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Weakly o-minimal structures

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1 Introduction

Let $(M,<)$ be a dense linear ordering without endpoints and A a subset of M. The set A is said to be *convex* if for all $a, b \in A$ and $c \in M$ with $a < c < b$ we have $c \in A$. A structure $(M, <, \ldots)$ equipped with a dense linear ordering $<$ without endpoints is said to be *o-minimal* (*weakly o-minimal*) if every definable[1] subset of M is a finite union of intervals (convex sets) in $(M,<)$, respectively. A theory T is said to be *weakly o-minimal* if every model of T is weakly o-minimal.

It is well-known that the monotonicity theorem of [3] fails in a weakly o-minimal structure. However Arfiev [1] showed that the "weaker" version of the monotonicity theorem of [3] holds in any weakly o-minimal structure. In this paper we survey Arfiev's results.

2 Preliminaries

Let M be a weakly o-minimal structure. For each $A, B \subseteq M$ we write $A < B$ if $a < b$ whenever $a \in A$ and $b \in B$. An ordered pair $\langle C, D\rangle$ of non-empty definable subsets in M is called a *definable cut* if $C < D$, $C \cup D = M$ and D has no lowest elements. The set of all definable cuts in M will be denoted by $\overline{M}$. Moreover we define a linear order on $\overline{M}$ by $\langle C_1, D_1\rangle < \langle C_2, D_2\rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M,<)$

[1]Throughout this paper 'definable' means 'definable with parameters'.

as a substructure of $(\overline{M}, <)$ by identifying an element $a \in M$ with the definable cut $\langle(-\infty, a], (a, \infty)\rangle$.

Let A be a definable subset of M^n. Then a function $f : A \to \overline{M}$ is said to be *definable* if the set $\{\langle \overline{x}, y\rangle \in A \times M : f(\overline{x}) > y\}$ is definable.

Remark 1 *Let A be a definable subset of M^n. Suppose that f is a function from A into $\overline{M}$. Then the following conditions are equivalent:*

1. *f is definable;*

2. *there exists a formula $\varphi(\overline{x}, y)$ with parameters such that $f(\overline{a}) = \sup\varphi(\overline{a}, M)$ whenever $\overline{a} \in A$.*

Definition 2 Let $f : A \to \overline{M}$ be a function, where A is a subset of M. Then f is said to be *tidy* if one of the following holds:

1. for each $a \in A$ there exists an open interval $I \subseteq A$ with $a \in I$ such that $f \restriction I$ is strictly increasing, in which case f is said to be *locally increasing* on A;

2. for each $a \in A$ there exists an open interval $I \subseteq A$ with $a \in I$ such that $f \restriction I$ is strictly decreasing, in which case f is said to be *locally decreasing* on A;

3. for each $a \in A$ there exists an open interval $I \subseteq A$ with $a \in I$ such that $f \restriction I$ is constant, in which case f is said to be *locally constant* on A.

Definition 3 Let $f : A \to \overline{M}$ be a function, where A is a subset of M. Then f is said to be *have the local minimum throughout* A if for each $a \in A$ there exist $b_0, b_1 \in A$ with $b_0 < a < b_1$ such that for each $c \in (b_0, b_1) \setminus \{a\}$ we have $f(a) < f(c)$. Similarly, we define that f *has the local maximum throughout* A.

Definition 4 A weakly o-minimal structure M is said to be *have monotonicity* if for each definable fuction $f : A \subseteq M \to \overline{M}$ there exists $n \in \mathbb{N}$ and a partition of A into definable sets $X, I_0, \ldots, I_n$ such that X is finite, $I_0, \ldots, I_n$ are open convex sets and for each $i \leq n$ the function $f \restriction I_i$ is tidy.

Arfiev showed the following.

Theorem 5 ([1]) *Every weakly o-minimal structure M has monotonicity.*

In the next section we give the proof for Theorem 5.

