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Kyoto University
Certain classes of analytic functions concerned with uniformly starlike and convex functions

Junichi Nishiwaki
Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail: nishiwaki@math.kindai.ac.jp

and

Shigeyoshi Owa
Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail: owa@math.kindai.ac.jp

Abstract
Applying the coefficient inequalities of functions $f(z)$ belonging to the subclasses $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{MD}^*(\alpha,\beta)$ and $\mathcal{ND}^*(\alpha,\beta)$ are introduced. The object of the present paper is to derive some convolution properties of functions $f(z)$ in the classes $\mathcal{MD}^*(\alpha,\beta)$ and $\mathcal{ND}^*(\alpha,\beta)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Shams, Kulkarni and Jahangiri [4] have studied the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

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for some $\alpha (\alpha \geq 0)$ and for some $\beta (0 \leq \beta < 1)$. The subclass $K\mathcal{D}(\alpha, \beta)$ is defined by 
$f(z) \in K\mathcal{D}(\alpha, \beta)$ if and only if $zf'(z) \in SD(\alpha, \beta)$. In view of the classes $SD(\alpha, \beta)$ and $K\mathcal{D}(\alpha, \beta)$, we introduce the subclass $M\mathcal{D}(\alpha, \beta)$ consisting of all functions $f(z) \in A$ which satisfy

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \quad (z \in \mathbb{U})$$

for some $\alpha (\alpha \leq 0)$ and for some $\beta (\beta > 1)$. The class $N\mathcal{D}(\alpha, \beta)$ is also considered as the subclass of $A$ consisting of all functions $f(z)$ which satisfy $zf'(z) \in M\mathcal{D}(\alpha, \beta)$. We discuss some properties of functions $f(z)$ belonging to the classes $M\mathcal{D}(\alpha, \beta)$ and $N\mathcal{D}(\alpha, \beta)$.

We note if $f(z) \in M\mathcal{D}(\alpha, \beta)$, then, for $\alpha < -1$, $\frac{zf'(z)}{f(z)}$ lies in the region $G \equiv G(\alpha, \beta) \equiv \{ w = u + iv : \text{Re} \, at < \alpha |w - 1| + \beta \}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$\left( u - \frac{\alpha^2 - \beta}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

with vertices at the points

$$\left( \alpha^2 - \beta, \frac{\beta - 1}{\alpha^2 - 1}, \frac{1 - \beta}{\sqrt{\alpha^2 - 1}} \right), \left( \frac{\alpha + \beta}{\alpha + 1}, 0 \right), \left( \frac{\alpha - \beta}{\alpha - 1}, 0 \right).$$

Since $\frac{\alpha + \beta}{\alpha + 1} < 1 < \frac{\alpha - \beta}{\alpha - 1} < \beta$, we have $M\mathcal{D}(\alpha, \beta) \subset M\mathcal{D}(0, \beta) \equiv M(\beta)$. For $\alpha = -1$, if $f(z) \in M\mathcal{D}(\alpha, \beta)$, then $\frac{zf'(z)}{f(z)}$ belongs to the region which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2 - \beta^2 + 1}{2(\beta - 1)}.$$

In the case of $f(z) \in N\mathcal{D}(\alpha, \beta)$, $\frac{zf''(z)}{f'(z)}$ lies in the region which contains $w = 0$ and is bounded by the ellipse

$$\left( u + \frac{\beta - 1}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2} \quad (\alpha < -1)$$

with vertices at the points

$$\left( 1 - \beta, \frac{\beta - 1}{\alpha^2 - 1}, \frac{\alpha + \beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left( \frac{1 - \beta}{\alpha^2 - 1}, \frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left( \frac{1 - \beta}{\alpha - 1}, 0 \right), \left( \frac{\beta - 1}{\alpha + 1}, 0 \right).$$

Since $\frac{\beta - 1}{\alpha + 1} < 0 < \frac{1 - \beta}{\alpha - 1} < \beta$, we have $N\mathcal{D}(\alpha, \beta) \subset N\mathcal{D}(0, \beta) \equiv M(\beta)$. And for $\alpha = -1$, $\frac{zf''(z)}{f'(z)}$ lies in the domain which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2}{2(\beta - 1)} + \frac{\beta - 1}{2}.$$
The classes $\mathcal{M} (\alpha)$ and $\mathcal{N} (\alpha)$ were considered by Uralegaddi, Ganigi and Sarangi [3], Nishiwaki and Owa [1], and Owa and Nishiwaki [2].

