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<tr>
<td>Author(s)</td>
<td>Nishimoto, Katsuyuki</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1470: 18-26</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-02</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/48103">http://hdl.handle.net/2433/48103</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On Some Doubly Infinite, Finite and Mixed Sums derived from The N-Fractional Calculus of A Power Function

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Abstract

In a previous paper, some doubly infinite, finite and mixed sums are reported using the N-fractional calculus \(((z - c)^{a+*\beta})_{\gamma}\) by the author and his colleagues.

In this article the same doubly infinite sums in a previous paper are discussed again using \(((z - c)^{\beta} \cdot (z - c)^{a})_{\gamma}\), the N-fractional calculus of products of power functions.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let \( D = \{D_{-}, D_{+}\} \), \( C = \{C_{-}, C_{+}\} \),

\( C_{-} \) be a curve along the cut joining two points \( z \) and \( -\infty + i \text{Im}(z) \),
\( C_{+} \) be a curve along the cut joining two points \( z \) and \( \infty + i \text{Im}(z) \),
\( D_{-} \) be a domain surrounded by \( C_{-} \), \( D_{+} \) be a domain surrounded by \( C_{+} \).

(Here \( D \) contains the points over the curve \( C \)).

Moreover, let \( f = f(z) \) be a regular function in \( D(z \in D) \),

\[ f_{\nu}(z) = (f)_{\nu} = \frac{\Gamma(\nu+1)}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta-z)^{\nu+1}} d\zeta \quad (\nu \not\in \mathbb{Z}), \]

\[ (f)_{\nu} = \lim_{\nu \to -m} (f)_{\nu} \quad (m \in \mathbb{Z}), \]

where \( -\pi \leq \text{arg}(\zeta - z) \leq \pi \) for \( C_{-} \), \( 0 \leq \text{arg}(\zeta - z) \leq 2\pi \) for \( C_{+} \),

\( \zeta \neq z \), \( z \in \mathbb{C} \), \( \nu \in \mathbb{R} \), \( \Gamma \); Gamma function,

then \((f)_{\nu}\) is the fractional differintegration of arbitrary order \( \nu \) (derivatives of order \( \nu \) for \( \nu > 0 \), and integrals of order \(-\nu\) for \( \nu < 0 \), with respect to \( z \), of the function \( f \), if \( |(f)_{\nu}| < \infty \).

(II) On the fractional calculus operator \( N^{\nu} \)
Theorem A. Let fractional calculus operator (Nishimoto's Operator) $N^\nu$ be

\[ N^\nu = \left( \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{\nu+i}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3) \]

with

\[ N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in \mathbb{Z}^*), \quad (4) \]

and define the binary operation $\circ$ as

\[ N^\beta \circ N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5) \]

then the set

\[ \{ N^\nu \} = \{ N^\nu | \nu \in \mathbb{R} \} \quad (6) \]

is an Abelian product group (having continuous index $\nu$ ) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator $N^\nu$, for the function $f$ such that $f \in F = \{ f; 0 \neq |f_0| < \infty, \nu \in \mathbb{R} \}$, where $f = f(z)$ and $z \in \mathbb{C}$.

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of $N^\beta$ and $N^\alpha$.)

Theorem B. "F.O.G. $\{N^\nu\}$" is an "Action product group which has continuous index $\nu$" for the set of $F$. (F.O.G.; Fractional calculus operator group)

Theorem C. Let

\[ S := \{ \pm N^\nu \} \cup \{ 0 \} = \{ N^\nu \} \cup \{- N^\nu \} \cup \{ 0 \} \quad (\nu \in \mathbb{R}). \quad (7) \]

Then the set $S$ is a commutative ring for the function $f \in F$, when the identity

\[ N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8) \]

holds. [5]

(III) Lemma. We have [1]

(1) \( ((z-c)^\beta)_\alpha = e^{-i\pi \alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty), \)

(1i) \( (\log(z-c))_\alpha = -e^{-i\pi \alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \)

(iii) \( ((z-c)^{-\alpha})_\alpha = -e^{i\pi \alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty), \)

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). ($\Gamma$; Gamma function),

(1v) \( (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad (u = u(z), \quad v = v(z)). \)
§ 1. Doubly Infinite, Finite and Mixed Infinite Sums

In the following $\alpha, \beta, \gamma \in R$.

Theorem 1. Let

$$L(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(k-\alpha+m)\Gamma(\gamma-\beta-m)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(k-\alpha)\Gamma(-\beta)}.$$  \hspace{1cm} (1)

(i) When $\alpha, \beta, \gamma \not\in \mathbb{Z}_0^+$, we have the following doubly infinite sums:

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} m \left( \frac{z-c}{z} \right)^\alpha,$$ \hspace{1cm} (2)

where

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi (\gamma-\alpha-\beta)}{\sin \pi (\alpha+\beta) \cdot \sin \pi (\gamma-\beta)} \quad (|Q|=M<\infty),$$ \hspace{1cm} (3)

and

$$\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right|, \quad \left| \frac{\Gamma(\gamma-\alpha-\beta-m)}{\Gamma(-\alpha-\beta)} \right| < \infty.$$ \hspace{1cm} (4)