3 Proof of Theorem 5

Throughout this section we assume that $(M, <, \ldots)$ is a weakly o-minimal structure and f is a definable function from definable subset A of M into $\overline{M}$. We now define the following formulas:

$$\begin{aligned}
\varphi_0(x) &:\equiv \exists x_1 > x(\forall y \in (x, x_1) f(y) < f(x));\\
\varphi_1(x) &:\equiv \exists x_1 > x(\forall y \in (x, x_1) f(y) = f(x));\\
\varphi_2(x) &:\equiv \exists x_1 > x(\forall y \in (x, x_1) f(y) > f(x));\\
\psi_0(x) &:\equiv \exists x_0 < x(\forall y \in (x_0, x) f(y) < f(x));\\
\psi_1(x) &:\equiv \exists x_0 < x(\forall y \in (x_0, x) f(y) = f(x));\\
\psi_2(x) &:\equiv \exists x_0 < x(\forall y \in (x_0, x) f(y) > f(x));\\
\theta_{ij}(x) &:\equiv \psi_i(x) \wedge \varphi_j(x) \text{ for each } i, j \leq 2.
\end{aligned}$$

To show Therem 5, we first prove some lemmas needed later.

Lemma 6 ([2]) *For each $x \in \mathrm{Int}(A)$, there exist $i, j \leq 2$ such that $\theta_{ij}(x)$ holds.*

Proof. Suppose that there exists some $x \in \mathrm{Int}(A)$ such that $\varphi_j(x)$ does not hold for some $j \leq 2$. Then the set $\{y \in A : f(y) \leq f(x)\}$ cannot be written as a union of finitely many convex sets, contradicting that M is weakly o-minimal. Thus, for each $x \in \mathrm{Int}(A)$ there exists some $j \leq 2$ such that $\varphi_j(x)$ holds. Similarly, for each $x \in \mathrm{Int}(A)$ there exists some $i \leq 2$ such that $\psi_i(x)$ holds. □

Lemma 7 ([2]) *There exists a partition of A into finitely many points and open convex sets such that each open convex set lies in the solution set of some formula θ_{ij}.*

Proof. By Lemma 6, there exists a finite subset X of A such that we have $X \cup \bigcup_{i,j \leq 2} \theta_{ij} = A$. For each $i, j \leq 2$, by weak o-minimality of M, the set θ_{ij} can be written as a union of finitely many points and open convex sets. This finishes the proof. □

Lemma 8 ([2, Lemma 3.6]) *Let I be an open interval of M. Then it cannot happen that one of the formulas $\theta_{01}, \theta_{10}, \theta_{12}, \theta_{21}$ holds throughout I.*

Proof. Suppose that the formula θ_{01} holds throughout I. Let x be an element of I. Since $\varphi_1(x)$ holds, there exists some $x_1 > x$ with $x_1 \in I$ such that for each $y \in (x, x_1)$ we have $f(x) = f(y)$. Let z be an element of open interval (x, x_1). Since $\psi_0(z)$ holds, there exists some w with $x < w < z$ such that for each $y \in (w, z)$ we have $f(y) < f(z)$, a contradiction.

The other cases are similar. □

We show the next lemma later.

Lemma 9 *Let I be an open interval of M. Suppose that $h : I \to \overline{M}$ has the local minimum or maximum throughout I. Then h is not definable.*

Lemma 10 *Let I be an open interval of M. Then it cannot happen that one of the formulas θ_{00} and θ_{22} holds throughout I.*

Proof. This lemma follows from Lemma 9. □

Lemma 11 *Let I be a non-empty open definable convex subset of M such that θ_{02} holds throughout I. Then there exists $n \in \mathbb{N}$ and a partition of I into definable sets $X, I_0, \ldots, I_n$ such that X is finite, $I_0, \ldots, I_n$ are open convex sets and for each $i \leq n$ the function $f \restriction I_i$ is locally increasing. Similarly, if θ_{20} holds throughout I, the same conclusion holds with 'locally incresing' replaced by 'locally decreasing'.*

Proof. Suppose that $\theta_{02}(x)$ holds throughout I. We define the following formulas:

$$\chi_0(x) :\equiv \forall x_1 > x[\exists y, z(x < y < z < x_1 \wedge f(z) \leq f(y))];$$
$$\chi_2(x) :\equiv \forall x_0 < x[\exists y, z(x_0 < y < z < x \wedge f(z) \leq f(y))].$$

Claim $\chi_0(x)$ and $\chi_2(x)$ cannot hold throughout a subinterval of I.