## 2 Coefficient inequalities for the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

**Theorem 2.1.** If $f(z) \in \mathcal{A}$ satisfies

$$
\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
$$

for some $\alpha (\alpha \leq 0)$ and for some $\beta (\beta > 1)$, then $f(z) \in \mathcal{MD}(\alpha, \beta)$.

**Proof.** Let us suppose that

$$
\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
$$

for $f(z) \in \mathcal{A}$. It suffices to show that

$$
\left| \frac{zf'(z)}{f(z)} - \alpha |zf'(z)| - \beta + 1 \right| < 1 \quad (z \in \mathbb{U}).
$$

We have

$$
\left| \frac{zf'(z)}{f(z)} - \alpha \frac{zf'(z)}{f(z)} - 1 - \beta \right| + 1
$$

$$
= \frac{zf'(z) - \alpha e^{i\theta} |zf'(z)| - \beta f(z) + f(z)}{zf'(z) - \alpha e^{i\theta} |zf'(z)| - \beta f(z) - f(z)}
$$

$$
= \left| \frac{z + \sum_{n=2}^{\infty} na_n z^n - \alpha \sum_{n=2}^{\infty} (n-1)a_n z^{n-1} - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n + z + \sum_{n=2}^{\infty} a_n z^n}{z + \sum_{n=2}^{\infty} na_n z^n - \alpha \sum_{n=2}^{\infty} (n-1)a_n z^{n-1} - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n - z - \sum_{n=2}^{\infty} a_n z^n} \right|
$$

$$
= \left| \frac{(2 - \beta) + \sum_{n=2}^{\infty} (n-\beta+1)a_n z^{n-1} - \alpha e^{i\theta} |\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}|}{(\beta - \sum_{n=2}^{\infty} (n-\beta-1)a_n z^{n-1} + \alpha e^{i\theta} |\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}|)} \right|
$$
The last expression is bounded above by 1 if
\[|2 - \beta| + \sum_{n=2}^{\infty} |n - \beta + 1| |a_n| - \alpha \sum_{n=2}^{\infty} (n-1) |a_n| \leq \beta - \sum_{n=2}^{\infty} |n - \beta - 1| |a_n| - \alpha \sum_{n=2}^{\infty} (n-1) |a_n|\]
which is equivalent to our condition
\[\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|\]
of the theorem. This completes the proof of the theorem.

By using Theorem 2.1, we have

**Corollary 2.1.** If \( f(z) \in A \) satisfies
\[\sum_{n=2}^{\infty} n \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|\]
for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \), then \( f(z) \in ND(\alpha, \beta) \)

**Proof.** From \( f(z) \in ND(\alpha, \beta) \) if and only if \( z f'(z) \in MD(\alpha, \beta) \), replacing \( a_n \) by \( na_n \) in Theorem 2.1, we have the corollary.

### 3 Relation for \( MD^*(\alpha, \beta) \) and \( ND^*(\alpha, \beta) \)

By Theorem 2.1, the class \( MD^*(\alpha, \beta) \) is considered as the subclass of \( MD(\alpha, \beta) \) consisting of \( f(z) \) satisfying
\[\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|\]
for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \). The class \( ND^*(\alpha, \beta) \) is also considered as the subclass of \( ND(\alpha, \beta) \) consisting of \( f(z) \) which satisfy
\[\sum_{n=2}^{\infty} n \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|\]
for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \). By the coefficient inequalities for the classes \( MD^*(\alpha, \beta) \) and \( ND^*(\alpha, \beta) \), we see
Theorem 3.1. If \( f(z) \in A \), then
\[
\mathcal{M}D^{*}(\alpha_1, \beta) \subset \mathcal{M}D^{*}(\alpha_2, \beta)
\]
for some \( \alpha_1 \) and \( \alpha_2 (\alpha_1 \leq \alpha_2 \leq 0) \).

Proof. For \( \alpha_1 \leq \alpha_2 \leq 0 \), we obtain
\[
\sum_{n=2}^{\infty} \{ |n-\beta+1| + |n-\beta-1| - 2\alpha_2(n-1) \} |a_n| 
\leq \sum_{n=2}^{\infty} \{ |n-\beta+1| + |n-\beta-1| - 2\alpha_1(n-1) \} |a_n|.
\]
Therefore, if \( f(z) \in \mathcal{M}D^{*}(\alpha_1, \beta) \), then \( f(z) \in \mathcal{M}D^{*}(\alpha_2, \beta) \). Hence we get the required result. \( \square \)

By using Theorem 3.1, we also have

Corollary 3.1. If \( f(z) \in A \), then
\[
\mathcal{N}D^{*}(\alpha_1, \beta) \subset \mathcal{N}D^{*}(\alpha_2, \beta)
\]
for some \( \alpha_1 \) and \( \alpha_2 (\alpha_1 \leq \alpha_2 \leq 0) \).

