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Kyoto University
On Sakaguchi type functions

Shigeyoshi Owa
Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail: owa@math.kindai.ac.jp

Tadayuki Sekine
Office of Mathematics, College of Pharmacy, Nihon University
Narashinodai, Funabashi, Chiba 274-8555, Japan
E-mail: tsekine@pha.nihon-u.ac.jp

and

Rikuo Yamakawa
Office of Mathematics, Shibaura Institute of Technology
Minuma, Saitama, Saitama 337-8570, Japan
E-mail: yamakawa@sic.shibaura-it.ac.jp

Abstract

Two subclasses $S(\alpha,t)$ and $\mathcal{T}(\alpha,t)$ are introduced concerning with Sakaguchi functions in the open unit disk $U$. Further, by using the coefficient inequalities for the classes $S(\alpha,t)$ and $\mathcal{T}(\alpha,t)$, two subclasses $S_0(\alpha,t)$ and $\mathcal{T}_0(\alpha,t)$ are defined. The object of the present paper is to discuss some properties of functions belonging to the classes $S_0(\alpha,t)$ and $\mathcal{T}_0(\alpha,t)$.

1 Introduction

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. A function $f(z) \in A$ is said to be in the class $S(\alpha,t)$ if it satisfies

$$\text{Re} \left\{ \frac{(1-t)zf'(z)}{f(z)-f(tz)} \right\} > \alpha, \quad |t| \leq 1, \ t \neq 1$$

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for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in \mathbb{U}$. The class $S(0, -1)$ was introduced by Sakaguchi [5]. Therefore, a function $f(z) \in S(\alpha, -1)$ is called Sakaguchi function of order $\alpha$. Incidentally the class of uniformly starlike functions introduced by Goodman [1] is following.

$$UST = \left\{ f(z) \in A : \text{Re} \left( \frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)} \right) > 0 \right\}, \quad (z, \zeta) \in \mathbb{U} \times \mathbb{U}. $$

As for $S(\alpha, t)$ and $UST$, Rønning [4] showed the following important result.

**Remark 1.1** $f(z) \in UST$ if and only if for every $z \in \mathbb{U}$, $|t| = 1$

$$\text{Re} \left( \frac{(1-t)zf'(z)}{f(z) - f(tz)} \right) - 1 < 1 - \alpha.$$ 

We also denote by $T(\alpha, t)$ the subclass of $A$ consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, t)$. Recently Cho, Kwon and Owa [2], and, recently, Owa, Sekine and Yamakawa [3] have discussed some properties for functions $f(z)$ in $S(\alpha, -1)$, $T(\alpha, -1)$. Now we show some results for functions belonging to the classes $S(\alpha, t)$ and $T(\alpha, t)$.

## 2 $S_0(\alpha, t)$ and $T_0(\alpha, t)$

We first prove the following two theorems which are similar to the results of Cho, Kwon and Owa [2].

**Theorem 2.1** If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{|n-u_n| + (1-\alpha)|u_n|\}|a_n| \leq 1 - \alpha, \quad u_n = 1 + t + t^2 + \cdots + t^{n-1} \quad (2.1)$$

for some $\alpha(0 \leq \alpha < 1)$, then $f(z) \in S(\alpha, t)$.

**Proof** For Theorem 1, we show that if $f(z)$ satisfies (2.1) then

$$\left| \frac{(1-t)zf'(z)}{f(z) - f(tz)} - 1 \right| < 1 - \alpha.$$ 

Evidently, since

$$\frac{(1-t)zf'(z)}{f(z) - f(tz)} - 1 = \frac{z + \sum_{n=2}^{\infty} na_nz^{n-1}}{z + \sum_{n=2}^{\infty} u_n a_n z^n} - 1 = \frac{\sum_{n=2}^{\infty} (n-u_n)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} u_n a_n z^{n-1}},$$

we see that

$$\left| \frac{(1-t)zf'(z)}{f(z) - f(tz)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{|n-u_n||a_n|}{1 - \sum_{n=2}^{\infty} |u_n||a_n|}.$$ 