Proof of Claim. Suppose for a contradiction that $\chi_0(x)$ holds throughout a subinterval $I_0 \subseteq I$. The argument for $\chi_2(x)$ is similar. For each $a \in I_0$, we define the following:

$$\begin{aligned} V_a &:= \{x \in I_0 : x < a \text{ and if } y \in [x,a), \text{then } f(y) < f(a)\} \\ &\quad \cup \{x \in I_0 : x > a \text{ and if } y \in (a,x], \text{then } f(y) > f(a)\} \cup \{a\}; \\ g(a) &:= \inf V_a. \end{aligned}$$

Since $\theta_{02}(x)$ holds throughout I_0, the set V_a is an infinite definable convex set and a is not a boundary point of V_a. Then, by Lemma 9, it suffices to show that g has the local minimum throughout I_0. We define the following formulas:

$$\begin{aligned} \mu_0(x,a) &:\equiv x < a \wedge g(x) \leq g(a); \\ \mu_1(x,a) &:\equiv x < a \wedge g(x) > g(a); \\ \nu_0(x,a) &:\equiv x > a \wedge g(x) \leq g(a); \\ \nu_1(x,a) &:\equiv x > a \wedge g(x) > g(a). \end{aligned}$$

By weak o-minimality of M, for each $a \in I_0$, there exist open interval $J \subseteq I_0$ and $K \subseteq I_0$ with $J < a < K$ such that a is a boundary point of J and K, either $\mu_0(x,a)$ or $\mu_1(x,a)$ holds throughout J, and either $\nu_0(x,a)$ or $\nu_1(x,a)$ holds throughout K. Then, it suffices to show that $\mu_1(x,a)$ holds throughout J and $\nu_1(x,a)$ holds throughout K. Suppose for a contradiction that $\mu_0(x,a)$ holds throughout J. The argument for $\nu_0(x,a)$ is similar. Since a is not a boundary point of V_a, there exists $b \in V_a \cap J$. Since $\chi_0(b)$ holds, there exist $c, d \in V_a \cap J$ such that $b < c < d < a$ and $f(d) \leq f(c)$. Hence, by the definition of g, we have $b < c \leq g(d)$. Now, since b is an element of V_a, we have $g(a) \leq b < g(d)$, contradicting that $\mu_0(d,a)$ holds. □

By the claim, the set $\{x \in I : \chi_0(x) \vee \chi_2(x)\}$ is finite. Hence, we finish the proof. □

Proof of Theorem 5. By Lemma 6 through 11, the theorem follows. □

Finally, we show Lemma 9.

Proof of Lemma 9. Suppose that $h : I \to \overline{M}$ has the local minimum throughout I. Suppose for a contradiction that h is definable.

Claim 1 We may assume that h is injective.

Proof of Claim 1. Define the following equivalence relation on I^2:

$$E(x,y) \iff h(x) = h(y).$$

We first verify that every equivalence class on E is finite. Let A be an infinite class. Then, by weak o-minimality of M, there exists an open subinterval J of A. Since h has the local minimum throughout J, for each $x \in J$ there exists $y \in J$ such that we have $h(y) > h(x)$, a contradiction. Hence, every equivalence class on E is finite. Therefore the set $Z := \{x \in I : \forall y(E(x,y) \to x \leq y)\}$ is infinite. By weak o-minimality of M, there exists an open subinterval J' of Z. We may assume $\mathrm{dom}(h) = J'$. $\square$

From now on, by Claim 1, we assume that the function h is injective. For each $a, b \in I$, we define the following:

$$\begin{aligned} U_a :=& \{x : x > a \wedge \forall y \in (a,x](h(y) > h(a))\} \\ & \cup \{x : x < a \wedge \forall y \in [x,a)(h(y) > h(a))\} \cup \{a\}; \\ a \prec b \iff& U_a \supsetneq U_b. \end{aligned}$$

Then, since h has the local minimum throughout I, for each $a \in I$ the set U_a is an infinite definable convex set. The predicate $\prec$ is a partial ordering.

