throughout $I$. Suppose for a contradiction that $h$ is definable.
Claim 1 We may assume that $h$ is injective.

Proof of Claim 1. Define the following equivalence relation on $I^2$:

$$E(x,y) \iff h(x) = h(y).$$

We first verify that every equivalence class on $E$ is finite. Let $A$ be an infinite class. Then, by weak o-minimality of $M$, there exists an open subinterval $J$ of $A$. Since $h$ has the local minimum throughout $J$, for each $x \in J$ there exists $y \in J$ such that we have $h(y) > h(x)$, a contradiction. Hence, every equivalence class on $E$ is finite. Therefore the set $Z := \{x \in I : \forall y(E(x,y) \rightarrow x \leq y)\}$ is infinite. By weak o-minimality of $M$, there exists an open subinterval $J'$ of $Z$. We may assume $\text{dom}(h) = J'$. \qed

From now on, by Claim 1, we assume that the function $h$ is injective. For each $a, b \in I$, we define the following:

$$U_a := \{x : x > a \land \forall y \in (a,x)(h(y) > h(a))\}$$

$$\cup \{x : x < a \land \forall y \in [x,a)(h(y) > h(a))\} \cup \{a\};$$

$$a < b \iff U_a \supset U_b.$$ Then, since $h$ has the local minimum throughout $I$, for each $a \in I$ the set $U_a$ is an infinite definable convex set. The predicate $<$ is a partial ordering.

Claim 2 Let $a, b, c \in I$. Suppose that $a, b, c$ are pairwise distinct. Then the following hold.

1. $U_a \neq U_b$;
2. $a$ is not a boundary point of $U_a$;
3. $b \in U_a \iff a < b$;
4. If $U_a \cap U_b \neq \emptyset$, then either $a < b$ or $b < a$;
5. If $a < b < c$ and $a < c$, then $a < b$;
6. If $a < b < c$ and $a < b$, then $a < c$;
7. If $b < a$ and $c < a$, then $b < c$ or $c < b$;
8. $C_a := \{x \in I : x < a\}$ is finite.
Proof of Claim 2.

(1): $h$ は単調増加, $h(a) 
eq h(b)$ となる. したがって $U_a 
eq U_b$ である.
(2): $h$ の仮定より, これはいえる.
(3): 明らか。

(⇒) $b \in U_a$ とする. このとき $h(b) > h(a)$ である. ここで, 一般性を失うことなしに $a < b$ とする. $c \in U_b$ を任意にとってこのとき, $a \leq c \leq b$ ならば $U_a$ は convex なので, $c \in U_a$ である. また, $b < c$ ならば $c \in U_b$ より, 任意の $d \in (b, c]$ に対して $h(d) > h(b) > h(a)$ である. よって $c \in U_a$ となる. 同様に, $c < a$ ならば $c \in U_a$ となる. このことから $U_a \supseteq U_b$ がいえる.

(4): 一般性を失うことなしに $a < b$ とする. 仮定より $c \in U_a \cap U_b$ かつ $a < c < b$ を満たす元が存在する. まず $h(a) < h(b)$ と思う. 任意に $d \in U_b$ をとる. $a \leq d \leq c$ ならば $U_a$ は convex なので, $d \in U_a$ である. また, $c < d$ ならば任意の $e \in (c, d]$ に対し $h(e) > h(b) > h(a)$ となる. よって, $d \in U_a$ となる. 同様に, $d < a$ ならば $d \in U_a$ となる. したがって $U_b \subseteq U_a$ が成り立つ. 同様に $h(b) < h(a)$ ならば $U_a \subseteq U_b$ が成り立つ.

(5): $b < a$ になったとする. すると仮定より, $U_a \supsetneq U_b \supsetneq U_c$ かつ $b < a < c$ となる. よって $b, c \in U_b$ となり, $U_b$ は convex なので $a \in U_b$ がいえる. これは矛盾する.

(6): (5) 同様に示せる.

(7): 仮定より $a \in U_b \cap U_c$ である. よって, $U_b \cap U_c \neq \emptyset$ が成り立つ. すると, (4) より結論がいえる.

