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Geometric Properties of Solutions of a Class of Ordinary Linear Differential Equations

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Abstract—The main object of this paper is to investigate several geometric properties of the solutions of the following second-order linear differential equation:

$$w''(z) + p(z)w(z) = 0,$$

where the function $p(z)$ is analytic in the open unit disk \mathbb{U} . Relevant connections of the results presented in this paper with those given earlier by, for example, Robertson, Miller and Saitoh are also considered.

Keywords—Starlike functions, Strongly starlike functions, Close-to-convex functions, Differential equations, Univalent function.

1 Introduction

Let \mathcal{A} denote the class of functions f normalized by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also, let S, S^* and $S^*(\alpha)$ denote the subclasses of \mathcal{A} consisting of functions which are, respectively, univalent, starlike with respect to the origin, and starlike of order α in \mathbb{U} ($0 \leq \alpha < 1$). Thus, by definition, we have (see, for detail, [1] [6] [8]).

$$(1.2) \quad S^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \ (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}$$

and

$$(1.3) \quad S^* := S^*(0).$$

Furthermore, $SS^*(\beta)$ denote the subclasses of \mathcal{A} consisting of functions which are strongly starlike of order β in \mathbb{U} ($0 < \beta \leq 1$). By definition, we have

$$(1.4) \quad SS^*(\beta) := \left\{ f : f \in \mathcal{A} \text{ and } \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \leq \frac{\pi}{2}\beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1) \right\}.$$

For functions $f \in \mathcal{A}$ with $f'(z) \neq 0$ ($z \in \mathbb{U}$), we define the Schwarzian derivative of $f(z)$ by

$$(1.5) \quad S(f, z) := \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

We begin by recalling the following result of Miller [3].

Theorem A. (See [3]) Let the function $p(z)$ be analytic in \mathbb{U} with

$$|zp(z)| < 1 \quad (z \in \mathbb{U}).$$

Also, let $v(z)$ denote the unique solution of the following initial-value problem:

$$(1.6) \quad v''(z) + p(z)v(z) = 0 \quad (v(0) = 0; v'(0) = 1)$$

in \mathbb{U} . Then

$$(1.7) \quad \left| \frac{zv'(z)}{v(z)} - 1 \right| < 1 \quad (z \in \mathbb{U})$$

and $v(z)$ is a starlike conformal map of the unit disk \mathbb{U} .

Theorem A is related rather closely to some earlier results of Nehari [5] and Robertson [9], which we recall here as Theorem B and C below.

Theorem B. (See [5]) If $f \in \mathcal{A}$ satisfies the following inequality involving its Schwarzian derivative defined by (1.5):

$$(1.8) \quad |S(f, z)| \leq \frac{\pi^2}{2} \quad (z \in \mathbb{U}),$$

then $f \in S$. The result is sharp for the function $f(z)$ given by

$$(1.9) \quad f(z) = \frac{e^{i\pi z} - 1}{i\pi}.$$

Theorem C. (See [9]) Let $zp(z)$ be analytic in \mathbb{U} and

$$(1.10) \quad \operatorname{Re} \{z^2 p(z)\} \leq \frac{\pi^2}{4} |z|^2 \quad (z \in \mathbb{U}).$$

Then the unique solution $W = W(z)$ of the following initial-value problem:

$$(1.11) \quad W''(z) + p(z)W(z) = 0 \quad (W(0) = 0; W'(0) = 1)$$

is univalent and starlike in \mathbb{U} . The constant $\frac{\pi^2}{4}$ in inequality (1.10) is the best possible.

Remark 1. By putting

$$(1.12) \quad p(z) = \frac{1}{2} S(f, z) \quad (z \in \mathbb{U})$$

and using (1.8), we obtain inequality (1.10). Obviously, therefore, the hypothesis in Theorem B is stronger than that in Theorem C.

2 A class of bounded functions

Let \mathcal{B}_J denote the class of bounded functions

$$(2.1) \quad w(z) = \sum_{n=1}^{\infty} c_n z^n,$$

analytic in \mathbb{U} , for which

$$(2.2) \quad |w(z)| < J \quad (z \in \mathbb{U}; J > 0).$$

If $g(z) \in \mathcal{B}_J$, then we can show that the function $w(z)$ defined by

$$(2.3) \quad w(z) := z^{-\frac{1}{2}} \int_0^z g(t) t^{-\frac{1}{2}} dt$$

is also in the class \mathcal{B}_J . Thus, in terms of derivatives, we have

$$(2.4) \quad \left| \frac{1}{2}w(z) + zw'(z) \right| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}).$$

Furthermore, by setting

$$(2.5) \quad h(u, v) := \frac{1}{2}u + v,$$

we can rewrite (2.4) in the form:

$$(2.6) \quad |h(w(z), zw'(z))| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}).$$

In this section, we show that implication (2.6) holds true for functions $h(u, v)$ in the class \mathcal{H}_J given by Definition 1 below (see also [4]).

