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Certain classes of analytic functions concerned with uniformly starlike and convex functions

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Abstract

Applying the coefficient inequalities of functions $f(z)$ belonging to the subclasses $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{MD}^*(\alpha,\beta)$ and $\mathcal{ND}^*(\alpha,\beta)$ are introduced. The object of the present paper is to derive some convolution properties of functions $f(z)$ in the classes $\mathcal{MD}^*(\alpha,\beta)$ and $\mathcal{ND}^*(\alpha,\beta)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}\,|\,|z|<1\}$. Shams, Kulkarni and Jahangiri [4] have studied the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \qquad (z\in\mathbb{U})$$

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for some $\alpha (\alpha \geqq 0)$ and for some $\beta (0 \leqq \beta < 1)$. The subclass $\mathcal{KD}(\alpha,\beta)$ is defined by $f(z) \in \mathcal{KD}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{SD}(\alpha,\beta)$. In view of the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$, we introduce the subclass $\mathcal{MD}(\alpha,\beta)$ consisting of all functions $f(z) \in \mathcal{A}$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \qquad (z \in \mathbb{U})$$

for some $\alpha(\alpha \leqq 0)$ and for some $\beta(\beta > 1)$. The class $\mathcal{ND}(\alpha,\beta)$ is also considered as the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy $zf'(z) \in \mathcal{MD}(\alpha,\beta)$. We discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$.

We note if $f(z) \in \mathcal{MD}(\alpha,\beta)$, then, for $\alpha < -1$, $\dfrac{zf'(z)}{f(z)}$ lies in the region $G \equiv G(\alpha,\beta) \equiv \{w = u + iv : \mathrm{Re}\, w < \alpha|w-1| + \beta\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$\left(u - \frac{\alpha^2-\beta}{\alpha^2-1}\right)^2 + \frac{\alpha^2}{\alpha^2-1}v^2 = \frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

with vertices at the points

$$\left(\frac{\alpha^2-\beta}{\alpha^2-1}, \frac{\beta-1}{\sqrt{\alpha^2-1}}\right), \left(\frac{\alpha^2-\beta}{\alpha^2-1}, \frac{1-\beta}{\sqrt{\alpha^2-1}}\right), \left(\frac{\alpha+\beta}{\alpha+1}, 0\right), \left(\frac{\alpha-\beta}{\alpha-1}, 0\right).$$

Since $\dfrac{\alpha+\beta}{\alpha+1} < 1 < \dfrac{\alpha-\beta}{\alpha-1} < \beta$, we have $\mathcal{MD}(\alpha,\beta) \subset \mathcal{MD}(0,\beta) \equiv \mathcal{M}(\beta)$. For $\alpha = -1$, if $f(z) \in \mathcal{MD}(\alpha,\beta)$, then $\dfrac{zf'(z)}{f(z)}$ belongs to the region which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2 - \beta^2 + 1}{2(\beta-1)}.$$

In the case of $f(z) \in \mathcal{ND}(\alpha,\beta)$, $\dfrac{zf''(z)}{f'(z)}$ lies in the region which contains $w = 0$ and is bounded by the ellipse

$$\left(u + \frac{\beta-1}{\alpha^2-1}\right)^2 + \frac{\alpha^2}{\alpha^2-1}v^2 = \frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2} \qquad (\alpha < -1)$$

with vertices at the points

$$\left(\frac{1-\beta}{\alpha^2-1}, \frac{\beta-1}{\sqrt{\alpha^2-1}}\right), \left(\frac{1-\beta}{\alpha^2-1}, -\frac{\beta-1}{\sqrt{\alpha^2-1}}\right), \left(\frac{1-\beta}{\alpha-1}, 0\right), \left(\frac{\beta-1}{\alpha+1}, 0\right).$$

Since $\dfrac{\beta-1}{\alpha+1} < 0 < \dfrac{1-\beta}{\alpha-1} < \beta$, we have $\mathcal{ND}(\alpha,\beta) \subset \mathcal{ND}(0,\beta) \equiv \mathcal{M}(\beta)$. And for $\alpha = -1$, $\dfrac{zf''(z)}{f'(z)}$ lies in the domain which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2}{2(\beta-1)} + \frac{\beta-1}{2}.$$

The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were considered by Uralegaddi, Ganigi and Sarangi [3], Nishiwaki and Owa [1], and Owa and Nishiwaki [2].