(8): $C_a$ が有限集合だったとする. このとき $M$ は weakly o-minimal より, ある開区間 $J$ が存在して $J \subseteq C_a$ となる. $b \in J$ を任意にとって, $b$ は $U_b$ の境界ではないので, $c, d \in U_b \cap J$ かつ $c < b < d$ を満たすものが存在する. すると, $c, d \in C_a$ より, $c < a$ かつ $d < a$ となる. ここで (7) より, $c < d$ または $d < c$ である. $c < d$ とする ($d < c$ の場合も同様に示せる). すると, $d \in U_c$ かつ $c < b < d$ となる. $U_c$ は convex より, $b \in U_c$ である. これは $c \in U_b$ に反する. □

Claim 2 の (8) より $<$ は離散順序である. ここで

$$ K := \{ x \in I : \text{任意の } y \in I \text{ に対し, } y \notin x \}; $$

$$ \tilde{a} := \{ x \in I : a < x \text{ かつ } a < y < x \text{ を満たす } y \text{ は存在しない } \} $$

と定義する.

Claim 3 The following conditions hold:
1. $I \setminus K = \bigcup_{a \in I} \tilde{a}$;

2. the set $K$ is finite;

3. the set $\tilde{a}$ is finite.

Proof of Claim 3.

(1): 任意に $a \in I$ と $b \in K$ をとる．すると $K$ の定義より，$a \not\in b$ である．
次に，任意に $c \in I \setminus K$ をとると，$c \not\in K$ より，ある元 $d \in I$ が存在し
て $d < c$ が成り立つ．ここで $\prec$ は離散順序より，ある元 $d' \in I$ が存在して
$c \in d'$ がいえる．
また $e_1 \neq e_2$ を任意にとる．もし $e_1 \cap e_2 \neq \emptyset$ であったとすると，ある
元 $x \in e_1 \cap e_2$ がとれる．すると $e_1 < x$ かつ $e_2 < x$ だから Claim 2 の (7)
より，$e_1 < e_2$ または $e_2 < e_1$ である．$e_1 < e_2$ と思う ($e_2 < e_1$ の場合も同
様)．すると，$e_1 < e_2 < x$ となるが，これは $x \not\in e_1$ に反する．したがって，
$e_1 \cap e_2 = \emptyset$ である．

(2): $K$ が無限集合だったとする．このとき $M$ は weakly o-minimal よ
り，ある開区間 $J$ が存在して $J \subseteq K$ となる．$b \in J$ を任意にとる．$b$ は $U_b$
の境界ではないので，ある元 $c \in U_b \cap J$ が存在する．すると，$c \not\in U_b$ より，
$b < c$ となる．これは $c \in J \subseteq K$ に反する．

(3): $\tilde{a}$ が無限集合だったとする．このとき $M$ は weakly o-minimal より，
ある開区間 $J$ が存在して $J \subseteq \tilde{a}$ となる．$b \in J$ を任意にとる．$b$ は $U_b$
の境界ではないので，ある元 $c \in U_b \cap J$ が存在する．$b \in \tilde{a}$ かつ $c \not\in U_b$ より，
$a < b < c$ となる．これは $c \in J \subseteq \tilde{a}$ に反する．

Claim 3 の (2) より，$I \setminus K$ は有限集合である．さて任意の $a, b \in I \setminus K$
に対して，

$$E'(a, b) \iff M \models \exists c \in I \ (a \in \tilde{c} \land b \in \tilde{c})$$

と定義すると，$E'(x, y)$ は Claim 3 の (1) より $(I \setminus K)^2$ 上の同値関係にな
る．また Claim 3 の (3) より，$E'(x, y)$ の各クラスは有限集合である．よっ
て $X := \{x \in I \setminus K : M \models \forall y \in I \setminus K \ (E'(x, y) \rightarrow x \leq y)\}$ は無限集合に
なる．

さて $X$ の definable convex な構成要素で $\prec$ に関して最大のものを $Y$
とする．$a \in Y$ をとる．$a$ は $U_a$ の境界ではないので，$b_1, b_2 \in U_a$ かつ
$b_1 < a < b_2$ となる元が存在する．すると $\prec$ は離散順序より，$a < b \leq b_2$ か
つ$a < b' \leq b_1$となる元たちが存在する．このとき，$E'(b, b')$である．また
$b_1 < a < b_2$だから，Claim 2 の (5) より $b' < a < b$となる．よって，$b \notin X$
がいえる．ところで$b$ は $U_b$ の境界ではないので，$c \in U_b$ かつ $b < c$となる
元が存在する．すると，$\tilde{c}$ は有限より，$d \in \tilde{c}$ かつ $d \in X$ となるものがとれ
る．よって $b < c < d$ かつ $b < c$なので，Claim 2 の (6) より $b < d$となる．
これは $Y$ の性質に反する．
したがって，$h$ は definable ではない．

参考文献