Definition 1. Let \mathcal{H}_J be the class of complex functions $h(u, v)$ satisfying each of the following conditions:

- (i) $h(u, v)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$;
- (ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| < J$ ($J > 0$);
- (iii) $|h(Je^{i\theta}, Ke^{i\theta})| \leq J$ whenever $(Je^{i\theta}, Ke^{i\theta}) \in \mathbb{D}$ ($\theta \in \mathbb{R}$; $K \geq J > 0$).

Example 1. It is easily seen that the function

$$(2.7) \quad h(u, v) = \gamma u + v \quad (\operatorname{Re}(\gamma) \geq 0; \mathbb{D} = \mathbb{C} \times \mathbb{C})$$

is in the class \mathcal{H}_J .

Definition 2. Let $h \in \mathcal{H}_J$ with the corresponding domain \mathbb{D} . We denote by $\mathcal{B}_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in \mathbb{U} and satisfy each of the following conditions:

- (i) $(w(z), zw'(z)) \in \mathbb{D}$;

(ii) $|h(w(z), zw(z))| < J \quad (z \in \mathbb{U}; J > 0)$. vspace1mm

The function class $\mathcal{B}_J(h)$ is not empty. Indeed, for any given function $h \in \mathcal{H}_J$, we have

$$(2.8) \quad w(z) = c_1 z \in \mathcal{B}_J(h),$$

for sufficiently small $|c_1|$ depending on h .

Theorem D. (See [10]) For any $h \in \mathcal{H}_J$,

$$\mathcal{B}_J(h) \subset \mathcal{B}_J \quad (h \in \mathcal{H}_J; J > 0)$$

Remark 2. Theorem D show that, if $h \in \mathcal{H}_J$ (with the corresponding domain \mathbb{D}) and if $w(z)$, given by (2.1), is analytic in \mathbb{U} and

$$(w(z), zw'(z)) \in \mathbb{D},$$

them the implication (2.4) holds true.

Theorem D leads us immediately to the following result, which was also given by [10].

Theorem E. (See [10]) Let $h \in \mathcal{H}_J$ and let the function $b(z)$ be analytic in \mathbb{U} with

$$|b(z)| < J \quad (z \in \mathbb{U}; J > 0).$$

If the following initial-value problem:

$$(2.9) \quad h(w(z), zw'(z)) = b(z) \quad (w(0) = 0)$$

has a solution $w(z)$ analytic in \mathbb{U} , then

$$(2.10) \quad |w(z)| < J \quad (z \in \mathbb{U}; J > 0)$$

Recently, using Theorem E, we prove the following Theorem F and Theorem G.

Theorem F. (See [10]) Let $a(z)$ and $b(z)$ be analytic in \mathbb{U} with

$$(2.11) \quad \left| z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right) \right| < \frac{1}{2}$$

and

$$(2.12) \quad |a(z)| \leq 1.$$

Let $w(z)$ denote the solution of the initial-value problem:

$$(2.13) \quad w''(z) + a(z)w'(z) + b(z)w(z) = 0 \quad (w(0) = 0; w'(0) = 1)$$

in \mathbb{U} . Then $w(z)$ is starlike in \mathbb{U} .

Theorem G. (See [7]) Let the functions $a(z)$ and $b(z)$ be analytic in \mathbb{U} with

$$(2.14) \quad \left| z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right) \right| < J \quad (z \in \mathbb{U}; J > 0)$$

and

$$(2.15) \quad \operatorname{Re}\{za(z)\} > -2J \quad (z \in \mathbb{U}; J > 0).$$

Also, let $w(z)$ denote the solution of the initial-value problem (2.13) in \mathbb{U} . Then

$$(2.16) \quad 1 - J - \frac{1}{2}\operatorname{Re}\{za(z)\} < \operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} < 1 + J - \frac{1}{2}\operatorname{Re}\{za(z)\} \quad (z \in \mathbb{U}; J > 0).$$

Example 2. Let $a(z) = 1$, $b(z) = 0$ in Theorem F, then the solution of

$$(2.17) \quad w''(z) + w'(z) = 0$$

is $w(z) = 1 - e^{-z} \in S^*$.

