Certain classes of analytic functions concerned with uniformly starlike and convex functions (Sakaguchi Functions in Univalent Function Theory and Its Applications)

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Certain classes of analytic functions concerned with uniformly starlike and convex functions

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Abstract

Applying the coefficient inequalities of functions $f(z)$ belonging to the subclasses $\mathcal{M}D(\alpha, \beta)$ and $\mathcal{N}D(\alpha, \beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{M}D^*(\alpha, \beta)$ and $\mathcal{N}D^*(\alpha, \beta)$ are introduced. The object of the present paper is to derive some convolution properties of functions $f(z)$ in the classes $\mathcal{M}D^*(\alpha, \beta)$ and $\mathcal{N}D^*(\alpha, \beta)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Shams, Kulkarni and Jahangiri [4] have studied the subclass $\mathcal{S}D(\alpha, \beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

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for some $\alpha (\alpha \geq 0)$ and for some $\beta (0 \leq \beta < 1)$. The subclass $KD(\alpha, \beta)$ is defined by $f(z) \in KD(\alpha, \beta)$ if and only if $zf'(z) \in SD(\alpha, \beta)$. In view of the classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$, we introduce the subclass $MD(\alpha, \beta)$ consisting of all functions $f(z) \in A$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in U)$$

for some $\alpha (\alpha \leq 0)$ and for some $\beta (\beta > 1)$. The class $\mathcal{N}D(\alpha, \beta)$ is also considered as the subclass of $A$ consisting of all functions $f(z)$ which satisfy $zf'(z) \in MD(\alpha, \beta)$. We discuss some properties of functions $f(z)$ belonging to the classes $MD(\alpha, \beta)$ and $\mathcal{N}D(\alpha, \beta)$.

We note if $f(z) \in MD(\alpha, \beta)$, then, for $\alpha < -1$, $zf'(z)$ lies in the region $G \equiv G(\alpha, \beta) \equiv \{ w = u + iv : \text{Re} w < \alpha |w-1| + \beta \}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$\left( u - \frac{\alpha^2 - \beta}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

with vertices at the points

$$\left( \frac{\alpha^2 - \beta}{\alpha^2 - 1}, \sqrt{\alpha^2 - 1} \right), \left( \frac{\alpha^2 - \beta}{\alpha^2 - 1}, -\sqrt{\alpha^2 - 1} \right), \left( \frac{\alpha + \beta}{\alpha + 1}, 0 \right), \left( \frac{\alpha - \beta}{\alpha - 1}, 0 \right).$$

Since $\frac{\alpha + \beta}{\alpha + 1} < 1 < \frac{\alpha}{\alpha - 1} < \beta$, we have $MD(\alpha, \beta) \subset MD(0, \beta) \equiv M(\beta)$. For $\alpha = -1$, if $f(z) \in MD(\alpha, \beta)$, then $zf'(z)$ belongs to the region which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2 - \beta^2 + 1}{2(\beta - 1)}.$$

In the case of $f(z) \in N\mathcal{D}(\alpha, \beta)$, $zf'(z)$ lies in the region which contains $w = 0$ and is bounded by the ellipse

$$\left( u + \frac{\beta - 1}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2} \quad (\alpha < -1)$$

with vertices at the points

$$\left( \frac{1 - \beta}{\alpha^2 - 1}, \sqrt{\alpha^2 - 1} \right), \left( \frac{1 - \beta}{\alpha^2 - 1}, -\sqrt{\alpha^2 - 1} \right), \left( \frac{1 - \beta}{\alpha - 1}, 0 \right), \left( \frac{\beta - 1}{\alpha - 1}, 0 \right).$$

Since $\frac{\beta - 1}{\alpha + 1} < 0 < \frac{1 - \beta}{\alpha - 1} < \beta$, we have $N\mathcal{D}(\alpha, \beta) \subset N\mathcal{D}(0, \beta) \equiv M(\beta)$. And for $\alpha = -1$, $zf''(z)$ lies in the domain which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2}{2(\beta - 1)} + \frac{\beta - 1}{2}.$$
The classes $M(\alpha)$ and $N(\alpha)$ were considered by Uralegaddi, Ganigi and Sarangi [3], Nishiwaki and Owa [1], and Owa and Nishiwaki [2].