Therefore, if $f(z)$ satisfies (2.1), then we have

$$\left| \frac{(1-t)zf'(z)}{f(z) - f(tz)} - 1 \right| < 1 - \alpha.$$ 

This completes the proof of Theorem 2.1.
Theorem 2.2 If $f(z) \in A$ satisfies
\[ \sum_{n=2}^{\infty} n \{|n-u_n| + (1-\alpha)|u_n|\}|a_n| \leq 1 - \alpha \] (2.2)
for some $\alpha$ ($0 \leq \alpha < 1$), then $f(z) \in \mathcal{T}(\alpha, t)$.

Proof Noting that $f(z) \in \mathcal{T}(\alpha, t)$ if and only if $zf'(z) \in \mathcal{S}(\alpha, t)$, we can prove Theorem 2.

We now define
\[
\mathcal{S}_0(\alpha, t) = \{f(z) \in A : f(z) \text{satisfies (2.1)}\}
\]
and
\[
\mathcal{T}_0(\alpha, t) = \{f(z) \in A : f(z) \text{satisfies (2.2)}\}
\]

In view of the above theorems, we see

Example 2.1 Let us consider a function $f(z)$ given by
\[
f(z) = z + (1-\alpha) \left( \frac{\lambda \delta_2}{2(2-\alpha)} z^2 + \frac{(1-\lambda)\delta_3}{7-3\alpha} z^3 \right), \quad 0 \leq \lambda \leq 1, \ |\delta_2| = |\delta_3| = 1
\] (2.3)
Then for any $t$ ($|t| \leq 1, t \neq 1$), $f(z) \in \mathcal{S}_0(\alpha, t) \subset \mathcal{S}(\alpha, t)$.

Example 2.2 Let us consider a function $f(z)$ given by
\[
f(z) = z + (1-\alpha) \left( \frac{\lambda \delta_2}{4(2-\alpha)} z^2 + \frac{(1-\lambda)\delta_3}{3(7-3\alpha)} z^3 \right), \quad 0 \leq \lambda \leq 1, \ |\delta_2| = |\delta_3| = 1
\] (2.4)
Then for any $t$ ($|t| \leq 1, t \neq 1$), $f(z) \in \mathcal{T}_0(\alpha, t) \subset \mathcal{T}(\alpha, t)$.

3 Coefficient inequalities

Next applying Carathéodory function $p(z)$ defined by
\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\] (3.1)
in $\mathbb{U}$, we discuss the coefficient inequalities for functions $f(z)$ in $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$.

Theorem 3.1 If $f(z) \in \mathcal{S}(\alpha, t)$, then
\[
|a_n| \leq \beta \left\{ 1 + \beta \sum_{j=2}^{n-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2>j_1} \sum_{j=2}^{n-2} \frac{|u_{j_2} u_{j_1}|}{|v_{j_1} v_{j_2}|} + \beta^3 \sum_{j_3>j_2>j_1} \sum_{j=2}^{n-3} \frac{|u_{j_3} u_{j_2} u_{j_1}|}{|v_{j_1} v_{j_2} v_{j_3}|} + \cdots + \beta^{n-2} \prod_{j=2}^{n-1} \frac{|u_j|}{|v_j|} \right\}
\] (3.1)
where
\[
\beta = 2(1-\alpha), \quad v_n = n - u_n.
\] (3.2)
Proof. We define the function $p(z)$ by
\[
p(z) = \frac{1}{1 - \alpha} \left( \frac{(1-t)zf'(z)}{f(z) - f(tz)} - \alpha \right) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{3.3}
\]
for $f(z) \in S(\alpha, t)$. Then $p(z)$ is a Carathéodory function and satisfies
\[
|p_n| \leq 2 \quad (n \geq 1). \tag{3.4}
\]
Since
\[
(1-t)zf'(z) = (f(z) - f(tz)) (\alpha + (1-\alpha)p(z)),
\]
we have
\[
z + \sum_{n=2}^{\infty} na_n z^n = \left( z + \sum_{n=2}^{\infty} u_n a_n z^n \right) \left( 1 + (1-\alpha) \sum_{n=2}^{\infty} p_n z^n \right)
\]
where
\[
u_n = 1 + t + t^2 + \cdots + t^{n-1}.
\]
So we get
\[
a_n = \frac{1 - \alpha}{n - u_n} \left( p_1 u_{n-1} a_{n-1} + p_2 u_{n-2} a_{n-2} + \cdots + p_{n-2} u_2 a_2 + p_{n-1} \right). \tag{3.5}
\]
From the equation (3.5), we easily have that
\[
|a_2| = \left| \frac{1 - \alpha}{2 - u_2} p_1 \right| \leq \frac{2(1-\alpha)}{|2 - u_2|},
\]
\[
|a_3| \leq \frac{2(1-\alpha)}{|3 - u_3|} (|u_2 a_2| + 1) \leq \frac{2(1-\alpha)}{|3 - u_3|} \left( 1 + 2(1-\alpha) \frac{|u_2|}{|2 - u_2|} \right),
\]
and
\[
|a_4| \leq \frac{2(1-\alpha)}{|4 - u_4|} \left\{ 1 + 2(1-\alpha) \left( \frac{|u_2|}{|2 - u_2|} + \frac{|u_3|}{|3 - u_3|} \right) + 2^2(1-\alpha)^2 \frac{|u_2 u_3|}{|2 - u_2||3 - u_3|} \right\}.
\]
Thus, using the mathematical induction, we obtain the inequality (3.1).