(ii) When $\alpha, \beta \not\in \mathbb{Z}^+$, we have the following mixed infinite sums:

$$\sum_{k=0}^{\infty} \sum_{m=0}^{k} L(\alpha, \beta, s; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q(\alpha, \beta, s) \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left( \frac{z-c}{z} \right)^\alpha,$$ \hspace{1cm} (5)

for $s \in \mathbb{Z}^+$ where

$$|c/z| < 1, \quad |(z-c)/z| < \infty,$$

and

$$\left| \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right|, \quad \left| \frac{\Gamma(s-m-\beta)}{\Gamma(-\beta)} \right| < \infty.$$ \hspace{1cm} (6)

Proof of (i). We have

$$(z-c)^\alpha = z^\alpha \left( 1 - \frac{c}{z} \right)^\alpha$$ \hspace{1cm} (7)

$$= z^\alpha \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} z^{-k} \quad (|z|>|c|)$$ \hspace{1cm} (8)

$$= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} z^{\alpha-k}.$$ \hspace{1cm} (9)
Next make \((z - c)^{\beta} \times (7)\), then operate \(N^\gamma\) to its both sides, we obtain

\[
((z - c)^{\beta} \cdot (z - c)^{\alpha})_{\gamma} = \sum_{k=0}^{\infty} \frac{(-c)^{k+1} \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} ((z - c)^{\beta} \cdot z^{a-k}),
\]

and

\[
= \sum_{k=0}^{\infty} \frac{(-c)^{k+1} \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} \sum_{\mu=0}^{\infty} \frac{\Gamma(\gamma + 1)}{\mu! \Gamma(\gamma + 1 - \mu)} ((z - c)^{\beta})_{\gamma-k} (z^{a-k})_{\mu},
\]

Now we have

\[
(z - c)^{\beta}_{\gamma-k} = e^{-i\pi(\gamma-k)} \frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)} (z - c)^{\beta-\gamma+k},
\]

and

\[
(z^{a-k})_{\mu} = e^{-i\pi \mu} \frac{\Gamma(\mu + k - \alpha)}{\Gamma(k - \alpha)} z^{\alpha-k-n}.
\]

respectively.

On the other hand we have

\[
((z - c)^{\beta} \cdot (z - c)^{\alpha})_{\gamma} = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + 1)}{k! \Gamma(\gamma + 1 - k)} ((z - c)^{\beta})_{\gamma-k} ((z - c)^{\alpha})_{k},
\]

\[
= e^{-i\pi \gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + 1) \Gamma(\gamma - \beta - k) \Gamma(k - \alpha)}{k! \Gamma(\gamma + 1 - k) \Gamma(-\beta) \Gamma(-\alpha)} (z - c)^{a+\beta-\gamma}.
\]

since

\[
(z - c)^{\beta}_{\gamma-k} = e^{-i\pi(\gamma-k)} \frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)} (z - c)^{\beta-\gamma+k},
\]

and

\[
((z - c)^{\alpha})_{k} = e^{-i\pi k} \frac{\Gamma(k - \alpha)}{\Gamma(-\alpha)} (z - c)^{a-k},
\]

and

\[
\Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{\Gamma(k - \lambda)}.
\]
where

$$\lambda_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \Gamma(\lambda + k) / \Gamma(\lambda), \text{ with } \lambda_0 = 1$$

(notation of Pochhammer).

Next we have the identity

$$\sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{k! [c]_k} = {}_2F_1(a, b ; c ; 1)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \left( \Re(c-a-b) > 0, \quad c \notin \mathbb{Z}_0 \right) \quad (18)$$

Therefore, we have

$$((z - c)^\alpha \cdot (z - c)^\beta) = e^{-i\pi \gamma} \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} (z - c)^{\alpha+\beta-\gamma} {}_2F_1(-\alpha, -\gamma; 1+\beta-\gamma; 1)$$

$$= e^{-im} \frac{\Gamma(\gamma - \beta)\Gamma(1+\alpha+\beta)\Gamma(1+\beta-\gamma)}{\Gamma(-\beta)\Gamma(1+\beta)\Gamma(1+\alpha+\beta-\gamma)} (z - c)^{\alpha+\beta-\gamma} \quad (19)$$

$$= e^{-im} \frac{\sin \pi \beta \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \beta)} \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z - c)^{\alpha+\beta-\gamma} \quad (20)$$

from (14), because we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbb{Z}) \quad (21)$$

Therefore, substituting (23), (10) and (11) into (9) we obtain

$$Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z - c)^{\alpha+\beta-\gamma}$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma + 1)\Gamma(\gamma - m - \beta)\Gamma(m + k - \alpha)}{m! \Gamma(\gamma + 1 - m)\Gamma(-\beta)\Gamma(k - \alpha)}$$

$$\times (z - c)^{\beta+\gamma+m} z^{\alpha-k-m} \quad (22)$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma ; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z - c}{z} \right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left( \frac{z - c}{z} \right)^\alpha \quad (23)$$
from (25), using the notation (1), under the conditions.