2 Coefficient inequalities for the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\{|n-\beta+1|+|n-\beta-1|-2\alpha(n-1)\}|a_n| \leqq \beta-|2-\beta|$$

for some $\alpha(\alpha \leqq 0)$ and for some $\beta(\beta > 1)$, then $f(z) \in \mathcal{MD}(\alpha,\beta)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty}\{|n-\beta+1|+|n-\beta-1|-2\alpha(n-1)\}|a_n| \leqq \beta-|2-\beta|$$

for $f(z) \in \mathcal{A}$. It sufficies to show that

$$\left|\frac{\left(\frac{zf'(z)}{f(z)}-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|-\beta\right)+1}{\left(\frac{zf'(z)}{f(z)}-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|-\beta\right)-1}\right| < 1 \qquad (z \in \mathbb{U}).$$

We have

$$\left|\frac{\left(\frac{zf'(z)}{f(z)}-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|-\beta\right)+1}{\left(\frac{zf'(z)}{f(z)}-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|-\beta\right)-1}\right| = \left|\frac{zf'(z)-\alpha e^{i\theta}|zf'(z)-f(z)|-\beta f(z)+f(z)}{zf'(z)-\alpha e^{i\theta}|zf'(z)-f(z)|-\beta f(z)-f(z)}\right|$$

$$= \left|\frac{z+\sum_{n=2}^{\infty} na_nz^n-\alpha e^{i\theta}|\sum_{n=2}^{\infty}(n-1)a_nz^n|-\beta z-\beta\sum_{n=2}^{\infty}a_nz^n+z+\sum_{n=2}^{\infty}a_nz^n}{z+\sum_{n=2}^{\infty} na_nz^n-\alpha e^{i\theta}|\sum_{n=2}^{\infty}(n-1)a_nz^n|-\beta z-\beta\sum_{n=2}^{\infty}a_nz^n-z-\sum_{n=2}^{\infty}a_nz^n}\right|$$

$$= \left|\frac{(2-\beta)+\sum_{n=2}^{\infty}(n-\beta+1)a_nz^{n-1}-\alpha e^{i\theta}|\sum_{n=2}^{\infty}(n-1)a_nz^{n-1}|}{-(\beta-\sum_{n=2}^{\infty}(n-\beta-1)a_nz^{n-1}+\alpha e^{i\theta}|\sum_{n=2}^{\infty}(n-1)a_nz^{n-1}|)}\right|$$

$$< \frac{|2-\beta| + \sum_{n=2}^{\infty}|n-\beta+1||a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n|}{\beta - \sum_{n=2}^{\infty}|n-\beta-1||a_n| + \alpha\sum_{n=2}^{\infty}(n-1)|a_n|}.$$

The last expression is bounded above by 1 if

$$|2-\beta| + \sum_{n=2}^{\infty}|n-\beta+1||a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n| \leqq \beta - \sum_{n=2}^{\infty}|n-\beta-1||a_n| - \alpha\sum_{n=2}^{\infty}(n-1)|a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty}\{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)\}|a_n| \leqq \beta - |2-\beta|$$

of the theorem. This completes the proof of the theorem. □

By using Theorem2.1, we have

Corollary 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n\{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)\}|a_n| \leqq \beta - |2-\beta|$$

for some $\alpha(\alpha \leqq 0)$ and for some $\beta(\beta > 1)$, then $f(z) \in \mathcal{ND}(\alpha,\beta)$

Proof. From $f(z) \in \mathcal{ND}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha,\beta)$, replacing a_n by na_n in Theorem2.1, we have the corollary. □