Example 3. Let $a(z) = -2Jz$ and $b(z) = J^2z^2$ in Theorem G ($J > 0$), then the solution of the following initial-value problem:

$$(2.18) \quad w''(z) - 2Jzw'(z) + J^2z^2w(z) = 0 \quad (w(0) = 0; w'(0) = 1)$$

is given by

$$(2.19) \quad w(z) = \frac{1}{\sqrt{J}} \exp\left(\frac{1}{2}Jz^2\right) \sin(\sqrt{J}z).$$

In this case, if the further assume that $0 < J \leq \frac{1}{2}$, then

$$w(z) \in S^*(1 - 2J) \quad \left(0 < J \leq \frac{1}{2}\right),$$

so that, in particular, we have

$$J = \frac{1}{2} : w(z) = \sqrt{2} \exp\left(\frac{z^2}{4}\right) \sin\left(\frac{z}{\sqrt{2}}\right) \in S^*,$$

$$J = \frac{1}{3} : w(z) = \sqrt{3} \exp\left(\frac{z^2}{6}\right) \sin\left(\frac{z}{\sqrt{3}}\right) \in S^*\left(\frac{1}{3}\right),$$

$$J = \frac{1}{4} : w(z) = 2 \exp\left(\frac{z^2}{8}\right) \sin\left(\frac{z}{2}\right) \in S^*\left(\frac{1}{2}\right),$$

and so on.

3 Main results and their consequences

Theorem 1. Let $P_n(z)$ be non-constant polynomial of degree $n \geq 1$ with $|P_n(z)| < J$ ($z \in \mathbb{U}; J > 0$). Let $w(z)$ be the solution of the initial-value problem:

$$(3.1) \quad w''(z) + P_n(z)w(z) = 0 \quad (w(0) = 0; w'(0) = 1)$$

in \mathbb{U} . Then we have

$$(3.2) \quad 1 - J < \operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} < 1 + J \quad (z \in \mathbb{U}).$$

Proof. If we put

$$(3.3) \quad u(z) = \frac{zw'(z)}{w(z)} - 1 \quad (z \in \mathbb{U}),$$

then $u(z)$ is analytic in \mathbb{U} , $u(0) = 0$ and (3.1) becomes

$$(3.4) \quad [u(z)]^2 + u(z) + zu'(z) = z^2 P_n(z),$$

or equivalently

$$(3.5) \quad h(u(z), zu'(z)) = z^2 P_n(z),$$

where $h(r, s) = r^2 + r + s$. It is easy to check $h(r, s) \in \mathcal{H}_J$, i.e.

- (i) $h(r, s)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$,
- (ii) $(0, 0) \in \mathbb{D}$, $|h(0, 0)| = 0 < J$,
- (iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J \quad (K \geq J)$.

From assumption, we have

$$|z^2 P_n(z)| < J \quad (z \in \mathbb{U}; J > 0).$$

By using Lemma 1, we have

$$|u(z)| < J \quad (z \in \mathbb{U}; J > 0),$$

which, in view of the relationship (3.3), yields

$$\left| \frac{zw'(z)}{w(z)} - 1 \right| < J \quad (z \in \mathbb{U}; J > 0),$$

that is,

$$(3.6) \quad 1 - J < \operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} < 1 + J \quad (z \in \mathbb{U}; J > 0).$$

□

Putting $J = 1$ in Theorem 1, we have the following Corollary.

Corollary 1. Let $P_n(z)$ be a non-constant polynomial of degree $n \geq 1$ with $|P_n(z)| < 1$ ($z \in \mathbb{U}$). Let $w(z)$ be the solution of the initial-value problem (3.1) in \mathbb{U} . Then $w(z)$ is starlike in \mathbb{U} .

Remark 3. It is well know that every solution $w(z)$ of the initial-value problem (3.1) is an entire function.

Example 4. Let $P_2(z) = \frac{3}{4} - \frac{z^2}{16}$ in Corollary 1. ($n=2$), the solution of the following initial-value problem:

$$(3.7) \quad w''(z) + \left(\frac{3}{4} - \frac{z^2}{16}\right)w(z) = 0 \quad (w(0) = 0; w'(z) = 1)$$

is given by

$$w(z) = z \exp\left(-\frac{z^2}{8}\right) \in S^*.$$

$$w''(z) + \left(\frac{6}{7} - \frac{4}{49}z^2\right)w(z) = 0 \implies w(z) = z \exp\left(-\frac{z^2}{7}\right) \in S^*.$$

$$w''(z) + \left(\frac{9}{10} - \frac{9}{100}z^2\right)w(z) = 0 \implies w(z) = z \exp\left(-\frac{3}{20}z^2\right) \in S^*.$$

Remark 4. Let $P_n(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ ($z \in \mathbb{U}$).

