数理解析研究所講究録 1470 巻 2006 年 18-26

On Some Doubly Infinite, Finite and Mixed Sums derived from The N- Fractional Calculus of A Power Function

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Abstract

In a previous paper, some doubly infinite, finite and mixed sums are reported using the N- fractional calculus $((z-c)^{\alpha+\beta})_\gamma$ by the author and his colleagues.

In this article the same doubly infinite sums in a previous paper are discussed again using $((z-c)^{\beta}\cdot(z-c)^{\alpha})_\gamma$, the N-fractional calculus of products of power functions.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_\nu(z)=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^-), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad (m\in Z^+), \tag{2}$$

where $-\pi\le\arg(\zeta-z)\le\pi$ for C_- , $0\le\arg(\zeta-z)\le 2\pi$ for C_+ ,

$\zeta\ne z$, $z\in C$, $\nu\in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z , of the function f , if $|(f)_\nu|<\infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z^-), \quad [\,Refer\ to\,(\,1\,)\,] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha\, , \beta \in R), \tag{5}$$

then the set

$$\left\{ N^\nu \right\} = \left\{ N^\nu \middle| \, \nu \in R \right\} \tag{6}$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^\nu)^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^ν *, for the function* f *such that* $f \in F = \left\{ f\,;\ 0 \neq |f_\nu| < \infty, \nu \in R \right\}$, *where* $f = f(z)$ *and* $z \in C$.
(*vis.* $-\infty < \nu < \infty$).
(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α.)

Theorem B. " F.O.G. $\{N^\nu\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group)

Theorem C. *Let*

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S *is a commutative ring for the function* $f \in F$, *when the identity*

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z-c)^{\beta - \alpha} \qquad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

(ii) $$(\log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad (|\Gamma(\alpha)| < \infty),$$

where $z - c \neq 0$ in (i), and $z - c \neq 0, 1$ in (ii) and (iii). (Γ; Gamma function),

(iv) $$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. Doubly Infinite, Finite and Mixed Infinite Sums

In the following $\alpha, \beta, \gamma \in \boldsymbol{R}$.

Theorem 1. *Let*

$$L(\alpha, \beta, \gamma ; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(k-\alpha+m)\Gamma(\gamma-\beta-m)}{k!\cdot m!\,\Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(k-\alpha)\Gamma(-\beta)}. \tag{1}$$

(i) *When* $\alpha, \beta, \gamma \notin \boldsymbol{Z}_0^+$, *we have the following doubly infinite sums* ;

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} L(\alpha, \beta, \gamma ; k, m)\left(\frac{-c}{z}\right)^k\left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta, \gamma)\ \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^\alpha , \tag{2}$$

where

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin\pi\beta\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\beta)} \quad (|Q| = M < \infty), \tag{3}$$

$$|-c/z| < 1, \quad |(z-c)/z| < 1,$$

and

$$\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(\gamma-\beta-m)}{\Gamma(-\beta)}\right| < \infty \ .$$

(ii) *When* $\alpha, \beta \notin \boldsymbol{Z}^+$, *we have the following mixed infinite sums* ;

$$\sum_{k=0}^{\infty}\sum_{m=0}^{s} L(\alpha, \beta, s ; k, m)\left(\frac{-c}{z}\right)^k\left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta ; s)\ \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^\alpha , \tag{4}$$

for $s \in \boldsymbol{Z}^+$ *where*

$$|-c/z| < 1, \quad |(z-c)/z| < \infty,$$

and

$$\left|\frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(s-m-\beta)}{\Gamma(-\beta)}\right| < \infty \ .$$

Proof of (i). We have

$$(z-c)^\alpha = z^\alpha\left(1-\frac{c}{z}\right)^\alpha \tag{5}$$

$$= z^\alpha \sum_{k=0}^{\infty} \frac{(-c)^k\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} z^{-k} \quad (|z| > |-c|) \tag{6}$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} z^{\alpha-k} \ . \tag{7}$$

Next make $(z-c)^{\beta}\times(7)$, then operate N^{γ} to its both sides, we obtain

$$((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma}=\sum_{k=0}^{\infty}\frac{(-c)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\left((z-c)^{\beta}\cdot z^{\alpha-k}\right)_{\gamma} \tag{8}$$

$$=\sum_{k=0}^{\infty}\frac{(-c)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\sum_{m=0}^{\infty}\frac{\Gamma(\gamma+1)}{m!\,\Gamma(\gamma+1-m)}\left((z-c)^{\beta}\right)_{\gamma-m}(z^{\alpha-k})_{m}\,. \tag{9}$$

