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Convolution properties for certain subclasses of analytic functions involving Silverman paper

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z\in\mathbb{C}||z|<1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are univalent in $\mathbb{U}$.
Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\quad(z\in\mathbb{U}),$$

for some $\alpha\quad(0\leqq\alpha<1)$. A function $f(z)\in\mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\quad(z\in\mathbb{U}),$$

for some $\alpha\quad(0\leqq\alpha<1)$. A function $f(z)\in\mathcal{K}(\alpha)$ is said to be convex of order α in $\mathbb{U}$. We note that

$$f(z)\in\mathcal{K}(\alpha)\Longleftrightarrow zf'(z)\in\mathcal{S}^*(\alpha).$$

In 1975, Silverman [1] gave the following coefficient inequalities for the functions in the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

Theorem A. *If $f(z)\in\mathcal{A}$ satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq 1-\alpha\quad(0\leqq\alpha<1),$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha \quad (z\in\mathbb{U}, 0\leqq\alpha<1),$$

that is, that $f(z)\in\mathcal{S}^*(\alpha)$.

Theorem B. *If* $f(z)\in\mathcal{A}$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\leqq 1-\alpha \quad (0\leqq\alpha<1),$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right|<1-\alpha \quad (z\in\mathbb{U}, 0\leqq\alpha<1),$$

that is, that $f(z)\in\mathcal{K}(\alpha)$.

In this paper, we consider a new subclass $\mathcal{M}(\alpha)$ of $\mathcal{A}$ consisting of functions $f(z)$ such that

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha} \quad (z\in\mathbb{U}),$$

for some α $(0<\alpha<1)$. We also introduce and investigate here the subclass $\mathcal{N}(\alpha)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inclusion relationship

$$zf'(z)\in\mathcal{M}(\alpha).$$

2 Properties of the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$

Theorem 1 *If* $f(z)\in\mathcal{A}$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)}-\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<1-2\alpha \quad (z\in\mathbb{U}) \tag{2.1}$$

for some α $\left(\frac{1}{4}\leqq\alpha<\frac{1}{2}\right)$, *then*

$$\left|\frac{f(z)}{zf'(z)}-1\right|<\frac{1}{2\alpha}-1 \quad (z\in\mathbb{U}),$$

therefore, $f(z)\in\mathcal{M}(\alpha)$.

Corollary 1 *If* $f(z)\in\mathcal{A}$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}-\frac{z(2f''(z)+zf'''(z))}{f'(z)+zf''(z)}\right|<1-2\alpha \quad (z\in\mathbb{U}) \tag{2.6}$$

for some α $\left(\frac{1}{4}\leqq\alpha<\frac{1}{2}\right)$, *then*

$$\left|\frac{f'(z)}{f'(z)+zf''(z)}-1\right|<\frac{1}{2\alpha}-1 \quad (z\in\mathbb{U}),$$

therefore, $f(z) \in \mathcal{N}(\alpha)$.

Theorem 2 *If* $f(z) \in \mathcal{M}(\alpha)$ $(\frac{1}{2} \leqq \alpha < 1)$, *then*

$$\left|\left(\frac{z}{f(z)}\right)^{\beta} - 1\right| < 1 - \gamma \quad (z \in \mathbb{U}),$$

where, $0 \leqq \gamma < 1$ *and* $0 < \beta \leqq 1 - \gamma$.

Corollary 2 *If* $f(z) \in \mathcal{M}(\alpha)$ $\left(\frac{1}{2} \leqq \alpha < 1\right)$, *then*

$$\left|\left(\frac{z}{f(z)}\right)^{\beta} - 1\right| < 1 \quad (z \in \mathbb{U})$$

where $0 < \beta \leqq 1$.

Theorem 3 *If* $f(z) \in \mathcal{N}(\alpha)$ $\left(\frac{1}{2} \leqq \alpha < 1\right)$, *then*

$$\left|\left(\frac{1}{f'(z)}\right)^{\beta} - 1\right| < 1 - \gamma \quad (z \in \mathbb{U}),$$

where $0 \leqq \gamma < 1$ *and* $0 < \beta \leqq 1 - \gamma$.

Corollary 3 *If* $f(z) \in \mathcal{N}(\alpha)$ $\left(\frac{1}{2} \leqq \alpha < 1\right)$, *then*

$$\left|\left(\frac{1}{f'(z)}\right)^{\beta} - 1\right| < 1 \quad (z \in \mathbb{U}),$$

where $0 < \beta \leqq 1$.