Remark 3.1 Equalities in Theorem 3.1 are attended for $f(z)$ given by
\[
\frac{zf'(z)}{f(z) - f(tz)} = \frac{1 + (1-2\alpha)z}{1 - z}.
\]

Remark 3.2 If we put $\alpha = 0, t = 0$ in Theorem 3.2, then we have well known result
\[
f(z) \in S^* \Rightarrow |a_n| \leq n
\]
where $S^*$ is usual starlike class. And if we put $\alpha = 0, t = -1$, then we have the result due to Sakaguchi [5]
\[
f(z) \in STS \Rightarrow |a_n| \leq 1,
\]
where $STS$ is Sakaguchi function class.

For functions $T(\alpha, t)$, similarly we have

**Theorem 3.2** If $f(z) \in T(\alpha, t)$, then

$$|a_n| \leq \frac{\beta}{n|v_n|} \left\{ 1 + \beta \sum_{j=2}^{n-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_3>j_1}^{n-2} \sum_{j_1=2}^{n-2} \frac{|u_{j_1}u_{j_2}|}{|v_{j_1}v_{j_2}|} + \beta^3 \sum_{j_3>j_2>j_1}^{n-3} \sum_{j_2=2}^{n-3} \sum_{j_1=2}^{n-3} \frac{|u_{j_1}u_{j_2}u_{j_3}|}{|v_{j_1}v_{j_2}v_{j_3}|} + \cdots + \beta^{n-2} \prod_{j=2}^{n-1} \frac{|u_j|}{|v_j|} \right\},$$

(3.6)

where

$$\beta = 2(1 - \alpha), \quad v_n = n - u_n.$$

### 4 Distortion inequalities

For functions $f(z)$ in the classes $S_0(\alpha, t)$ and $T_0(\alpha, t)$, we derive

**Theorem 4.1** If $f(z) \in S_0(\alpha, t)$, then

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - A_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z|^n + A_j|z|^{j+1}$$

(4.1)

where

$$A_j = \frac{1 - \alpha - \sum_{n=2}^{j} \{|n - u_n| + (1 - \alpha)|u_n|\}|a_n|}{j + 1 - \alpha|u_{j+1}|} \quad (j \geq 2).$$

(4.2)

**Proof** From the inequality (2.1), we know that

$$\sum_{n=j+1}^{\infty} \{|n - u_n| + (1 - \alpha)|u_n|\}|a_n| \leq 1 - \alpha - \sum_{n=2}^{j} \{|n - u_n| + (1 - \alpha)|u_n|\}|a_n|.$$

