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# On Some Doubly Infinite, Finite and Mixed Sums derived from The N- Fractional Calculus of A Power Function

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### Abstract

In a previous paper, some doubly infinite, finite and mixed sums are reported using the N- fractional calculus $((z-c)^{\alpha+\beta})_\gamma$ by the author and his colleagues.

In this article the same doubly infinite sums in a previous paper are discussed again using $((z-c)^\beta \cdot (z-c)^\alpha)_\gamma$, the N-fractional calculus of products of power functions.


## § 0. Introduction ( Definition of Fractional Calculus )

**( I )** Definition. ( by K. Nishimoto ) ( [ 1 ] Vol. 1 )

Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$,

$C_-$ be a curve along the cut joining two points $z$ and $-\infty + i\,\mathrm{Im}(z)$,

$C_+$ be a curve along the cut joining two points $z$ and $\infty + i\,\mathrm{Im}(z)$,

$D_-$ be a domain surrounded by $C_-$, $D_+$ be a domain surrounded by $C_+$.

( Here $D$ contains the points over the curve $C$ ).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu \notin Z^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu\to -m}(f)_\nu \qquad (m\in Z^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for $C_-$, $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for $C_+$,

$\zeta \neq z$, $z \in C$, $\nu \in R$, $\Gamma$ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order $\nu$ ( derivatives of order $\nu$ for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$ ), with respect to $z$, of the function $f$, if $|(f)_\nu| < \infty$.

**(II)** On the fractional calculus operator $N^\nu$ [ 3 ]

**Theorem A.** *Let fractional calculus operator ( Nishimoto's Operator ) $N^{\nu}$ be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right)\qquad(\nu\notin Z^{-}),\quad [\,Refer\ to\,(1)\,]\tag{3}$$

*with*

$$N^{-m}=\lim_{\nu\to -m}N^{\nu}\qquad(m\in Z^{+}),\tag{4}$$

*and define the binary operation $\circ$ as*

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in R),\tag{5}$$

*then the set*

$$\left\{N^{\nu}\right\}=\left\{N^{\nu}\,|\,\nu\in R\right\}\tag{6}$$

*is an Abelian product group ( having continuous index $\nu$ ) which has the inverse transform operator $(N^{\nu})^{-1}=N^{-\nu}$ to the fractional calculus operator $N^{\nu}$ , for the function $f$ such that $f\in F=\left\{f;\ 0\neq|f_{\nu}|<\infty,\nu\in R\right\}$, where $f=f(z)$ and $z\in C$.*
*( vis. $-\infty<\nu<\infty$ ).*
( For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of $N^{\beta}$ and $N^{\alpha}$. )

**Theorem B.** " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index $\nu$ " for the set of $F$ .* ( *F.O.G.* ; Fractional calculus operator group )

**Theorem C.** *Let*

$$S:=\{\pm N^{\nu}\}\cup\{0\}=\{N^{\nu}\}\cup\{-N^{\nu}\}\cup\{0\}\quad(\nu\in R).\tag{7}$$

*Then the set $S$ is a commutative ring for the function $f\in F$, when the identity*

$$N^{\alpha}+N^{\beta}=N^{\gamma}\quad(N^{\alpha},N^{\beta},N^{\gamma}\in S)\tag{8}$$

*holds.* [ 5 ]

**( III ) Lemma.** We have [ 1 ]

( i ) $$((z-c)^{\beta})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty\right),$$

( i i ) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad(|\Gamma(\alpha)|<\infty),$$

( i i i ) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad(|\Gamma(\alpha)|<\infty),$$

where $z-c\neq 0$ in ( i ), and $z-c\neq 0,1$ in ( i i ) and ( i i i ). ( $\Gamma$; Gamma function ),

( i v ) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\left(\begin{array}{l}u=u(z),\\ v=v(z)\end{array}\right).$$

### § 1. Doubly Infinite, Finite and Mixed Infinite Sums

In the following $\alpha, \beta, \gamma \in \boldsymbol{R}$ .