3 Coefficient inequalities

Theorem 4. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty}(n - \alpha)|a_n| \leqq \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha & (0 < \alpha \leqq \frac{1}{2}) \\ 1 - \alpha & (\frac{1}{2} \leqq \alpha < 1) \end{cases} \tag{3.1}$$

for some α $(0 < \alpha < 1)$, *then* $f(z) \in \mathcal{M}(\alpha)$.

Theorem 5. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leqq \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha & (0 < \alpha \leqq \frac{1}{2}) \\ 1 - \alpha & (\frac{1}{2} \leqq \alpha < 1) \end{cases} \tag{3.2}$$

for some α $(0 < \alpha < 1)$, *then* $f(z) \in \mathcal{N}(\alpha)$.

4 Convolution

Definition *If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$ are given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}.$$

Then the convolution $(f * g)(z)$ of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

In this section, we define

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, 3, \cdots)$$

and

$$g_k(z) = z + \sum_{n=2}^{\infty} b_{n,k} z^n \quad (k = 1, 2, 3, \cdots).$$

Theorem 6 *If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $\frac{1}{2} \leqq \alpha_j < 1$ for each $j = 1, 2$, then*

$$(f_1 * f_2)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{(1-\alpha_1)(1-\alpha_2)}{(2-\alpha_1)(2-\alpha_2) - (1-\alpha_1)(1-\alpha_2)}.$$

Theorem 7 *If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $\frac{1}{2} \leqq \alpha_j < 1$ for each $j = 1, 2, \cdots, m$, then*

$$(f_1 * f_2 * \cdots * f_m)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{\prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}$$

Theorem 8 *If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $\frac{1}{2} \leqq \alpha_k < 1$ for each $k = 1, 2$, then*

$$(g_1 * g_2)(z) \in \mathcal{N}^*(\alpha),$$

where

$$\alpha = 1 - \frac{(1-\alpha_1)(1-\alpha_2)}{2(2-\alpha_1)(2-\alpha_2) - (1-\alpha_1)(1-\alpha_2)}.$$

Theorem 9 *If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $\frac{1}{2} \leqq \alpha_k < 1$ for each $k = 1, 2, 3, \cdots, m$, then*

$$(g_1 * g_2 * \cdots * g_m)(z) \in \mathcal{N}^*(\alpha),$$

where

$$\alpha = 1 - \frac{\prod_{k=1}^{m}(1-\alpha_k)}{2^{m-1}\prod_{k=1}^{m}(2-\alpha_k) - \prod_{k=1}^{m}(1-\alpha_k)}$$

Theorem 10 *If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $0 < \alpha_j \leqq \frac{1}{2}$ for each $j = 1, 2$, then*

$$(f_1 * f_2)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = \frac{2\alpha_1\alpha_2}{\alpha_1\alpha_2 + (2-\alpha_1)(2-\alpha_2)}.$$

Theorem 11 *If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $0 < \alpha_j \leqq \frac{1}{2}$ for each $j = 1, 2, 3, \cdots, m$, then*

$$(f_1 * f_2 * \cdots * f_m)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = \frac{2\prod_{j=1}^{m}\alpha_j}{\prod_{j=1}^{m}\alpha_j + \prod_{j=1}^{m}(2-\alpha_j)}$$

Theorem 12 *If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leqq \frac{1}{2}$ for each $k = 1, 2$, then*

$$(g_1 * g_2)(z) \in \mathcal{N}^*(\alpha)$$

where

$$\alpha = \frac{2\alpha_1\alpha_2}{\alpha_1\alpha_2 + 2(2-\alpha_1)(2-\alpha_2)}.$$

Theorem 13 *If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leqq \frac{1}{2}$ for each $k = 1, 2, 3, \cdots, m$, then*

$$(g_1 * g_2 * \cdots * g_m)(z) \in \mathcal{N}^*(\alpha),$$

where

$$\alpha = \frac{2\prod_{k=1}^{m}\alpha_k}{\prod_{k=1}^{m}\alpha_k + 2^{m-1}\prod_{k=1}^{m}(2-\alpha_k)}$$