3 Relation for $\mathcal{MD}^*(\alpha,\beta)$ and $\mathcal{ND}^*(\alpha,\beta)$

By Theorem2.1, the class $\mathcal{MD}^*(\alpha,\beta)$ is considered as the subclass of $\mathcal{MD}(\alpha,\beta)$ consisting of $f(z)$ satisfying

$$\sum_{n=2}^{\infty}\{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)\}|a_n| \leqq \beta - |2-\beta| \tag{3.1}$$

for some $\alpha(\alpha \leqq 0)$ and for some $\beta(\beta > 1)$. The class $\mathcal{ND}^*(\alpha,\beta)$ is also considered as the subclass of $\mathcal{ND}(\alpha,\beta)$ consisting of $f(z)$ which satisfy

$$\sum_{n=2}^{\infty} n\{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)\}|a_n| \leqq \beta - |2-\beta| \tag{3.2}$$

for some $\alpha(\alpha \leqq 0)$ and for some $\beta(\beta > 1)$. By the coefficient inequalities for the classes $\mathcal{MD}^*(\alpha,\beta)$ and $\mathcal{ND}^*(\alpha,\beta)$, we see

Theorem 3.1. *If $f(z) \in \mathcal{A}$, then*

$$\mathcal{MD}^*(\alpha_1, \beta) \subset \mathcal{MD}^*(\alpha_2, \beta)$$

for some α_1 and $\alpha_2 (\alpha_1 \leqq \alpha_2 \leqq 0)$.

Proof. For $\alpha_1 \leqq \alpha_2 \leqq 0$, we obtain

$$\sum_{n=2}^{\infty}\{|n-\beta+1|+|n-\beta-1|-2\alpha_2(n-1)\}|a_n|$$
$$\leqq \sum_{n=2}^{\infty}\{|n-\beta+1|+|n-\beta-1|-2\alpha_1(n-1)\}|a_n|.$$

Therefore, if $f(z) \in \mathcal{MD}^*(\alpha_1, \beta)$, then $f(z) \in \mathcal{MD}^*(\alpha_2, \beta)$. Hence we get the required result. □

By using Theorem3.1, we also have

Corollary 3.1. *If $f(z) \in \mathcal{A}$, then*

$$\mathcal{ND}^*(\alpha_1, \beta) \subset \mathcal{ND}^*(\alpha_2, \beta)$$

for some α_1 and $\alpha_2 (\alpha_1 \leqq \alpha_2 \leqq 0)$.

4 Convolution of the classes $\mathcal{MD}^*(\alpha, \beta)$ and $\mathcal{ND}^*(\alpha, \beta)$

For analytic functions $f_j(z)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (j = 1, 2, \cdots, p),$$

the Hadamard product (or convolution) of $f_1(z), f_2(z), \cdots, f_p(z)$ is defined by

$$(f_1 * f_2 * \cdots * f_p)(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^{p} a_{n,j} \right) z^n.$$

Thus we have

Theorem 4.1. *If $f_1(z) \in \mathcal{MD}^*(\alpha, \beta_1)$ and $f_2(z) \in \mathcal{MD}^*(\alpha, \beta_2)$ for some $\alpha\,(\alpha \leqq 2 - \sqrt{5})$ and $\beta_1, \beta_2\,(1 < \beta_1, \beta_2 \leqq 2)$, then $(f_1 * f_2) \in \mathcal{MD}^*(\alpha, \beta)$, where*

$$\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.$$

Proof. From (3.1), for $f(z) \in \mathcal{MD}^*(\alpha,\beta)$ with $1 < \beta \leqq 2$, we have

$$\sum_{n=2}^{\infty}\{(n+1-\beta)+(n-1-\beta)-2\alpha(n-1)\}|a_n| \leqq \sum_{n=2}^{\infty}\{(n+1-\beta)+|n-1-\beta|-2\alpha(n-1)\}|a_n|$$
$$\leqq 2(\beta-1),$$

that is, if $f(z) \in \mathcal{MD}^*(\alpha,\beta)$, then

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha)-\beta+\alpha}{\beta-1}|a_n| \leqq 1. \tag{4.1}$$