4 Convolution of the classes \( \mathcal{M}D^{*}(\alpha, \beta) \) and \( \mathcal{N}D^{*}(\alpha, \beta) \)

For analytic functions \( f_j(z) \) given by
\[
f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, \cdots, p),
\]
the Hadamard product (or convolution) of \( f_1(z) \), \( f_2(z) \), \cdots, \( f_p(z) \) is defined by
\[
(f_1 \ast f_2 \ast \cdots \ast f_p)(z) = z + \sum_{n=2}^{\infty} \left( \prod_{j=1}^{p} a_{n,j} \right) z^n.
\]
Thus we have

Theorem 4.1. If \( f_1(z) \in \mathcal{M}D^{*}(\alpha, \beta_1) \) and \( f_2(z) \in \mathcal{M}D^{*}(\alpha, \beta_2) \) for some \( \alpha (\alpha \leq 2 - \sqrt{5}) \) and \( \beta_1, \beta_2 (1 < \beta_1, \beta_2 \leq 2) \), then \( (f_1 \ast f_2) \in \mathcal{M}D^{*}(\alpha, \beta) \), where
\[
\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.
\]
Proof. From (3.1), for $f(z) \in \mathcal{MD}^*(\alpha, \beta)$ with $1 < \beta \leq 2$, we have

\[
\sum_{n=2}^{\infty} \{(n+1-\beta)+(n-1-\beta)-2\alpha(n-1)\}|a_n| \leq \sum_{n=2}^{\infty} \{(n+1-\beta)+|n-1-\beta|-2\alpha(n-1)\}|a_n| \\
\leq 2(\beta-1),
\]

that is, if $f(z) \in \mathcal{MD}^*(\alpha, \beta)$, then

\[
(4.1) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta+\alpha}{\beta-1}|a_n| \leq 1.
\]

Conversely, if $f(z)$ satisfies

\[
(4.2) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_n| \leq 1,
\]

then $f(z) \in \mathcal{MD}^*(\alpha, \beta)$. From (4.1), if $f_1(z) \in \mathcal{MD}^*(\alpha, \beta_1)$, then

\[
(4.3) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta_1+\alpha}{\beta_1-1}|a_{n,1}| \leq 1,
\]

and also if $f_2(z) \in \mathcal{MD}^*(\alpha, \beta_2)$, then

\[
(4.4) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta_2+\alpha}{\beta_2-1}|a_{n,2}| \leq 1.
\]

Applying the Shwarz's inequality, we have the following inequality

\[
(4.5) \quad \sum_{n=2}^{\infty} \sqrt{\frac{(n-\alpha)-\beta_1+\alpha}{(n-\alpha)-\beta_2+\alpha}} \sqrt{|a_{n,1}||a_{n,2}|} \leq 1
\]

by (4.3) and (4.4). From (4.2) and (4.5), if the following inequality

\[
(4.6) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_{n,1}||a_{n,2}|
\]

\[
\leq \sum_{n=2}^{\infty} \sqrt{\frac{(n-\alpha)-\beta_1+\alpha}{(n-\alpha)-\beta_2+\alpha}} \sqrt{|a_{n,1}||a_{n,2}|}
\]

is satisfied, then we say that $f(z) \in \mathcal{MD}^*(\alpha, \beta)$. This inequality holds true if

\[
(4.7) \quad \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \sqrt{|a_{n,1}|a_{n,2}|} \leq \sqrt{\frac{(n-\alpha)-\beta_1+\alpha}{(n-\alpha)-\beta_2+\alpha}} \frac{(n-\alpha)-\beta_2+\alpha}{\beta_1-1)(\beta_2-1)}
\]

for all $n \geq 2$. Therefore, we have

\[
(4.8) \quad \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \leq \frac{(n-\alpha)-\beta_1+\alpha}{(n-\alpha)-\beta_2+\alpha} \frac{(n-\alpha)-\beta_2+\alpha}{\beta_1-1)(\beta_2-1)}
\]
which is equivalent to

\[(4.9) \quad \beta \geq 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(n(1 - \alpha) + \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + n(1 - \alpha) - \beta_1 + \alpha}(n(1 - \alpha) - \beta_2 + \alpha)} \]

for all \( n \geq 2 \).