On the other hand

$$|n - u_n| + (1 - \alpha)|u_n| \geq n - \alpha|u_n|,$$

and hence $n - \alpha|u_n|$ is monotonically increasing with respect to $n$. Thus we deduce

$$(j + 1 - \alpha|u_{j+1}|) \sum_{n=j+1}^{\infty} |a_n| \leq 1 - \alpha - \sum_{n=2}^{j} \{|n - u_n| + (1 - \alpha)|u_n|\}|a_n|,$$

which implies that

$$\sum_{n=j+1}^{\infty} |a_n| \leq A_j.$$
Therefore we have that

\[ |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z^n| + A_j|z|^{j+1} \]

and

\[ |f(z)| \geq |z| - \sum_{n=2}^{j} |a_n||z^n| - A_j|z|^{j+1}. \]

This completes the proof of the theorem.

Similarly we have

**Theorem 4.2** If \( f(z) \in \mathcal{T}_0(\alpha, t) \), then

\[ |z| - \sum_{n=2}^{j} n|a_n||z^n| - B_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} n|a_n||z^n| + B_j|z|^{j+1} \]  \hspace{1cm} (4.4)

and

\[ 1 - \sum_{n=2}^{j} n|a_n||z^{n-1}| - C_j|z|^{j-1} \leq |f'(z)| \leq 1 + \sum_{n=2}^{j} n|a_n||z^{n-1}| + C_j|z|^{j-1} \]  \hspace{1cm} (4.5)

where

\[ B_j = \frac{1 - \alpha - \sum_{n=2}^{j} n\{|n-u_n|+(1-\alpha)|u_n|\}|a_n|}{(j+1)\{j+1-\alpha|u_{j+1}|\}} \]  \hspace{1cm} (j \geq 2). \hspace{1cm} (4.6)

and

\[ C_j = \frac{1 - \alpha - \sum_{n=2}^{j} n\{|n-u_n|+(1-\alpha)|u_n|\}|a_n|}{j+1-\alpha|u_{j+1}|} \]  \hspace{1cm} (j \geq 2). \hspace{1cm} (4.7)

## 5 Relation between the classes

By the definitions for the classes \( \mathcal{S}_0(\alpha, t) \), and \( \mathcal{T}_0(\alpha, t) \), evidently we have

\[ \mathcal{S}_0(\alpha, t) \subset \mathcal{S}_0(\beta, t) \] \hspace{1cm} (0 \leq \beta \leq \alpha < 1)

and

\[ \mathcal{T}_0(\alpha, 1) \subset \mathcal{T}_0(\beta, t) \] \hspace{1cm} (0 \leq \beta \leq \alpha < 1).

Let us consider a relation between \( \mathcal{S}_0(\beta, t) \) and \( \mathcal{T}_0(\alpha, t) \).

**Theorem 5.1** If \( f(z) \in \mathcal{T}_0(\alpha, t) \), then \( f(z) \in \mathcal{S}_0\left(\frac{1+\alpha}{2}, t\right) \).

**Proof** Let \( f(z) \in \mathcal{T}_0(\alpha, t) \). Then, if \( \beta \) satisfies

\[ \frac{|n-u_n|+(1-\beta)|u_n|}{1-\beta} \leq n \frac{|n-u_n|+(1-\alpha)|u_n|}{1-\alpha} \]  \hspace{1cm} (5.1)
for all $n \geq 2$, then we have that $f(z) \in S_0(\beta, t)$. From (5.1), we have

$$\beta \leq 1 - \frac{(1 - \alpha)|n - u_n|}{n|n - u_n| + (1 - \alpha)(n - 1)|u_n|}.$$  \hspace{1cm} (5.2)

Furthermore, since for all $n \geq 2$

$$\frac{|n - u_n|}{n|n - u_n| + (1 - \alpha)(n - 1)|u_n|} \leq \frac{1}{n} \leq \frac{1}{2},$$  \hspace{1cm} (5.3)

we obtain

$$f(z) \in S_0\left(\frac{1 + \alpha}{2}, t\right).$$

References