If $\sum_{k=0}^n |a_k| \leq 1$, then $|P_n(z)| < 1$ ($z \in \mathbb{U}$).

Theorem 2. Let $P_n(z)$ be a non-constant polynomial of degree $n \geq 1$ with $|P_n(z)| < J$ ($z \in \mathbb{U}; 0 < J \leq 1$). Let $w(z)$ ($z \in \mathbb{U}$) be the solution of the initial-value problem (3.1) in \mathbb{U} . Then $w(z)$ is strongly starlike of order α , that is,

$$(3.8) \quad \left| \arg \left\{ \frac{zw'(z)}{w(z)} \right\} \right| < \frac{\pi}{2} \alpha$$

for some α ($0 < \alpha \leq 1$) and

$$(3.9) \quad \alpha = \frac{2}{\pi} \sin^{-1} J \quad (0 < J \leq 1)$$

Proof. If we put

$$(3.10) \quad u(z) = \frac{zw'(z)}{w(z)} - 1 \quad (z \in \mathbb{U}),$$

then $u(z)$ is analytic in \mathbb{U} , $u(0) = 0$ and (3.1) becomes

$$(3.11) \quad [u(z)]^2 + u(z) + zu'(z) = z^2 P_n(z),$$

or equivalently

$$(3.12) \quad h(u(z), zu'(z)) = z^2 P_n(z),$$

where $h(r, s) = r^2 + r + s$. It is easy to check $h(r, s) \in \mathcal{H}_J$, that is,

- (i) $h(r, s)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$;
- (ii) $(0, 0) \in \mathbb{D}$, $|h(0, 0)| = 0 < J$;
- (iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ ($1 \geq K \geq J > 0$).

From assumption, we have

$$|z^2 P_n(z)| < J \quad (z \in \mathbb{U}; 0 < J \leq 1).$$

By using Theorem E, we obtain

$$|u(z)| < J \quad (z \in \mathbb{U}; 0 < J \leq 1).$$

Therefore, we have

$$\left| \arg \left\{ \frac{zw'(z)}{w(z)} \right\} \right| < \frac{\pi}{2} \alpha$$

for some α ($0 < \alpha \leq 1$) and $\alpha = \frac{2}{\pi} \sin^{-1} J$ ($0 < J \leq 1$).

□

Remark 5. Putting $\alpha = 1$ in Theorem 2, we have Corollary 1.

For $a(z) = -z$ and $b(z) = \lambda$ ($\lambda \in \mathbb{C}$) in Theorem D, the initial-value problem (2.13) becomes

$$(3.13) \quad w''(z) - zw'(z) + \lambda w(z) = 0 \quad (w(0) = 0; w'(0) = 1),$$

which, under the following transformation:

$$(3.14) \quad w(z) = \exp\left(\frac{z^2}{4}\right) v(z),$$

assumes the normal form as given below

$$(3.15) \quad v''(z) + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) v(z) = 0 \quad (v(0) = 0; v'(0) = 1).$$

These differential equations (3.13) and (3.15) are well-known, so called respectively Hermite's differential equation and Weber's differential equation.

Next, we prove

Theorem 3. We consider Weber's differential equation (3.12). If

$$(3.16) \quad \left| \lambda + \frac{1}{2} - \frac{z^2}{4} \right| < J \quad (z \in \mathbb{U}; 0 < J \leq 1),$$

then $v(z)$ is strongly starlike of order α , that is,

$$(3.17) \quad \left| \arg \left\{ \frac{zv'(z)}{v(z)} \right\} \right| < \frac{\pi}{2} \alpha$$

for some α ($0 < \alpha \leq 1$), and α is satisfies (3.9).

Proof of Theorem 3 is similar to the proof of Theorem 2. Taking $\alpha = 1$ in Theorem 3, we have

Corollary 2. (See [10]) We consider Weber's differential equation (3.15). If

$$(3.18) \quad \left| \lambda + \frac{1}{2} - \frac{z^2}{4} \right| < 1 \quad (z \in \mathbb{U}),$$

then the solution $v(z)$ is starlike in \mathbb{U} .

We need the following lemma to prove next result.