Let \( G(n) \) be the right hand side of the last inequality. Then \( G(n) \) is decreasing for \( n \geq 2 \) for \( \alpha \leq 2 - \sqrt{5} \). Thus \( G(2) \) is the maximum of \( G(n) \) for \( \alpha(\alpha \leq 2 - \sqrt{5}) \). This completes the proof of the theorem. \( \square \)

For the functions \( f(z) \) belonging to the class \( N^D^*(\alpha, \beta) \), we also have

**Corollary 4.1.** If \( f_1(z) \in N^D^*(\alpha, \beta_1) \) and \( f_2(z) \in N^D^*(\alpha, \beta_2) \) for some \( \alpha \) and \( \beta_1, \beta_2, \)

\[ (1 < \beta_1, \beta_2 \leq 2) \text{ then } (f_1 * f_2)(z) \in N^D^*(\alpha, \beta), \]

where

\[ \beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}. \]

By virtue of Theorem 4.1, we have the following theorem.

**Theorem 4.2.** If \( f_j \in M^D^*(\alpha, \beta_j) \) \( (j = 1, 2, \cdots, p) \) for some \( \alpha(\alpha \leq 2 - \sqrt{5}) \) and \( \beta_j(1 < \beta_j \leq 2) \), then \( (f_1 * f_2 * \cdots * f_p) \in M^D^*(\alpha, \beta) \), where

\[ \beta = 1 + \frac{A_p}{B_p - C_pD_p + E_p} \quad (p \geq 2), \]

\[ A_p = \Pi_{j=1}^{p}(\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2}\Pi_{j=1}^{p}(\beta_j - 1), \]

\[ C_p = \sum_{m=1}^{p-2}(2 - \alpha)^{p-m-2}(1 - \alpha)^{m-1}, \quad D_p = \Pi_{j=1}^{p-2}(\beta_j - 1)\Pi_{j=p-m+1}^{p}(2 - \alpha - \beta_l), \]

and

\[ E_p = (1 - \alpha)^{p-2}\Pi_{j=1}^{p}(2 - \alpha - \beta_j). \]

**Proof.** When \( p = 2 \), we have

\[ \beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}. \]

Let us suppose that \( (f_1 * \cdots * f_k) \in M^D^*(\alpha, \beta_0) \) and \( f_{k+1} \in M^D^*(\alpha, \beta_{k+1}) \), where

\[ \beta_0 = 1 + \frac{A_k}{B_k - C_kD_k + E_k} \quad (k \geq 2). \]
Using Theorem 4.1 and replacing $\beta_1$ by $\beta_0$ and $\beta_2$ by $\beta_{k+1}$, we see that
\[
\beta = 1 + \frac{(\beta_0 - 1)(\beta_{k+1} - 1)(2 - \alpha)}{(\beta_0 - 1)(\beta_{k+1} - 1) + (2 - \alpha - \beta_0)(2 - \alpha - \beta_{k+1})}
\]
\[
= 1 + \frac{A_{k+1}}{B_{k+1} - \{B_k(2 - \alpha - \beta_{k+1}) + (1 - \alpha)C_kD_k(2 - \alpha - \beta_{k+1})\} + E_{k+1}}
\]
\[
= 1 + \frac{A_{k+1}}{B_{k+1} - C_{k+1}D_{k+1} + E_{k+1}}
\]
where
\[
C_k^+ = \sum_{m=2}^{k-1} (2 - \alpha)^{k-m-1}(1 - \alpha)^{m-1}.
\]
This completes the proof of the Theorem. \hfill \Box

Finally we have

**Corollary 4.2.** If $f_j \in \mathcal{N}D^*(\alpha, \beta_j)$ $(j = 1, 2, \cdots, p)$ for some $\alpha$ and $\beta_j(1 < \beta_j \leq 2)$, then $(f_1 * f_2 * \cdots * f_p) \in \mathcal{N}D^*(\alpha, \beta)$, where
\[
\beta = 1 + \frac{A_p}{B_p - C_p^*D_p + 2^{n-1}E_p} \quad (p \geq 2),
\]
\[
A_p = \prod_{j=1}^{p} (\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2}\Pi_{j=1}^{p}(\beta_j - 1),
\]
\[
C_p^* = \sum_{m=1}^{p-2} 2^m(2 - \alpha)^{p-m-2}(1 - \alpha)^{m-1}, \quad D_p = \Pi_{j=1}^{p-m} (\beta_j - 1)\Pi_{i=p-m+1}^{p}(2 - \alpha - \beta_i),
\]
and
\[
E_p = (1 - \alpha)^{p-2}\Pi_{j=1}^{p}(2 - \alpha - \beta_j).
\]

**References**