Conversely, if $f(z)$ satisfies

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_n| \leqq 1, \tag{4.2}$$

then $f(z) \in \mathcal{MD}^*(\alpha,\beta)$. From (4.1), if $f_1(z) \in \mathcal{MD}^*(\alpha,\beta_1)$, then

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha)-\beta_1+\alpha}{\beta_1-1}|a_{n,1}| \leqq 1, \tag{4.3}$$

and also if $f_2(z) \in \mathcal{MD}^*(\alpha,\beta_2)$, then

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha)-\beta_2+\alpha}{\beta_2-1}|a_{n,2}| \leqq 1. \tag{4.4}$$

Applying the Shwarz's inequality, we have the following inequality

$$\sum_{n=2}^{\infty}\sqrt{\frac{\{n(1-\alpha)-\beta_1+\alpha\}\{n(1-\alpha)-\beta_2+\alpha\}}{(\beta_1-1)(\beta_2-1)}}\sqrt{|a_{n,1}||a_{n,2}|} \leqq 1 \tag{4.5}$$

by (4.3) and (4.4). From (4.2) and (4.5), if the following inequality

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_{n,1}||a_{n,2}| \tag{4.6}$$
$$\leqq \sum_{n=2}^{\infty}\sqrt{\frac{\{n(1-\alpha)-\beta_1+\alpha\}\{n(1-\alpha)-\beta_2+\alpha\}}{(\beta_1-1)(\beta_2-1)}}\sqrt{|a_{n,1}||a_{n,2}|}$$

is satisfied, then we say that $f(z) \in \mathcal{MD}^*(\alpha,\beta)$. This inequality holds true if

$$\frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}\sqrt{|a_{n,1}||a_{n,2}|} \leqq \sqrt{\frac{\{n(1-\alpha)-\beta_1+\alpha\}\{n(1-\alpha)-\beta_2+\alpha\}}{(\beta_1-1)(\beta_2-1)}} \tag{4.7}$$

for all $n \geqq 2$. Therefore, we have

$$\frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \leqq \frac{\{n(1-\alpha)-\beta_1+\alpha\}\{n(1-\alpha)-\beta_2+\alpha\}}{(\beta_1-1)(\beta_2-1)} \tag{4.8}$$

which is equivalent to

$$\beta \geqq 1+\frac{(\beta_1-1)(\beta_2-1)\{n(1-\alpha)+\alpha\}}{(\beta_1-1)(\beta_2-1)+\{n(1-\alpha)-\beta_1+\alpha\}\{n(1-\alpha)-\beta_2+\alpha\}} \tag{4.9}$$

for all $n \geqq 2$.
Let $G(n)$ be the right hand side of the last inequality. Then $G(n)$ is decreasing for $n \geqq 2$ for $\alpha \leqq 2-\sqrt{5}$. Thus $G(2)$ is the maximum of $G(n)$ for $\alpha(\alpha \leqq 2-\sqrt{5})$. This completes the proof of the theorem. □

For the functions $f(z)$ belonging to the class $\mathcal{ND}^*(\alpha,\beta)$, we also have

Corollary 4.1. *If $f_1(z) \in \mathcal{ND}^*(\alpha,\beta_1)$ and $f_2(z) \in \mathcal{ND}^*(\alpha,\beta_2)$ for some α and β_1, β_2, $(1<\beta_1,\beta_2 \leqq 2)$ then $(f_1 * f_2)(z) \in \mathcal{ND}^*(\alpha,\beta)$, where*

$$\beta = 1+\frac{(\beta_1-1)(\beta_2-1)(2-\alpha)}{(\beta_1-1)(\beta_2-1)+2(2-\alpha-\beta_1)(2-\alpha-\beta_2)}.$$

By virtue of Theorem4.1, we have the following theorem.