Lemma 1. (See [4]) Let $h(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C}$ such that

(i) $h(r, s, t)$ is conti. in a domain $\mathbb{D} \subset \mathbb{C}^3$;

(ii) $(0, 0, 0) \in \mathbb{D}$ and $|h(0, 0, 0)| < J$ ($J > 0$);

(iii) $|h(Je^{i\theta}, Ke^{i\theta}, L)| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}, L) \in \mathbb{D}$, $K \geq J$ and $\text{Re}[Le^{-i\theta}] \geq 0$.

Let $w(z) = w_1z + w_2z^2 + \dots$ be analytic in \mathbb{U} . If $(w(z), zw'(z), z^2w''(z)) \in \mathbb{D}$ ($z \in \mathbb{U}$) and

$$(3.19) \quad |h(w(z), zw'(z), z^2w''(z))| < J \quad (z \in \mathbb{U}),$$

then $|w(z)| < J$ ($z \in \mathbb{U}$).

Applying Lemma 1, we prove the following theorem.

Theorem 4. We consider the Weber's differential equation (3.15). Let $|\lambda + \frac{1}{2} - \frac{z^2}{4}| < J$ ($z \in \mathbb{U}$; $0 < J \leq 1$), then we have

$$(3.20) \quad \left| \arg \left\{ \frac{v(z)}{z} \right\} \right| < \frac{\pi}{2} \alpha$$

for some α ($0 < \alpha \leq 1$), and satisfies (3.9).

Proof. We put

$$(3.21) \quad u(z) = \frac{v(z)}{z} - 1 \quad (z \in \mathbb{U})$$

Then $u(z)$ is analytic in \mathbb{U} , $u(0) = 0$ and

$$(3.22) \quad \frac{2zu'(z)}{1+u(z)} + \frac{z^2u''(z)}{1+u(z)} = -z^2 \left(\lambda + \frac{1}{2} - \frac{z^2}{4} \right)$$

or equivalently

$$(3.23) \quad h(u(z), zu'(z), z^2u''(z)) = -z^2 \left(\lambda + \frac{1}{2} - \frac{z^2}{4} \right),$$

where $h(r, s, t) = \frac{2s}{1+r} + \frac{t}{1+r}$.

It is easy to check the following conditions, that is, that

(i) $h(r, s, t)$ is continuous in $\mathbb{D} = \mathbb{C} \setminus \{-1\} \times \mathbb{C} \times \mathbb{C}$;

(ii) $(0, 0, 0) \in \mathbb{D}$ and $|h(0, 0, 0)| = 0 < J$ ($0 < J \leq 1$);

(iii) $|h(Je^{i\theta}, Ke^{i\theta}, L)| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}, L) \in \mathbb{D}$, $1 \geq K \geq J$ and $\text{Re}[Le^{-i\theta}] \geq 0$.

From assumption of Theorem, we have

$$\left| -z^2 \left(\lambda + \frac{1}{2} - \frac{z^2}{4} \right) \right| < J \quad (z \in \mathbb{U}; 0 < J \leq 1).$$

By using Lemma 1, we obtain

$$(3.24) \quad |u(z)| < J \quad (z \in \mathbb{U}; 0 < J \leq 1).$$

Therefore, we have

$$\left| \arg \left\{ \frac{v(z)}{z} \right\} \right| < \frac{\pi}{2} \alpha$$

for some α ($0 < \alpha \leq 1$) and α is satisfies (3.9).

□

Putting $\alpha = 1$ in Theorem 4, we obtain

Corollary 3. We consider the Weber's differential equation (3.15). Let $|\lambda + \frac{1}{2} - \frac{z^2}{4}| < 1$ ($z \in \mathbb{U}$), then we have

$$(3.25) \quad \operatorname{Re} \left\{ \frac{v(z)}{z} \right\} > 0.$$

Now, we recall next lemma by Yamaguchi.

Lemma 2. ([12]) Let $f(z) = z + a_2 z^2 + \dots$ be analytic in \mathbb{U} . If $\operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > 0$ ($z \in \mathbb{U}$), then we have

$$(3.26) \quad \operatorname{Re} \{f'(z)\} > 0$$

for $|z| < \sqrt{2} - 1$.

Applying Lemma 2, we have the following Corollary.

Corollary 4. We consider the Weber's differential equation (3.15). Let $|\lambda + \frac{1}{2} - \frac{z^2}{4}| < 1$ ($z \in \mathbb{U}$), then $v(z)$ is close-to-convex in $|z| < \sqrt{2} - 1$.

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