**Theorem 1.** *Let*

$$L(\alpha, \beta, \gamma ; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(k-\alpha+m)\Gamma(\gamma-\beta-m)}{k! \cdot m!\, \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(k-\alpha)\Gamma(-\beta)}. \tag{1}$$

(i) *When* $\alpha, \beta, \gamma \notin \boldsymbol{Z}_0^+$ , *we have the following doubly infinite sums* ;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma ; k, m)\left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}, \tag{2}$$

*where*

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi\beta \cdot \sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta) \cdot \sin\pi(\gamma-\beta)} \quad (|Q| = M < \infty), \tag{3}$$

$$|-c/z| < 1, \quad |(z-c)/z| < 1,$$

*and*

$$\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(\gamma-\beta-m)}{\Gamma(-\beta)}\right| < \infty .$$

(ii) *When* $\alpha, \beta \notin \boldsymbol{Z}^+$ , *we have the following mixed infinite sums* ;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{s} L(\alpha, \beta, s ; k, m)\left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta ; s) \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}, \tag{4}$$

*for* $s \in \boldsymbol{Z}^+$ *where*

$$|-c/z| < 1, \quad |(z-c)/z| < \infty,$$

*and*

$$\left|\frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(s-m-\beta)}{\Gamma(-\beta)}\right| < \infty .$$

**Proof of** (i). We have

$$(z-c)^{\alpha} = z^{\alpha}\left(1-\frac{c}{z}\right)^{\alpha} \tag{5}$$

$$= z^{\alpha} \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k!\, \Gamma(\alpha+1-k)} z^{-k} \quad \left(|z| > |-c|\right) \tag{6}$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k!\, \Gamma(\alpha+1-k)} z^{\alpha-k} . \tag{7}$$

Next make $(z-c)^{\beta}\times(7)$, then operate $N^{\gamma}$ to its both sides, we obtain

$$((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma}=\sum_{k=0}^{\infty}\frac{(-c)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\left((z-c)^{\beta}\cdot z^{\alpha-k}\right)_{\gamma} \tag{8}$$

$$=\sum_{k=0}^{\infty}\frac{(-c)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\sum_{m=0}^{\infty}\frac{\Gamma(\gamma+1)}{m!\,\Gamma(\gamma+1-m)}\left((z-c)^{\beta}\right)_{\gamma-m}(z^{\alpha-k})_{m}\,. \tag{9}$$

Now we have

$$\left((z-c)^{\beta}\right)_{\gamma-m}=e^{-i\pi(\gamma-m)}\frac{\Gamma(\gamma-m-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\gamma+m}, \tag{10}$$

$$\left(\left|\frac{\Gamma(\gamma-m-\beta)}{\Gamma(-\beta)}\right|<\infty\right)$$

and

$$(z^{\alpha-k})_{m}=e^{-i\pi m}\frac{\Gamma(m+k-\alpha)}{\Gamma(k-\alpha)}z^{\alpha-k-m}\quad. \tag{11}$$

respectively.

On the other hand we have

$$((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma}=\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+1)}{k!\,\Gamma(\gamma+1-k)}\left((z-c)^{\beta}\right)_{\gamma-k}\left((z-c)^{\alpha}\right)_{k} \tag{12}$$

$$=e^{-i\pi\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+1)\Gamma(\gamma-\beta-k)\Gamma(k-\alpha)}{k!\,\Gamma(\gamma+1-k)\Gamma(-\beta)\Gamma(-\alpha)}(z-c)^{\alpha+\beta-\gamma}\,. \tag{13}$$

$$=e^{-i\pi\gamma}\frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}(z-c)^{\alpha+\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_{k}[-\gamma]_{k}}{k!\;[1+\beta-\gamma]_{k}} \tag{14}$$

since

$$\left((z-c)^{\beta}\right)_{\gamma-k}=e^{-i\pi(\gamma-k)}\frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\gamma+k}, \tag{15}$$

$$\left(\left|\frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)}\right|<\infty\right),$$

$$((z-c)^{\alpha})_{k}=e^{-i\pi k}\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)}(z-c)^{\alpha-k}\quad, \tag{16}$$

and

$$\Gamma(\lambda+1-k)=(-1)^{-k}\frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)} \tag{17}$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \text{ with } [\lambda]_0 = 1$$

( notation of Pochhammer ).