2 Coefficient inequalities for the classes $MD(\alpha, \beta)$ and $ND(\alpha, \beta)$

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|$$

for some $\alpha (\alpha \leq 0)$ and for some $\beta (\beta > 1)$, then $f(z) \in MD(\alpha, \beta)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|$$

for $f(z) \in A$. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) - \beta \right| + 1 \leq \beta - |2 - \beta|$$

$(z \in \mathbb{U})$.

We have

$$\left| \frac{zf'(z)}{f(z)} - \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) - \beta \right| + 1 = \left| z + \sum_{n=2}^{\infty} \frac{n a_n z^n - \alpha e^{i\theta} | \sum_{n=2}^{\infty} (n-1) a_n z^n | - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n + z + \sum_{n=2}^{\infty} a_n z^n - \beta \sum_{n=2}^{\infty} a_n z^n - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n - z - \sum_{n=2}^{\infty} a_n z^n \right|$$

$$= (2 - \beta) + \sum_{n=2}^{\infty} (n - \beta + 1) a_n z^{n-1} - \alpha e^{i\theta} | \sum_{n=2}^{\infty} (n-1) a_n z^{n-1} |$$

$$- \beta z - \beta \sum_{n=2}^{\infty} a_n z^n - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n - z - \sum_{n=2}^{\infty} a_n z^n$$

$$= (2 - \beta) + \sum_{n=2}^{\infty} (n - \beta + 1) a_n z^{n-1} - \alpha e^{i\theta} | \sum_{n=2}^{\infty} (n-1) a_n z^{n-1} |$$

$\leq \beta - |2 - \beta|$.
\[
< \frac{|2 - \beta| + \sum_{n=2}^{\infty} |n - \beta + 1||a_n| - \alpha \sum_{n=2}^{\infty} (n-1)|a_n|}{\beta - \sum_{n=2}^{\infty} |n - \beta - 1||a_n| + \alpha \sum_{n=2}^{\infty} (n-1)|a_n|}.
\]

The last expression is bounded above by 1 if
\[
|2 - \beta| + \sum_{n=2}^{\infty} |n - \beta + 1||a_n| - \alpha \sum_{n=2}^{\infty} (n-1)|a_n| \leq \beta - \sum_{n=2}^{\infty} |n - \beta - 1||a_n| + \alpha \sum_{n=2}^{\infty} (n-1)|a_n|
\]
which is equivalent to our condition
\[
\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
\]
of the theorem. This completes the proof of the theorem.

By using Theorem 2.1, we have

**Corollary 2.1.** If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n\{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
\]
for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \), then \( f(z) \in ND(\alpha, \beta) \)

**Proof.** From \( f(z) \in ND(\alpha, \beta) \) if and only if \( zf'(z) \in MD(\alpha, \beta) \), replacing \( a_n \) by \( na_n \) in Theorem 2.1, we have the corollary.

### 3 Relation for \( MD^*(\alpha, \beta) \) and \( ND^*(\alpha, \beta) \)

By Theorem 2.1, the class \( MD^*(\alpha, \beta) \) is considered as the subclass of \( MD(\alpha, \beta) \) consisting of \( f(z) \) satisfying
\[
(3.1) \quad \sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
\]
for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \). The class \( ND^*(\alpha, \beta) \) is also considered as the subclass of \( ND(\alpha, \beta) \) consisting of \( f(z) \) which satisfy
\[
(3.2) \quad \sum_{n=2}^{\infty} n\{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
\]
for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \). By the coefficient inequalities for the classes \( MD^*(\alpha, \beta) \) and \( ND^*(\alpha, \beta) \), we see
Theorem 3.1. If \( f(z) \in A \), then
\[
\mathcal{M}D^*(\alpha_1, \beta) \subset \mathcal{M}D^*(\alpha_2, \beta)
\]
for some \( \alpha_1 \) and \( \alpha_2 (\alpha_1 \leq \alpha_2 \leq 0) \).

Proof. For \( \alpha_1 \leq \alpha_2 \leq 0 \), we obtain
\[
\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha_2 (n-1)|a_n| \leq \sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha_1 (n-1)|a_n| \}.
\]
Therefore, if \( f(z) \in \mathcal{M}D^*(\alpha_1, \beta) \), then \( f(z) \in \mathcal{M}D^*(\alpha_2, \beta) \). Hence we get the required result. \( \square \)

By using Theorem 3.1, we also have

Corollary 3.1. If \( f(z) \in A \), then
\[
\mathcal{N}D^*(\alpha_1, \beta) \subset \mathcal{N}D^*(\alpha_2, \beta)
\]
for some \( \alpha_1 \) and \( \alpha_2 (\alpha_1 \leq \alpha_2 \leq 0) \).

