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On Some Doubly Infinite, Finite and Mixed Sums derived from The N-Fractional Calculus of A Power Function

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Abstract

In a previous paper, some doubly infinite, finite and mixed sums are reported using the N-fractional calculus \(((z-c)^{a\cdot*\beta})_{\gamma}\) by the author and his colleagues.

In this article the same doubly infinite sums in a previous paper are discussed again using \(((z-c)^{\beta} \cdot (z-c)^{a})_{\gamma}\), the N-fractional calculus of products of power functions.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition (by K. Nishimoto) ([1] Vol. 1)

Let \(D = \{D_{-}, D_{+}\}\), \(C = \{C_{-}, C_{+}\}\),

\(C_{-}\) be a curve along the cut joining two points \(z\) and \(-\infty + i\text{Im}(z)\),

\(C_{+}\) be a curve along the cut joining two points \(z\) and \(\infty + i\text{Im}(z)\),

\(D_{-}\) be a domain surrounded by \(C_{-}\), \(D_{+}\) be a domain surrounded by \(C_{+}\).

(Here \(D\) contains the points over the curve \(C\).)

Moreover, let \(f = f(z)\) be a regular function in \(D(\in D)\),

\[ f_{v}(z) = (f)_{v} = \frac{\Gamma(v+1)}{2\pi i} \int_{c}^{\zeta}(\zeta-z)^{v+1}d\zeta \quad (v \not\in \mathbb{Z}), \]

\[ (f)_{-m} = \lim_{v \to -m} (f)_{v} \quad (m \in \mathbb{Z}) \]

where \(-\pi \leq \arg(\zeta-z) \leq \pi\) for \(C_{-}\), \(0 \leq \arg(\zeta-z) \leq 2\pi\) for \(C_{+}\),

\(\zeta = z, \ z \in C, \ v \in \mathbb{R}, \ \Gamma;\) Gamma function,

then \((f)_{v}\) is the fractional differintegration of arbitrary order \(v\) (derivatives of order \(v\) for \(v > 0\), and integrals of order \(-v\) for \(v < 0\)), with respect to \(z\), of the function \(f\), if \(|(f)_{v}| < \infty\).

(II) On the fractional calculus operator \(N^{\nu}\) [3]
Theorem A. Let fractional calculus operator (Nishimoto's Operator) $N^\nu$ be
\[
N^\nu = \left( \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{\nu+1}} \right) (\nu \notin \mathbb{Z}), \quad \text{[Refer to (1)]} \tag{3}
\]
with
\[
N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in \mathbb{Z}^+), \tag{4}
\]
and define the binary operation $\circ$ as
\[
N^\beta \circ N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \tag{5}
\]
then the set
\[
\{N^\nu\} = \{N^\nu | \nu \in \mathbb{R}\} \tag{6}
\]
is an Abelian product group (having continuous index $\nu$) which has the inverse transform operator $(N^\nu)^{-1} = N^{\nu'}$ to the fractional calculus operator $N^\nu$, for the function $f$ such that $f \in F = \{f ; 0 \neq |f| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in C$. (For our convenience, we call $N^\beta \circ N^\alpha$ as product of $N^\beta$ and $N^\alpha$.)

Theorem B. "F.O.G. $\{N^\nu\}$ " is an "Action product group which has continuous index $\nu$" for the set of $F$. (F.O.G.; Fractional calculus operator group)

Theorem C. Let
\[
S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \tag{7}
\]
Then the set $S$ is a commutative ring for the function $f \in F$, when the identity
\[
N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}
\]
holds. [5]

(III) Lemma. We have [1]

(i) \((z-c)^\beta\alpha = e^{-is\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left( \frac{|\Gamma(\alpha-\beta)|}{\Gamma(-\beta)} < \infty \right), \tag{1}
\]

(ii) \((\log(z-c))_\alpha = -e^{-is\alpha} \Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \tag{i1}
\]

(iii) \(((z-c)^{-\alpha})_\alpha = -e^{is\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty), \tag{iii}
\]
where $z-c \neq 0$ in (i), and $z-c \neq 0,1$ in (ii) and (iii). ($\Gamma$; Gamma function),

(iv) \((u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \left( u = u(z), \quad v = v(z) \right). \tag{iv}
\]
§ 1. Doubly Infinite, Finite and Mixed Infinite Sums

In the following $\alpha, \beta, \gamma \in \mathbb{R}$.