Now we have

$$\left((z-c)^{\beta}\right)_{\gamma-m}=e^{-i\pi(\gamma-m)}\frac{\Gamma(\gamma-m-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\gamma+m}\,, \tag{10}$$

$$\left(\left|\frac{\Gamma(\gamma-m-\beta)}{\Gamma(-\beta)}\right|<\infty\right)$$

and

$$(z^{\alpha-k})_{m}=e^{-i\pi m}\frac{\Gamma(m+k-\alpha)}{\Gamma(k-\alpha)}z^{\alpha-k-m}\,. \tag{11}$$

respectively.

On the other hand we have

$$((z-c)^{\beta}\cdot(z-c)^{\alpha})_{\gamma}=\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+1)}{k!\,\Gamma(\gamma+1-k)}\left((z-c)^{\beta}\right)_{\gamma-k}\left((z-c)^{\alpha}\right)_{k} \tag{12}$$

$$=e^{-i\pi\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+1)\Gamma(\gamma-\beta-k)\Gamma(k-\alpha)}{k!\,\Gamma(\gamma+1-k)\Gamma(-\beta)\Gamma(-\alpha)}(z-c)^{\alpha+\beta-\gamma}\,. \tag{13}$$

$$=e^{-i\pi\gamma}\frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}(z-c)^{\alpha+\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_{k}[-\gamma]_{k}}{k!\;[1+\beta-\gamma]_{k}} \tag{14}$$

since

$$\left((z-c)^{\beta}\right)_{\gamma-k}=e^{-i\pi(\gamma-k)}\frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\gamma+k}\,, \tag{15}$$

$$\left(\left|\frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)}\right|<\infty\right),$$

$$((z-c)^{\alpha})_{k}=e^{-i\pi k}\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)}(z-c)^{\alpha-k}\,, \tag{16}$$

and

$$\Gamma(\lambda+1-k)=(-1)^{-k}\frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)} \tag{17}$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \text{ with } [\lambda]_0 = 1$$

(notation of Pochhammer).

Next we have the identity

$$\sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{k!\,[c]_k} = {}_2F_1(a, b\,; c\,; 1) \qquad (18)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \begin{pmatrix} \mathrm{Re}(c-a-b) > 0 \\ c \notin \mathbf{Z}_0^- \end{pmatrix}. \qquad (19)$$

Therefore, we have

$$((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} (z-c)^{\alpha+\beta-\gamma}\, {}_2F_1(-\alpha, -\gamma\,; 1+\beta-\gamma\,; 1) \qquad (20)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(\gamma-\beta)\Gamma(1+\alpha+\beta)\Gamma(1+\beta-\gamma)}{\Gamma(-\beta)\Gamma(1+\beta)\,\Gamma(1+\alpha+\beta-\gamma)} (z-c)^{\alpha+\beta-\gamma} \qquad (21)$$

$$\begin{pmatrix} \mathrm{Re}(\alpha+\beta+1) > 0 \\ (1+\beta-\gamma) \notin \mathbf{Z}_0^- \end{pmatrix}$$

$$= e^{-i\pi\gamma} \frac{\sin\pi\beta \cdot \sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\beta)} \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma} \qquad (22)$$

$$= e^{-i\pi\gamma}\, Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma} \qquad (23)$$

from (14), because we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin\pi\lambda} \qquad (\lambda \notin \mathbf{Z}). \qquad (24)$$

Therefore, substituting (23), (10) and (11) into (9) we obtain

$$Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma}$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(\gamma-m-\beta)\Gamma(m+k-\alpha)}{m!\,\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}$$

$$\times (z-c)^{\beta-\gamma+m} z^{\alpha-k-m}\,, \qquad (25)$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma\,; k, m) \left(\frac{-c}{z}\right)^{k} \left(\frac{z-c}{z}\right)^{m} = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\alpha}, \qquad (2)$$

from (25), using the notation (1) ,under the conditions.

Proof of (i i). Set $\gamma = s \in \mathbf{Z}^{+}$ in (2), we have then (4) clearly ubder the conditions.