Next we have the identity

$$\sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{k!\,[c]_k} = {}_2F_1(a, b\,; c\,; 1) \tag{18}$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \begin{pmatrix} \mathrm{Re}(c-a-b) > 0 \\ c \notin \mathbf{Z}_0^- \end{pmatrix}. \tag{19}$$

Therefore, we have

$$((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}\,\frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}(z-c)^{\alpha+\beta-\gamma}\,{}_2F_1(-\alpha,-\gamma\,;1+\beta-\gamma\,;1) \tag{20}$$

$$= e^{-i\pi\gamma}\,\frac{\Gamma(\gamma-\beta)\Gamma(1+\alpha+\beta)\Gamma(1+\beta-\gamma)}{\Gamma(-\beta)\Gamma(1+\beta)\,\Gamma(1+\alpha+\beta-\gamma)}(z-c)^{\alpha+\beta-\gamma} \tag{21}$$

$$\begin{pmatrix} \mathrm{Re}(\alpha+\beta+1) > 0 \\ (1+\beta-\gamma) \notin \mathbf{Z}_0^- \end{pmatrix}$$

$$= e^{-i\pi\gamma}\,\frac{\sin\pi\beta\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\beta)}\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma} \tag{22}$$

$$= e^{-i\pi\gamma}\,Q(\alpha,\beta,\gamma)\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma} \tag{23}$$

from ( 14 ), because we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin\pi\lambda} \qquad (\lambda \notin \mathbf{Z}). \tag{24}$$

Therefore, substituting ( 23 ), ( 10 ) and ( 11 ) into ( 9 ) we obtain

$$Q(\alpha,\beta,\gamma)\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}$$

$$= \sum_{k=0}^{\infty}\frac{(-c)^k\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\sum_{m=0}^{\infty}\frac{\Gamma(\gamma+1)\Gamma(\gamma-m-\beta)\Gamma(m+k-\alpha)}{m!\,\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}$$

$$\times(z-c)^{\beta-\gamma+m}\,z^{\alpha-k-m}\ , \tag{25}$$

we have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} L(\alpha,\beta,\gamma\,;k,m)\left(\frac{-c}{z}\right)^k\left(\frac{z-c}{z}\right)^m = Q(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}, \tag{2}$$

from ( 25 ), using the notation ( 1 ) ,under the conditions.

**Proof of** ( i i ). Set $\gamma = s \in \mathbf{Z}^+$ in ( 2 ), we have then ( 4 ) clearly ubder the conditions.

**Corollary 1.** *When* $r, s \in \mathbf{Z}^+$ *we have the following doubly finite sums;*

$$\sum_{k=0}^{r}\sum_{m=0}^{s} L(r,\beta,s\,;k,m)\left(\frac{-c}{z}\right)^k\left(\frac{z-c}{z}\right)^m = Q(r,\beta,s)\frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)}\left(\frac{z-c}{z}\right)^r, \quad (26)$$

*where*

$$|-c/z|,\quad |(z-c)/z| < \infty,$$

*and*

$$\left|\frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)}\right|,\quad \left|\frac{\Gamma(s-\beta-m)}{\Gamma(-\beta)}\right| < \infty\ .$$

**Proof.** Set $\alpha = r$ and $\gamma = s$ in ( 2 ) we have then this corollary clearly.

## § 2. Direct calculation of the doubly infinite sums

The direct calculation ( without the use of N- fractional calculus ) of the doubly infinite sum in the LHS of § 1. ( 2 ) is shown as follows.