Theorem 4.2. *If $f_j \in \mathcal{MD}^*(\alpha,\beta_j)$ $(j=1,2,\cdots,p)$ for some $\alpha(\alpha \leqq 2-\sqrt{5})$ and $\beta_j(1<\beta_j \leqq 2)$, then $(f_1 * f_2 * \cdots * f_p) \in \mathcal{MD}^*(\alpha,\beta)$, where*

$$\beta = 1+\frac{A_p}{B_p-C_pD_p+E_p} \qquad (p \geqq 2),$$

$$A_p = \Pi_{j=1}^{p}(\beta_j-1)(2-\alpha)^{p-1},\ B_p=(2-\alpha)^{p-2}\Pi_{j=1}^{p}(\beta_j-1),$$

$$C_p = \sum_{m=1}^{p-2}(2-\alpha)^{p-m-2}(1-\alpha)^{m-1},\ D_p = \Pi_{j=1}^{p-m}(\beta_j-1)\Pi_{l=p-m+1}^{p}(2-\alpha-\beta_l),$$

and

$$E_p = (1-\alpha)^{p-2}\Pi_{j=1}^{p}(2-\alpha-\beta_j).$$

Proof. When $p=2$, we have

$$\beta = 1+\frac{(\beta_1-1)(\beta_2-1)(2-\alpha)}{(\beta_1-1)(\beta_2-1)+(2-\alpha-\beta_1)(2-\alpha-\beta_2)}.$$

Let us suppose that $(f_1 * \cdots * f_k) \in \mathcal{MD}^*(\alpha,\beta_0)$ and $f_{k+1} \in \mathcal{MD}^*(\alpha,\beta_{k+1})$, where

$$\beta_0 = 1+\frac{A_k}{B_k-C_kD_k+E_k} \qquad (k \geqq 2).$$

Using Theorem4.1 and replacing β_1 by β_0 and β_2 by β_{k+1}, we see that

$$\begin{aligned}\beta &= 1+\frac{(\beta_0-1)(\beta_{k+1}-1)(2-\alpha)}{(\beta_0-1)(\beta_{k+1}-1)+(2-\alpha-\beta_0)(2-\alpha-\beta_{k+1})}\\ &= 1+\frac{A_{k+1}}{B_{k+1}-\{B_k(2-\alpha-\beta_{k+1})+(1-\alpha)C_kD_k(2-\alpha-\beta_{k+1})\}+E_{k+1}}\\ &= 1+\frac{A_{k+1}}{B_{k+1}-\{B_k(2-\alpha-\beta_{k+1})+C_k^+D_{k+1}\}+E_{k+1}}\\ &= 1+\frac{A_{k+1}}{B_{k+1}-C_{k+1}D_{k+1}+E_{k+1}},\end{aligned}$$

where

$$C_k^+ = \sum_{m=2}^{k-1}(2-\alpha)^{k-m-1}(1-\alpha)^{m-1}.$$

This completes the proof of the Theorem. □

Finally we have

Corollary 4.2. *If* $f_j \in \mathcal{ND}^*(\alpha,\beta_j)$ $(j=1,2,\cdots,p)$ *for some* α *and* $\beta_j(1<\beta_j \leqq 2)$, *then* $(f_1 * f_2 * \cdots * f_p) \in \mathcal{ND}^*(\alpha,\beta)$, *where*

$$\beta = 1+\frac{A_p}{B_p-C_p^*D_p+2^{n-1}E_p} \qquad (p \geqq 2),$$

$$A_p = \Pi_{j=1}^p(\beta_j-1)(2-\alpha)^{p-1},\ B_p=(2-\alpha)^{p-2}\Pi_{j=1}^p(\beta_j-1),$$

$$C_p^* = \sum_{m=1}^{p-2}2^m(2-\alpha)^{p-m-2}(1-\alpha)^{m-1},\ D_p = \Pi_{j=1}^{p-m}(\beta_j-1)\Pi_{l=p-m+1}^p(2-\alpha-\beta_l),$$

and

$$E_p = (1-\alpha)^{p-2}\Pi_{j=1}^p(2-\alpha-\beta_j).$$

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