4 Convolution of the classes \( \mathcal{M}D^*(\alpha, \beta) \) and \( \mathcal{N}D^*(\alpha, \beta) \)

For analytic functions \( f_j(z) \) given by
\[
f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, \ldots, p),
\]
the Hadamard product (or convolution) of \( f_1(z), f_2(z), \ldots, f_p(z) \) is defined by
\[
(f_1 * f_2 * \cdots * f_p)(z) = z + \sum_{n=2}^{\infty} \left( \prod_{j=1}^{p} a_{n,j} \right) z^n.
\]
Thus we have

Theorem 4.1. If \( f_1(z) \in \mathcal{M}D^*(\alpha_1, \beta_1) \) and \( f_2(z) \in \mathcal{M}D^*(\alpha_2, \beta_2) \) for some \( \alpha (\alpha \leq 2 - \sqrt{5}) \) and \( \beta_1, \beta_2 (1 < \beta_1, \beta_2 \leq 2) \), then \( (f_1 * f_2) \in \mathcal{M}D^*(\alpha, \beta) \), where
\[
\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.
\]
Proof. From (3.1), for $f(z) \in \mathcal{MD}^*(\alpha, \beta)$ with $1 < \beta \leq 2$, we have
\[
\sum_{n=2}^{\infty} \{(n+1-\beta)+(n-1-\beta)-2\alpha(n-1)\}|a_n| \leq \sum_{n=2}^{\infty} \{(n+1-\beta)+|n-1-\beta|-2\alpha(n-1)\}|a_n|
\leq 2(\beta-1),
\]
that is, if $f(z) \in \mathcal{MD}^*(\alpha, \beta)$, then
\[
\sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta+\alpha}{\beta-1}|a_n| \leq 1.
\]
(4.1)

Conversely, if $f(z)$ satisfies
\[
\sum_{n=2}^{\infty} \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_n| \leq 1,
\]
then $f(z) \in \mathcal{MD}^*(\alpha, \beta)$. From (4.1), if $f_1(z) \in \mathcal{MD}^*(\alpha, \beta_1)$, then
\[
\sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta_1+\alpha}{\beta_1-1}|a_{n,1}| \leq 1,
\]
and also if $f_2(z) \in \mathcal{MD}^*(\alpha, \beta_2)$, then
\[
\sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta_2+\alpha}{\beta_2-1}|a_{n,2}| \leq 1.
\]
(4.2)

Applying the Schwarz's inequality, we have the following inequality
\[
\sum_{n=2}^{\infty} \sqrt{\frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}} |a_{n,1}||a_{n,2}| \leq 1
\]
by (4.3) and (4.4). From (4.2) and (4.5), if the following inequality
\[
\sum_{n=2}^{\infty} \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_{n,1}||a_{n,2}|
\leq \sum_{n=2}^{\infty} \sqrt{\frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}} |a_{n,1}||a_{n,2}|
\]
is satisfied, then we say that $f(z) \in \mathcal{MD}^*(\alpha, \beta)$. This inequality holds true if
\[
\frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \sqrt{|a_{n,1}||a_{n,2}|} \leq \sqrt{\frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}}
\]
for all $n \geq 2$. Therefore, we have
\[
\frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \leq \frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}
\]
(4.7)
which is equivalent to
\[
\beta \geq 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)\{n(1 - \alpha) + \alpha\}}{(\beta_1 - 1)(\beta_2 - 1) + \{n(1 - \alpha) - \beta_1 + \alpha\}\{n(1 - \alpha) - \beta_2 + \alpha\}}
\]
for all \( n \geq 2 \).