**Theorem 2.** *Let*

$$L = L(\alpha,\beta,\gamma\,;k,m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\gamma-\beta-m)\Gamma(k-\alpha+m)}{k!\cdot m!\,\Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}, \quad (1)$$

*and*

$$Q = Q(\alpha,\beta,\gamma) := \frac{\sin\pi\beta\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\beta)} \qquad (|Q(\alpha,\beta,\gamma)| = M < \infty). \quad (2)$$

*We have then*

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} L\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{-c}{z}\right)^k = Q\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^\alpha, \quad (3)$$

*where*

$$|c/z| < 1,$$

*and*

$$(\alpha+\beta),\ (\gamma-\beta),\ (\gamma-\alpha-\beta) \notin \mathbf{Z}\ .$$

**Proof.** Now we have

$$L\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{-c}{z}\right)^k = \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}\cdot\frac{[-\alpha]_{k+m}[-\gamma]_m}{k!\cdot m!\,[1+\beta-\gamma]_m}\left(\frac{c}{z}\right)^k\left(\frac{z-c}{z}\right)^m \quad (4)$$

using the identity

$$\Gamma(\lambda+1-k) = (-1)^{-k}\frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)} \tag{5}$$

and

$$[-\alpha]_{k+m} = [-\alpha]_m[-\alpha+m]_k \ . \tag{6}$$

We have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} L\cdot\left(\frac{-c}{z}\right)^k\left(\frac{z-c}{z}\right)^m = \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}$$

$$\times\sum_{m=0}^{\infty}\frac{[-\alpha]_m[-\gamma]_m}{m!\,[1+\beta-\gamma]_m}\left(\frac{z-c}{z}\right)^m\sum_{k=0}^{\infty}\frac{[-\alpha+m]_k}{k\,!}\left(\frac{c}{z}\right)^k \tag{7}$$

$$= \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}\left(\frac{z-c}{z}\right)^\alpha\sum_{m=0}^{\infty}\frac{[-\alpha]_m[-\gamma]_m}{m!\,[1+\beta-\gamma]_m} \tag{8}$$

$$= \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}\left(\frac{z-c}{z}\right)^\alpha {}_2F_1(-\alpha\,,-\gamma\,;1+\beta-\gamma\,;1) \tag{9}$$

$$= \frac{\Gamma(\gamma-\beta)\Gamma(1+\beta-\gamma)\Gamma(1+\alpha+\beta)}{\Gamma(-\beta)\Gamma(1+\beta)\Gamma(1+\alpha+\beta-\gamma)}\left(\frac{z-c}{z}\right)^\alpha \ , \tag{10}$$

where

$$\left|\frac{-c}{z}\right|<1,\ \left|\frac{z-c}{z}\right|<1,\ \operatorname{Re}(\alpha+\beta)>-1 \ .$$

Because we ahave

$$\sum_{k=0}^{\infty}\frac{[-\alpha+m]_k}{k!}\left(\frac{c}{z}\right)^k = \left(\frac{z-c}{z}\right)^{\alpha-m} \tag{11}$$

since

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k = (1-z)^{-\lambda}, \tag{12}$$

and

$${}_2F_1(a,b;c;1) = \sum_{m=0}^{\infty}\frac{[a]_m[b]_m}{m!\,[c]_m} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad\begin{pmatrix}\operatorname{Re}(c-a-b)>0\\ c\notin \boldsymbol{Z}_0^-\end{pmatrix}. \tag{13}$$

Moreover we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin\pi\lambda}\qquad(\lambda\notin\boldsymbol{Z}), \tag{14}$$

then applying ( 14 ) to ( 10 ) we obtain

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} L\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{-c}{z}\right)^k = Q\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^\alpha . \tag{15}$$

## § 3. Commentary

[ I ] In a previous paper, the results obtained by the author are derived by the use of $\left((z-c)^{\alpha+\beta}\right)_\gamma$ , however the results shown in this article, the N- fractional calculus $\left((z-c)^{\beta}\cdot(z-c)^{\alpha}\right)_\gamma$ is used

[ II ] When $Q=Q(\alpha\,,\beta,\gamma)=1$ , § 1. ( 2 ) overlaps Theorem 2 obtained in a previous paper .[ 11 ]

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