Claim 2 Let $a, b, c \in I$. Suppose that a, b, c are pairwise distinct. Then the following hold.

1. $U_a \neq U_b$;
2. a is not a boundary point of U_a;
3. $b \in U_a \iff a \prec b$;
4. If $U_a \cap U_b \neq \emptyset$, then either $a \prec b$ or $b \prec a$;
5. If $a \prec b \prec c$ and $a < c$, then $a < b$;
6. If $a \prec b \prec c$ and $a < b$, then $a < c$;
7. If $b \prec a$ and $c \prec a$, then $b \prec c$ or $c \prec b$;
8. $C_a := \{x \in I : x \prec a\}$ is finite.

Proof of Claim 2.

(1): h は単謝より, $h(a) \neq h(b)$ となる. したがって $U_a \neq U_b$ である.

(2): h の仮定より, これはいえる.

(3): ($\Leftarrow$) 明らか.

($\Rightarrow$) $b \in U_a$ とする. このとき $h(b) > h(a)$ である. ここで, 一般性を失うことなしに $a < b$ とする. $c \in U_b$ を任意にとる. このとき, $a \leq c \leq b$ ならば U_a は convex なので, $c \in U_a$ である. また, $b < c$ ならば $c \in U_b$ より, 任意の $d \in (b, c]$ に対して $h(d) > h(b) > h(a)$ である. よって $c \in U_a$ となる. 同様に, $c < a$ ならば $c \in U_a$ となる. このことから $U_a \supsetneq U_b$ がいえる.

(4): 一般性を失うことなしに $a < b$ とする. 仮定より $c \in U_a \cap U_b$ かつ $a < c < b$ を満たす元が存在する. まず $h(a) < h(b)$ と思う. 任意に $d \in U_b$ をとる. $a \leq d \leq c$ ならば U_a は convex なので, $d \in U_a$ である. また, $c < d$ ならば任意の $e \in (c, d]$ に対し $h(e) \geq h(b) > h(a)$ となる. よって, $d \in U_a$ となる. 同様に, $d < a$ ならば $d \in U_a$ となる. したがって $U_b \subsetneq U_a$ が成り立つ. 同様に $h(b) < h(a)$ ならば $U_a \subsetneq U_b$ が成り立つ.

(5): $b < a$ になったとする. すると仮定より, $U_a \supsetneq U_b \supsetneq U_c$ かつ $b < a < c$ となる. よって $b, c \in U_b$ となり, U_b は convex なので $a \in U_b$ がいえる. これは矛盾する.

(6): (5) と同様に示せる.

(7): 仮定より $a \in U_b \cap U_c$ である. よって, $U_b \cap U_c \neq \emptyset$ が成り立つ. すると, (4) より結論がいえる.

(8): C_a が無限集合だったとする. このとき M は weakly o-minimal より, ある開区間 J が存在して $J \subseteq C_a$ となる. $b \in J$ を任意にとる. b は U_b の境界ではないので, $c, d \in U_b \cap J$ かつ $c < b < d$ を満たすものが存在する. すると $c, d \in C_a$ より, $c \prec a$ かつ $d \prec a$ となる. ここで (7) より, $c \prec d$ または $d \prec c$ である. $c \prec d$ とする ($d \prec c$ の場合も同様に示せる). すると, $d \in U_c$ かつ $c < b < d$ となる. U_c は convex より, $b \in U_c$ である. これは $c \in U_b$ に反する. □

Claim 2 の (8) より $\prec$ は離散順序である. ここで

$$K := \{x \in I : 任意の\ y \in I\ に対し, y \not\prec x\};$$
$$\tilde{a} := \{x \in I : a \prec x\ かつ\ a \prec y \prec x\ を満たす\ y\ は存在しない\}$$

と定義する.