Corollary 1. *When* $r, s \in \mathbf{Z}^{+}$ *we have the following doubly finite sums*;

$$\sum_{k=0}^{r}\sum_{m=0}^{s} L(r,\beta,s;k,m)\left(\frac{-c}{z}\right)^{k}\left(\frac{z-c}{z}\right)^{m} = Q(r,\beta,s)\frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)}\left(\frac{z-c}{z}\right)^{r}, \quad (26)$$

where

$$|-c/z|, \quad |(z-c)/z| < \infty,$$

and

$$\left|\frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)}\right|, \quad \left|\frac{\Gamma(s-\beta-m)}{\Gamma(-\beta)}\right| < \infty .$$

Proof. Set $\alpha = r$ and $\gamma = s$ in (2) we have then this corollary clearly.

§ 2. Direct calculation of the doubly infinite sums

The direct calculation (without the use of N- fractional calculus) of the doubly infinite sum in the LHS of § 1. (2) is shown as follows.

Theorem 2. *Let*

$$L = L(\alpha,\beta,\gamma;k,m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\gamma-\beta-m)\Gamma(k-\alpha+m)}{k!\cdot m!\,\Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}, \quad (1)$$

and

$$Q = Q(\alpha,\beta,\gamma) := \frac{\sin\pi\beta\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\beta)} \qquad (|Q(\alpha,\beta,\gamma)| = M < \infty). \quad (2)$$

We have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} L\cdot\left(\frac{z-c}{z}\right)^{m}\left(\frac{-c}{z}\right)^{k} = Q\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}, \quad (3)$$

where

$$|c/z| < 1,$$

and

$$(\alpha+\beta),\ (\gamma-\beta),\ (\gamma-\alpha-\beta) \notin \mathbf{Z} .$$

Proof. Now we have

$$L\cdot\left(\frac{z-c}{z}\right)^{m}\left(\frac{-c}{z}\right)^{k} = \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}\cdot\frac{[-\alpha]_{k+m}[-\gamma]_{m}}{k!\cdot m!\,[1+\beta-\gamma]_{m}}\left(\frac{c}{z}\right)^{k}\left(\frac{z-c}{z}\right)^{m} \quad (4)$$

using the identity

$$\Gamma(\lambda+1-k)=(-1)^{-k}\frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)} \tag{5}$$

and

$$[-\alpha]_{k+m}=[-\alpha]_m[-\alpha+m]_k \ . \tag{6}$$

We have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}L\cdot\left(\frac{-c}{z}\right)^k\left(\frac{z-c}{z}\right)^m=\frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}$$

$$\times\sum_{m=0}^{\infty}\frac{[-\alpha]_m[-\gamma]_m}{m!\,[1+\beta-\gamma]_m}\left(\frac{z-c}{z}\right)^m\sum_{k=0}^{\infty}\frac{[-\alpha+m]_k}{k\,!}\left(\frac{c}{z}\right)^k \tag{7}$$

$$=\frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}\sum_{m=0}^{\infty}\frac{[-\alpha]_m[-\gamma]_m}{m!\,[1+\beta-\gamma]_m} \tag{8}$$

$$=\frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}{}_2F_1(-\alpha,-\gamma;1+\beta-\gamma;1) \tag{9}$$

$$=\frac{\Gamma(\gamma-\beta)\Gamma(1+\beta-\gamma)\Gamma(1+\alpha+\beta)}{\Gamma(-\beta)\Gamma(1+\beta)\Gamma(1+\alpha+\beta-\gamma)}\left(\frac{z-c}{z}\right)^{\alpha}, \tag{10}$$

where

$$\left|\frac{-c}{z}\right|<1,\ \left|\frac{z-c}{z}\right|<1,\ \mathrm{Re}(\alpha+\beta)>-1 \ .$$

Because we ahave

$$\sum_{k=0}^{\infty}\frac{[-\alpha+m]_k}{k!}\left(\frac{c}{z}\right)^k=\left(\frac{z-c}{z}\right)^{\alpha-m} \tag{11}$$

since

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k=(1-z)^{-\lambda}, \tag{12}$$

and

$${}_2F_1(a,b;c;1)=\sum_{m=0}^{\infty}\frac{[a]_m[b]_m}{m!\,[c]_m}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad\begin{pmatrix}\mathrm{Re}(c-a-b)>0\\ c\notin\mathbf{Z}_0^-\end{pmatrix}. \tag{13}$$