Let \( G(n) \) be the right hand side of the last inequality. Then \( G(n) \) is decreasing for \( n \geq 2 \) for \( \alpha \leq 2 - \sqrt{5} \). Thus \( G(2) \) is the maximum of \( G(n) \) for \( \alpha(\alpha \leq 2 - \sqrt{5}) \). This completes the proof of the theorem. \qed

For the functions \( f(z) \) belonging to the class \( \mathcal{N}D^*(\alpha, \beta) \), we also have

**Corollary 4.1.** If \( f_1(z) \in \mathcal{N}D^*(\alpha, \beta_1) \) and \( f_2(z) \in \mathcal{N}D^*(\alpha, \beta_2) \) for some \( \alpha \) and \( \beta_1, \beta_2 \), \( (1 < \beta_1, \beta_2 \leq 2) \) then \( (f_1 \ast f_2)(z) \in \mathcal{N}D^*(\alpha, \beta) \), where
\[
\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.
\]

By virtue of Theorem 4.1, we have the following theorem.

**Theorem 4.2.** If \( f_j \in \mathcal{M}D^*(\alpha, \beta_j) \) \((j = 1, 2, \cdots, p)\) for some \( \alpha(\alpha \leq 2 - \sqrt{5}) \) and \( \beta_1(1 < \beta_j \leq 2) \), then \( (f_1 \ast f_2 \ast \cdots \ast f_p) \in \mathcal{M}D^*(\alpha, \beta) \), where
\[
\beta = 1 + \frac{A_p}{B_p - C_p D_p + E_p} \quad (p \geq 2),
\]
\[
A_p = \Pi_{j=1}^{p}(\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2}\Pi_{j=1}^{p}(\beta_j - 1),
\]
\[
C_p = \sum_{m=1}^{p-2}(2 - \alpha)^{p-m-2}(1 - \alpha)^{m-1}, \quad D_p = \Pi_{j=1}^{p-m}(\beta_j - 1)\Pi_{j=p-m+1}^{p}(2 - \alpha - \beta_j),
\]
and
\[
E_p = (1 - \alpha)^{p-2}\Pi_{j=1}^{p}(2 - \alpha - \beta_j).
\]

**Proof.** When \( p = 2 \), we have
\[
\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.
\]

Let us suppose that \( (f_1 \ast \cdots \ast f_k) \in \mathcal{M}D^*(\alpha, \beta_0) \) and \( f_{k+1} \in \mathcal{M}D^*(\alpha, \beta_{k+1}) \), where
\[
\beta_0 = 1 + \frac{A_k}{B_k - C_k D_k + E_k} \quad (k \geq 2).
\]
Using Theorem 4.1 and replacing $\beta_1$ by $\beta_0$ and $\beta_2$ by $\beta_{k+1}$, we see that

$$\beta = 1 + \frac{(\beta_0 - 1)(\beta_{k+1} - 1)(2 - \alpha)}{(\beta_0 - 1)(\beta_{k+1} - 1) + (2 - \alpha - \beta_0)(2 - \alpha - \beta_{k+1})}$$

$$= 1 + \frac{A_{k+1}}{B_{k+1} - \{B_k(2 - \alpha - \beta_{k+1}) + C_kD_k(2 - \alpha - \beta_{k+1})\} + E_{k+1}}$$

$$= 1 + \frac{A_{k+1}}{B_{k+1} - C_{k+1}D_{k+1} + E_{k+1}}$$

where

$$C_k^+ = \sum_{m=2}^{k-1} (2 - \alpha)^{k-m-1}(1 - \alpha)^{m-1}.$$

This completes the proof of the Theorem.

Finally we have

**Corollary 4.2.** If $f_j \in ND^*(\alpha, \beta_j)$ ($j = 1, 2, \cdots, p$) for some $\alpha$ and $\beta_j(1 < \beta_j \leq 2)$, then $(f_1 * f_2 * \cdots * f_p) \in ND^*(\alpha, \beta)$, where

$$\beta = 1 + \frac{A_p}{B_p - C_pD_p + 2^{n-1}E_p} \quad (p \geq 2),$$

$$A_p = \Pi_{j=1}^{p}(\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2}\Pi_{j=1}^{p}(\beta_j - 1),$$

$$C_p^* = \sum_{m=1}^{p-2} 2^m(2 - \alpha)^{p-m-2}(1 - \alpha)^{m-1}, \quad D_p = \Pi_{j=1}^{p-m}(\beta_j - 1)\Pi_{i=m+1}^{p}(2 - \alpha - \beta_i),$$

and

$$E_p = (1 - \alpha)^{p-2}\Pi_{j=1}^{p}(2 - \alpha - \beta_j).$$

**References**