Theorem 14 *If* $g_k(z) \in \mathcal{N}^*(\alpha_k)$ *with* $\frac{1}{2} \leqq \alpha_k < 1$ *for each* $k = 1, 2$, *then*

$$(g_1 * g_2)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{(1-\alpha_1)(1-\alpha_2)}{4(2-\alpha_1)(2-\alpha_2) - (1-\alpha_1)(1-\alpha_2)}.$$

Theorem 15 *If* $g_k(z) \in \mathcal{N}^*(\alpha_k)$ *with* $\frac{1}{2} \leqq \alpha_k < 1$ *for each* $k = 1, 2, 3, \cdots, m$, *then*

$$(g_1 * g_2 * \cdots * g_m)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{\prod_{k=1}^{m}(1-\alpha_k)}{4^{m-1}\prod_{k=1}^{m}(2-\alpha_k) - \prod_{k=1}^{m}(1-\alpha_k)}.$$

Theorem 16 *If* $f_1(z) \in \mathcal{M}^*(\alpha_1)$ *with* $\frac{1}{2} \leqq \alpha_1 < 1$, *and if* $g_1(z) \in \mathcal{N}^*(\beta_1)$ *with* $\frac{1}{2} \leqq \beta_1 < 1$, *then*

$$(f_1 * g_1)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{(1-\alpha_1)(1-\beta_1)}{2(2-\alpha_1)(2-\beta_1) - (1-\alpha_1)(1-\beta_1)}.$$

Theorem 17 *If* $f_j(z) \in \mathcal{M}^*(\alpha_j)$ *with* $\frac{1}{2} \leqq \alpha_j < 1$ *for each* $j = 1, 2, 3, \cdots, m$, *and if* $g_k(z) \in \mathcal{N}^*(\beta_k)$ *with* $\frac{1}{2} \leqq \beta_k < 1$ *for each* $k = 1, 2, 3, \cdots, p$, *then*

$$(f_1 * \cdots * f_m * g_1 * \cdots * g_p)(z) \in \mathcal{M}^*(\gamma)$$

where

$$\gamma = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)\prod_{k=1}^{p}(1-\beta_k)}{2^p\prod_{j=1}^{m}(2-\alpha_j)\prod_{k=1}^{p}(2-\beta_k) - \prod_{j=1}^{m}(1-\alpha_j)\prod_{k=1}^{p}(1-\beta_k)}$$

Theorem 18 *If* $g_k(z) \in \mathcal{N}^*(\alpha_k)$ *with* $0 < \alpha_k \leqq \frac{1}{2}$ *for each* $k = 1, 2$, *then*

$$(g_1 * g_2)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = \frac{2\alpha_1\alpha_2}{4(2-\alpha_1)(2-\alpha_2) + \alpha_1\alpha_2}.$$

Theorem 19 *If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leqq \frac{1}{2}$ for each $k = 1, 2, 3, \cdots, m$, then*

$$(g_1 * g_2 * \cdots * g_m)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = \frac{2\prod_{k=1}^{m} \alpha_k}{\prod_{k=1}^{m} \alpha_k + 4^{m-1} \prod_{k=1}^{m} (2 - \alpha_k)}.$$

Theorem 20 *If $f_1(z) \in \mathcal{M}^*(\alpha_1)$ with $0 < \alpha_1 \leqq \frac{1}{2}$, and if $g_1(z) \in \mathcal{N}^*(\beta_1)$ with $0 < \beta \leqq \frac{1}{2}$, then*

$$(f_1 * g_1)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = \frac{2\alpha_1\beta_1}{2(2 - \alpha_1)(2 - \beta_1) + \alpha_1\beta_1}.$$

Theorem 21 *If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $0 < \alpha_j \leqq \frac{1}{2}$, for each $j = 1, 2, 3, \cdots, m$, and if $g_k(z) \in \mathcal{N}^*(\beta_k)$ with $0 < \beta_k \leqq \frac{1}{2}$ for each $k = 1, 2, 3, \cdots, p$, then*

$$(f_1 * \cdots * f_m * g_1 * \cdots * g_p)(z) \in \mathcal{M}^*(\gamma),$$

where

$$\gamma = \frac{2\prod_{j=1}^{m} \alpha_j \prod_{k=1}^{p} \beta_k}{2^p \prod_{j=1}^{m} (2 - \alpha_j) \prod_{k=1}^{p} (2 - \beta_k) + \prod_{j=1}^{m} \alpha_j \prod_{k=1}^{p} \beta_k}.$$

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