Moreover we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda)=\frac{\pi}{\sin\pi\lambda}\qquad(\lambda\notin\mathbf{Z}), \tag{14}$$

then applying (14) to (10) we obtain

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}L\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{-c}{z}\right)^k=Q\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\alpha}. \tag{15}$$

§ 3. Commentary

[I] In a previous paper, the results obtained by the author are derived by the use of $\left((z-c)^{\alpha+\beta}\right)_\gamma$, however the results shown in this article, the N- fractional calculus $\left((z-c)^{\beta}\cdot(z-c)^{\alpha}\right)_\gamma$ is used

[I I] When $Q = Q(\alpha, \beta, \gamma) = 1$, § 1. (2) overlaps Theorem 2 obtained in a previous paper .[11]

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field Produced from The Set of N- Fractional Calculus Operator, J. Frac Calc. Vol. 24, Nov. (2003),29 - 36.

[6] K. Nishimoto, Ding- Kuo Chyan, Shy- Der Lin and Shih- Tong Tu ; On some infinite suns derived by N- fractional calculus, J. Frac. Calc. Vol.20 (2001), 91 - 97.

[7] Pin Yu Wang, Tsu- Chen Wu and Shih- Tong Tu; Some Infinite Sums via N- fractional calculus, J. Frac. Calc. Vol.21, May (2002), 71 -77.

[8] Shy- Der Lin, Shih- Tong Tu , Tsai- Ming Hsieh and H.M. Srivastava ; Some Finite and Infinite Sums Associated with the Digamma and Related Functions, J. Frac.Calc.Vol.22, Nov. (2002), 103- 114.

[9] K. Nishimoto ; N- Fractional Calculus of the Power and Logarithmic Functions and Some Identities (Continue), J. Frac. Calc. Vol.22, Nov. (2002), 59 - 65.

[10] K. Nishimoto and Susana S. de Romero ; Some Multiple Infinite Sums derived from The N-Fractional Calculus of some Power Functions, J. Frac. Calc.Vol.24, Nov. (2003), 67 - 76.

[11] K. Nishimoto ; Examinations for Some Doubly Infinite, Finite and Mixed Sums, J. Frac. Calc. Vol. 25, May (2004), 25 - 32.

[12] Shy- Der Lin and H. M. Srivastava ; Fractional Calculus and Its Applications Involving Bilateral Expansions and Multiple Infinite Sums, J. Frac. Calc. Vol. 25, May (2004), 47 - 58.

[13] K. Nishimoto, Susana S. de Romero and Ana I. Prieto ; Examinations for Some Doubly Infinite Sums derived by Means of N-Fractional Calculus, J. Frac. Calc.Vol.26, Nov. (2004), 1 - 8.

[14] K. Nishimoto ; Some Multiply Infinite, Mixed and Finite Sums derived from The N- Fractional Calculus of Some Power Functions, J. Frac. Calc.Vol.26, Nov. (2004), 9 - 23.

[15] K. Nishimoto ; Examinations for Some Doubly Infinite Sums derived from The N- Fractional Calculus of A Logarithmic Function, J. Frac.Calc. Vol. 26, Nov. (2004), 25 - 34.

[16] K. Nishimoto ; On Some $(q+1)$ Multiply Infinite Sums ($q \in Z^+$) derived from the N- Fractional Calculus of Some Power Functions (Part I), J. Frac.Calc.Vol. 26, Nov. (2004), 53 - 60.

[17] K. Nishimoto and Susana S. de Romero ; Numerical Examinations for Mixed and Double Finite Sums obtained by Means of N- Fractional Calculus, J. Frac.Calc.Vol. 26, Nov. (2004), 91 - 99.

[18] K.S. Miller and B. Ross ; An Introduction to The Fractional Calculus and Fractional Differtial Equations, John Wiley & Sons, (1993).

[19] Igor Podlubny ; Fractional Differential Equations (1999), Academic Press.

[20] R. Hilfer (Ed.) ; Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapor, New Jersey, London, Hong Kong.
[21] A.P. Prudnikov, Yu. A. Bryckov and O.I. Marichev ; Integrals and Series, Vol. I, Gordon and Breach, New York, (1986).
[22] S. Moriguchi, K. Udagawa and S. Hitotsumatsu ; Mathematical Formulae, Vol.2, Iwanami Zensho, (1957), Iwanami, Japan.

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