Proof of (ii). Set $\gamma = s \in \mathbb{Z}^+$ in (2), we have then (4) clearly under the conditions.

Corollary 1. When $r, s \in \mathbb{Z}^+$ we have the following doubly finite sums:

$$\sum_{k=0}^{r} \sum_{m=0}^{s} \frac{L(r, \beta, s; k, m)(-c/z)^k(z-c/z)^m}{\Gamma(s-r-\beta)(z-c/z)^{s-r-\beta}} = Q(r, \beta, s) \frac{\Gamma(s-r-\beta)(z-c/z)^{s-r-\beta}}{\Gamma(-r-\beta)},$$

where

$$|c/z|, \ |z-c/z| < \infty,$$

and

$$\frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)}, \ \frac{\Gamma(s-\beta-m)}{\Gamma(-\beta)} < \infty.$$

Proof. Set $\alpha = r$ and $\gamma = s$ in (2) we have then this corollary clearly.

§ 2. Direct calculation of the doubly infinite sums

The direct calculation (without the use of N-fractional calculus) of the doubly infinite sum in the LHS of § 1. (2) is shown as follows.

Theorem 2. Let

$$L = L(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\gamma-\beta-m)\Gamma(k-\alpha+m)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(\gamma-\beta)},$$

and

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \beta)} \quad (|Q(\alpha, \beta, \gamma)| = M < \infty).$$

We have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \cdot \frac{(z-c/z)^m}{\Gamma(\gamma-\beta)} \cdot \frac{(-c/z)^k}{\Gamma(-\alpha-\beta)} = Q \cdot \frac{\Gamma(\gamma-\beta)}{\Gamma(-\alpha-\beta)} \cdot \frac{(z-c/z)^\alpha}{\Gamma(-\gamma)},$$

where

$$|c/z| < 1,$$

and

$$(\alpha + \beta), \ (\gamma - \beta), \ (\gamma - \alpha - \beta) \notin \mathbb{Z}.$$
using the identity

$$\Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{\Gamma(k - \lambda)}$$  \hspace{1cm} (5)

and

$$[-\alpha]_{k+m} = [-\alpha]_m [-\alpha + m]_k . \hspace{1cm} (6)$$

We have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \cdot \left( \frac{z-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)}$$

$$\times \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m}{m! [1 + \beta - \gamma]_m} \left( \frac{z-c}{z} \right)^m \sum_{k=0}^{\infty} \frac{[-\alpha + m]_k}{k!} \left( \frac{c}{z} \right)^k$$  \hspace{1cm} (7)

$$= \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \left( \frac{z-c}{z} \right)^e \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m}{m! [1 + \beta - \gamma]_m}$$  \hspace{1cm} (8)

$$= \frac{\Gamma(\gamma - \beta) \Gamma(1 + \beta - \gamma) \Gamma(1 + \alpha + \beta)}{\Gamma(-\beta) \Gamma(1 + \beta) \Gamma(1 + \alpha + \beta - \gamma)} \left( \frac{z-c}{z} \right)^a \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m}{m! [1 + \beta - \gamma]_m}$$  \hspace{1cm} (9)

where

$$\left| \frac{z-c}{z} \right| < 1, \left| \frac{z-c}{z} \right| < 1, \text{Re}(\alpha + \beta) > -1 .$$

Because we have

$$\sum_{k=0}^{\infty} \frac{[-\alpha + m]_k}{k!} \left( \frac{c}{z} \right)^k = \left( \frac{z-c}{z} \right)^{a-m} \hspace{1cm} (11)$$

since

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1 - z)^{1-k},$$  \hspace{1cm} (12)

and

$$\text{_{2}F_{1}}(a, b; c; 1) = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m}{m! [c]_m} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left( \text{Re}(c-a-b) > 0 \right) \hspace{1cm} (13)$$

Moreover we have the identity

$$\Gamma(\lambda) \Gamma(1 - \lambda) = \frac{\pi}{\sin \pi \lambda} \hspace{1cm} (\lambda \notin \mathbb{Z}), \hspace{1cm} (14)$$

then applying \((14)\) to \((10)\) we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \cdot \left( \frac{z-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q \frac{\Gamma(\gamma - \alpha - \beta) \Gamma(z-c)}{\Gamma(-\alpha - \beta) \Gamma(z)} . \hspace{1cm} (15)$$
§ 3. Commentary

[1] In a previous paper, the results obtained by the author are derived by the use of \( (z-c)^{\alpha+\beta} \), however the results shown in this article, the N-fractional calculus \( (z-c)^\alpha \cdot (z-c)^\beta \) is used.

[11] When \( Q = Q(\alpha, \beta, \gamma) = 1 \), § 1.2 overlaps Theorem 2 obtained in a previous paper.

References

[16] K. Nishimoto ; On Some \((q+1)\) Multiply Infinite Sums \((q \in \mathbb{Z}^+)\) derived from the N-Fractional Calculus of Some Power Functions (Part I), J. Frac.Calc.Vo1.26, Nov. (2004), 53 - 60.

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