Claim 3 The following conditions hold:

1. $I \setminus K = \bigsqcup_{a\in I} \tilde{a}$;

2. the set K is finite;

3. the set $\tilde{a}$ is finite.

Proof of Claim 3.
(1): 任意に$a \in I$と$b \in K$をとる. するとKの定義より, $a \not\prec b$である. よって$b \notin \tilde{a}$がいえる.

次に, 任意に$c \in I \setminus K$をとると, $c \notin K$より, ある元$d \in I$が存在して$d \prec c$が成り立つ. ここで$\prec$は離散順序より, ある元$d' \in I$が存在して$c \in \tilde{d'}$がいえる.

また$e_1 \neq e_2$を任意にとる. もし$\tilde{e_1} \cap \tilde{e_2} \neq \emptyset$であったとすると, ある元$x \in \tilde{e_1} \cap \tilde{e_2}$がとれる. すると$e_1 \prec x$かつ$e_2 \prec x$だからClaim 2の(7)より, $e_1 \prec e_2$または$e_2 \prec e_1$である. $e_1 \prec e_2$と思う($e_2 \prec e_1$の場合も同様). すると, $e_1 \prec e_2 \prec x$となるが, これは$x \in \tilde{e_1}$に反する. したがって, $\tilde{e_1} \cap \tilde{e_2} = \emptyset$である.

(2): Kが無限集合だったとする. このときMはweakly o-minimalより, ある開区間Jが存在して$J \subseteq K$となる. $b \in J$を任意にとる. bはU_bの境界ではないので, ある元$c \in U_b \cap J$が存在する. すると, $c \in U_b$より, $b \prec c$となる. これは$c \in J \subseteq K$に反する.

(3): $\tilde{a}$が無限集合だったとする. このときMはweakly o-minimalより, ある開区間Jが存在して$J \subseteq \tilde{a}$となる. $b \in J$を任意にとる. bはU_bの境界ではないので, ある元$c \in U_b \cap J$が存在する. $b \in \tilde{a}$かつ$c \in U_b$より, $a \prec b \prec c$となる. これは$c \in J \subseteq \tilde{a}$に反する. □

Claim 3の(2)より, $I \setminus K$は無限集合である. さて任意の$a, b \in I \setminus K$に対して,

$$E'(a,b) \iff M \models \exists c \in I\ (a \in \tilde{c} \wedge b \in \tilde{c})$$

と定義すると, $E'(x,y)$はClaim 3の(1)より$(I \setminus K)^2$上の同値関係になる. またClaim 3の(3)より, $E'(x,y)$の各クラスは有限集合である. よって$X := \{x \in I \setminus K : M \models \forall y \in I \setminus K\ (E'(x,y) \rightarrow x \leq y)\}$は無限集合になる.

さてXのdefinable convexな構成要素で$<$に関して最大のものをYとする. $a \in Y$をとる. aはU_aの境界ではないので, $b_1, b_2 \in U_a$かつ$b_1 < a < b_2$となる元が存在する. すると$\prec$は離散順序より, $a \prec b \preceq b_2$か

つ $a \prec b' \preceq b_1$ となる元たちが存在する．このとき，$E'(b,b')$ である．また $b_1 < a < b_2$ だから，Claim 2 の (5) より $b' < a < b$ となる．よって，$b \notin X$ がいえる．ところで b は U_b の境界ではないので，$c \in U_b$ かつ $b < c$ となる元が存在する．すると，$\bar{c}$ は有限より，$d \in \bar{c}$ かつ $d \in X$ となるものがとれる．よって $b \prec c \prec d$ かつ $b < c$ なので，Claim 2 の (6) より $b < d$ となる．これは Y の性質に反する．

したがって，h は definable ではない． □

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