Theorem 1. Let

$$L(\alpha, \beta, \gamma ; k, m) := \frac{\Gamma(\alpha + 1)\Gamma(\gamma + 1)\Gamma(k - \alpha + m)\Gamma(\gamma - \beta - m)}{k! \cdot m! \Gamma(\alpha + 1 - k)\Gamma(\gamma + 1 - m)\Gamma(k - \alpha)\Gamma(-\beta)}.$$  \hspace{1cm} (1)

(i) When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^*$, we have the following doubly infinite sums;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma ; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left( \frac{z-c}{z} \right)^{\alpha},$$  \hspace{1cm} (2)

where

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi (\gamma - \alpha - \beta)}{\sin \pi (\alpha + \beta) \cdot \sin \pi (\gamma - \beta)} \quad (|Q| = M < \infty),$$ \hspace{1cm} (3)

$$\left| -\frac{c}{z} \right| < 1, \quad \left| \frac{z-c}{z} \right| < 1,$$

and

$$\left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right|, \quad \left| \frac{\Gamma(\gamma - \beta - m)}{\Gamma(-\beta)} \right| < \infty.$$

(ii) When $\alpha, \beta \notin \mathbb{Z}^*$, we have the following mixed infinite sums;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{k} L(\alpha, \beta, s ; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q(\alpha, \beta, s) \frac{\Gamma(s - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left( \frac{z-c}{z} \right)^{s\alpha},$$  \hspace{1cm} (4)

for $s \in \mathbb{Z}$ where

$$\left| -\frac{c}{z} \right| < 1, \quad \left| \frac{z-c}{z} \right| < \infty,$$

and

$$\left| \frac{\Gamma(s - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right|, \quad \left| \frac{\Gamma(s - m - \beta)}{\Gamma(-\beta)} \right| < \infty.$$

Proof of (i). We have

$$(z-c)^{\alpha} = z^{\alpha} \left( \frac{1}{z} \right)^{\alpha}$$ \hspace{1cm} (5)

$$= z^{\alpha} \sum_{k=0}^{\infty} \frac{(-c)^{k}\Gamma(\alpha + 1)}{k!\Gamma(\alpha + 1 - k)} z^{-k} \quad (|z| > |c|),$$ \hspace{1cm} (6)

$$= \sum_{k=0}^{\infty} \frac{(-c)^{k}\Gamma(\alpha + 1)}{k!\Gamma(\alpha + 1 - k)} z^{\alpha-k}.$$ \hspace{1cm} (7)
Next make \((z - c)^{\beta} \times (7)\), then operate \(N^{\gamma}\) to its both sides, we obtain

\[
\begin{align*}
((z - c)^{\beta} \cdot (z - c)^{\alpha})_{\gamma} &= \sum_{k=0}^{\infty} \frac{(-c)^{k} \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} ((z - c)^{\beta} \cdot z^{\alpha-k})_{\gamma}. \\
&= \sum_{k=0}^{\infty} \frac{(-c)^{k} \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma + 1)}{m! \Gamma(\gamma + 1 - m)} ((z - c)^{\beta})_{\gamma-m} (z^{\alpha-k})_{\gamma}. \\
\end{align*}
\]

(8)

Now we have

\[
\begin{align*}
((z - c)^{\beta})_{\gamma-m} &= e^{-i\pi \gamma-m} \frac{\Gamma(\gamma - m - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \gamma + m}, \\
&= \left( \frac{\Gamma(\gamma - m - \beta)}{\Gamma(-\beta)} \right) < \infty
\end{align*}
\]

(10)

and

\[
(z^{\alpha-k})_{\gamma-m} = e^{-i\pi \gamma} \frac{\Gamma(m + k - \alpha)}{\Gamma(k - \alpha)} z^{\alpha-k-m}.
\]

(11)

respectively.

On the other hand we have

\[
\begin{align*}
((z - c)^{\beta} \cdot (z - c)^{\alpha})_{\gamma} &= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + 1)}{k! \Gamma(\gamma + 1 - k)} ((z - c)^{\beta})_{\gamma-k} ((z - c)^{\alpha})_{\gamma-k} \\
&= e^{-i\pi \gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + 1) \Gamma(\gamma - \beta - k) \Gamma(k - \alpha)}{k! \Gamma(\gamma + 1 - k) \Gamma(-\beta) \Gamma(-\alpha)} (z - c)^{\beta - \gamma + k} \\
&= e^{-i\pi \gamma} \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_{k} [-\gamma]_{k}}{k! [1 + \beta - \gamma]_{k}}
\end{align*}
\]

(12)

since

\[
\begin{align*}
((z - c)^{\beta})_{\gamma-k} &= e^{-i\pi (\gamma-k)} \frac{\Gamma(\gamma - k - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \gamma + k}, \\
&= \left( \frac{\Gamma(\gamma - k - \beta)}{\Gamma(-\beta)} \right) < \infty
\end{align*}
\]

(15)

and

\[
\begin{align*}
((z - c)^{\alpha})_{\gamma-k} &= e^{-i\pi \gamma} \frac{\Gamma(k - \alpha)}{\Gamma(-\alpha)} (z - c)^{\alpha - k}, \\
\Gamma(\lambda + 1 - k) &= (-1)^{-k} \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{\Gamma(k - \lambda)}
\end{align*}
\]

(16)

and

\[
\begin{align*}
\Gamma(\lambda + 1 - k) &= (-1)^{-k} \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{\Gamma(k - \lambda)}
\end{align*}
\]

(17)
where
\[
\lambda_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \Gamma(\lambda + k) / \Gamma(\lambda), \text{ with } \lambda_0 = 1
\]
(notation of Pochhammer).

Next we have the identity
\[
\sum_{k=0}^{\infty} \left[ \frac{[a]}{[b]} \right]_k \frac{[c]}{[d]}_k = \binom{a+b}{c+d} F_1(a, b ; c ; 1)
\]
(18)

Therefore, we have
\[
((z - c)^\beta \cdot (z - c)^\gamma) = e^{-i\pi\gamma} \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} (z - c)^{\alpha + \beta - \gamma} F_1(-\alpha, -\gamma; 1 + \beta - \gamma; 1)
\]
(20)

\[
= e^{-i\pi\gamma} \frac{\sin(\pi\beta) \cdot \sin(\pi(\gamma - \alpha - \beta))}{\sin(\pi(\alpha + \beta)) \cdot \sin(\pi(\gamma - \beta))} (z - c)^{\alpha + \beta - \gamma}
\]
(21)

\[
= e^{-i\pi\gamma} Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z - c)^{\alpha + \beta - \gamma}
\]
(22)

from (14), because we have the identity
\[
\Gamma(\lambda) \Gamma(1 - \lambda) = \frac{\pi}{\sin(\pi\lambda)} \quad (\lambda \notin \mathbb{Z}).
\]
(24)

Therefore, substituting (23), (10) and (11) into (9) we obtain
\[
Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k) \Gamma(\gamma - m - \beta) \Gamma(m + k - \alpha)}
\]
\[
\times (z - c)^{\beta + \gamma + m} z^{-\alpha - k - m},
\]
(25)

we have then
\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma; k, m) \left( \frac{z - c}{z} \right)^k \left( \frac{z - c}{z} \right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left( \frac{z - c}{z} \right)^{\alpha}
\]
(2)
from (25), using the notation (1), under the conditions.

**Proof of (ii).** Set \( \gamma = s \in \mathbb{Z}^* \) in (2), we have then (4) clearly under the conditions.

**Corollary 1.** When \( r, s \in \mathbb{Z}^* \) we have the following doubly finite sums:

\[
\sum_{k=0}^{r} \sum_{m=0}^{s} L(r, \beta, s; k, m) \left( \frac{-c}{z} \right)^{k} \left( \frac{z-c}{z} \right)^{m} = Q(r, \beta, s) \frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)} \left( \frac{z-c}{z} \right)^{r},
\]

where

\[ |-c/z|, |(z-c)/z| < \infty, \]

and

\[
\left| \frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)} \right| < \infty.
\]

**Proof.** Set \( \alpha = r \) and \( \gamma = s \) in (2) we have then this corollary clearly.

**§ 2. Direct calculation of the doubly infinite sums**

The direct calculation (without the use of N-fractional calculus) of the doubly infinite sum in the LHS of § 1. (2) is shown as follows.

**Theorem 2.** Let

\[
L = L(\alpha, \beta, \gamma; k, m) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\gamma-\beta-m)\Gamma(k-\alpha+m)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}
\]

and

\[
Q = Q(\alpha, \beta, \gamma) = \frac{\sin \pi \beta \cdot \sin \pi(\gamma-\alpha-\beta)}{\sin \pi(\alpha+\beta) \cdot \sin \pi(\gamma-\beta)} \quad (|Q(\alpha, \beta, \gamma)| = M < \infty).
\]

We have then

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \left( \frac{z-c}{z} \right)^{m} \left( \frac{-c}{z} \right)^{k} = Q \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left( \frac{z-c}{z} \right)^{\alpha},
\]

where

\[ |c/z| < 1, \]

and

\( (\alpha + \beta), (\gamma - \beta), (\gamma - \alpha - \beta) \notin \mathbb{Z} \).

**Proof.** Now we have

\[
L \left( \frac{z-c}{z} \right)^{m} \left( \frac{-c}{z} \right)^{k} = \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} \cdot \frac{(-\alpha \cdot \beta \cdot \gamma \cdot m)! \cdot (\gamma - \alpha - \beta)_{m}}{k! \cdot m! \cdot (1 + \beta - \gamma)_{m}} \left( \frac{c}{z} \right)^{k} \left( \frac{z-c}{z} \right)^{m}
\]
using the identity
\[ \Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{\Gamma(k - \lambda)} \] (5)
and
\[ [-\alpha]_{k+n} = [-\alpha]_{n} [-\alpha + m]_{k} \] (6)
We have then
\[ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \left( \frac{-c}{z} \right)^{k} \left( \frac{z-c}{z} \right)^{m} = \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \] (7)
\[ \times \sum_{m=0}^{\infty} \frac{[-\alpha]_{m} [-\gamma]_{m}}{m! [1 + \beta - \gamma]_{m}} \left( \frac{z-c}{z} \right)^{m} \sum_{k=0}^{\infty} \frac{[-\alpha + m]_{k}}{k!} \left( \frac{c}{z} \right)^{k} \] (8)
\[ = \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \left( \frac{z-c}{z} \right)^{\alpha} {}_{2}F_{1}(\alpha, -\gamma; 1 + \beta - \gamma; 1) \] (9)
\[ = \frac{\Gamma(\gamma - \beta) \Gamma(1 + \beta - \gamma) \Gamma(1 + \alpha + \beta)}{\Gamma(-\beta) \Gamma(1 + \beta) \Gamma(1 + \alpha + \beta - \gamma)} \left( \frac{z-c}{z} \right)^{\alpha} \] (10)
where \[ \left| \frac{-c}{z} \right| < 1, \left| \frac{z-c}{z} \right| < 1, \text{Re}(\alpha + \beta) > -1. \]

Because we have
\[ \sum_{k=0}^{\infty} \frac{[-\alpha + m]_{k}}{k!} \left( \frac{c}{z} \right)^{k} = \left( \frac{z-c}{z} \right)^{\alpha-m} \] (11)
and
\[ \sum_{k=0}^{\infty} \frac{[\lambda]_{k}}{k!} z^{k} = (1-z)^{-\lambda}, \] (12)
and
\[ {}_{2}F_{1}(a, b; c; 1) = \sum_{m=0}^{\infty} \frac{[a]_{m} [b]_{m}}{m! [c]_{m}} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \] (13)
Moreover we have the identity
\[ \Gamma(\lambda) \Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbb{Z}), \] (14)
then applying (14) to (10) we obtain
\[ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \left( \frac{z-c}{z} \right)^{m} \left( \frac{-c}{z} \right)^{k} = Q \frac{\Gamma(\gamma - \alpha - \beta) \left( \frac{z-c}{z} \right)^{\alpha}}{\Gamma(-\alpha - \beta) \left( \frac{z-c}{z} \right)} \] (15)
§ 3. Commentary

[1] In a previous paper, the results obtained by the author are derived by the use of \( ((z - c)^{a + \beta})_r \), however the results shown in this article, the N- fractional calculus \( ((z - c)^{\alpha} \cdot (z - c)^{\beta})_r \) is used

[11] When \( Q = Q(\alpha, \beta, \gamma) = 1 \), § 1. (2) overlaps Theorem 2 obtained in a previous paper. \[ \{11\] 

References

[16] K. Nishimoto; On Some \( (q + 1) \) Multiply Infinite Sums \( (q \in \mathbb{Z}^+) \) derived from the N-Fractional Calculus of Some Power Functions (Part I), J. Frac.Calc.Vol. 26, Nov. (2004), 53 - 60.

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