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<thead>
<tr>
<th>項目</th>
<th>内容</th>
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<tbody>
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<td>資源タイプ</td>
<td>発行者</td>
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A counterpart of strong normality

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Abstract

For non-inaccessible \( \kappa \) we try to define an ideal with the property between normality and strong normality, which is expected to be a natural one.

1 Introduction

Throughout \( \kappa \) is regular uncountable and \( \lambda \) a cardinal \( > \kappa \). Let \( \mathcal{P}_\kappa\lambda \) denote the set of the subsets of \( \lambda \) with the cardinality less than \( \kappa \), that is, \( \mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa \} \). All the proofs are easily given by the reader.

**Definition 1.1.** Let \( X \subseteq \mathcal{P}_\kappa\lambda \).

We say \( X \) is *unbounded* if for every \( x \in \mathcal{P}_\kappa\lambda \) there exists \( y \in X \) such that \( x \subseteq y \).

\( X \) is said to be *closed* if it is closed under \( \subseteq \)-increasing sequence of length \( < \kappa \).

\( X \) is a *club* if it is closed and unbounded.

\( X \) is *stationary* if \( X \cap C \neq \emptyset \) for any club \( C \).

Let \( I_{\kappa,\lambda} = \{X \subseteq \mathcal{P}_\kappa\lambda : X \text{ is not unbounded}\} \) and \( \mathrm{NS}_{\kappa,\lambda} = \{X \subseteq \mathcal{P}_\kappa\lambda : X \text{ is not stationary}\} \).

Usually a large cardinal property is characterized by a normal ideal whose members are the sets without the property (or its dual filter):

- supercompactness \( \leftrightarrow \) normal measure
- partition property \( \leftrightarrow \) \( \mathrm{NP}_{\kappa,\lambda} \)
- ineffability \( \leftrightarrow \) \( \mathrm{NI}_{\kappa,\lambda} \)
- Shelah property \( \leftrightarrow \) \( \mathrm{NSh}_{\kappa,\lambda} \)
- subtlety \( \leftrightarrow \) nonsubtle ideal
Definition 1.2. We say $I$ is an ideal if the following hold:

1. $I \subseteq \mathcal{P}(\mathcal{P}_\kappa \lambda)$,
2. $\emptyset \in I$ and $\mathcal{P}_\kappa \lambda \notin I$,
3. if $X \subseteq Y \in I$, then $X \in I$,
4. $I$ is closed under the union of less than $\kappa$ many members (we say $I$ is $\kappa$ complete),
5. $I_{\kappa,\lambda} \subseteq I$ (we say $I$ is fine).

Let $I^+ = \mathcal{P}(\mathcal{P}_\kappa \lambda) \setminus I$ and $I^* = \{X \subseteq \mathcal{P}_\kappa \lambda : \mathcal{P}_\kappa \lambda \setminus X \in I \}$.

A function $f : \mathcal{P}_\kappa \lambda \rightarrow \lambda$ is regressive if $f(x) \in x$ for any $x \in \mathcal{P}_\kappa \lambda$.

An ideal $I$ on $\mathcal{P}_\kappa \lambda$ is normal if for any $X \in I^+$ and a regressive function $f$ on $X$ there exists $Y \in \mathcal{P}(X) \cap I^+$ such that $f \upharpoonright Y$ is constant.

Note that $I_{\kappa,\lambda}$ is the minimal, and $NS_{\kappa,\lambda}$ is the minimal normal ideal on $\mathcal{P}_\kappa \lambda$.

Forementioned ideals have a stronger property:

Definition 1.3. For $x, y \in \mathcal{P}_\kappa \lambda$, $y \prec x$ denotes $y \in \mathcal{P}_{x \cap \kappa} = \{s \subseteq x : |s| < |x \cap \kappa|\}$.

We say a function $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$ is set-regressive if $f(x) \prec x$ for any $x \in \mathcal{P}_\kappa \lambda$.

An ideal $I$ on $\mathcal{P}_\kappa \lambda$ is strongly normal if for any $X \in I^+$ and set-regressive function $f$ on $X$ there exists $Y \in \mathcal{P}(X) \cap I^+$ such that $f \upharpoonright Y$ is constant.

Let $WNS_{\kappa,\lambda}$ denote the minimal strongly normal ideal on $\mathcal{P}_\kappa \lambda$.

Fact 1.4. $\mathcal{P}_\kappa \lambda \notin WNS_{\kappa,\lambda}$ if and only if $\kappa$ is Mahlo or $\kappa = \nu^+$ with $\nu < \nu = \nu$ [6].

The following figure is known:

nonsubtite ideal

$I_{\kappa,\lambda} \subseteq NS_{\kappa,\lambda} \subseteq WNS_{\kappa,\lambda}$

$NSh_{\kappa,\lambda}$

2 Motivation

As is shown strong normality gives some limitation to $\kappa$. It seems natural to ask:
Can we define a natural strengthening of normality without assuming inaccessibility?

We consider several aspects of this question.

(1) Reflection.

Usual type of reflection is as follows:

if $\kappa$ has property $P$, we can find $\alpha < \kappa$ which has property $P$.

The stationary reflection of $P_{\omega_1}\lambda$ is:

if $S \subset P_{\omega_1}\lambda$ is stationary, then we can find $A$ of cardinality $\omega_1$ such that $\omega_1 \subset A \subset \lambda$ and $S \cap P_{\omega_1} A$ is stationary in $P_{\omega_1} A$.

The stationary reflection of $P_{\kappa}\lambda$ is false for $\kappa > \omega_1$ [11]. While the following holds[5][9] :

if $\kappa$ is $\lambda$ Shelah, then for any stationary $S \subset P_{\kappa}\lambda$ we can find $x \in P_{\kappa}\lambda$ such that $S \cap P_{x\cap \kappa} x$ is stationary in $P_{x\cap \kappa} x$.

(2) Diamond and subtlety.

It is known that $\diamondsuit_{\kappa}$ holds is $\kappa$ is subtle. Eliminating inaccessibility, this assumption can be weaken to "$\kappa$ is ethereal with $2^{<\kappa} = \kappa$.

While we have:

if $\kappa$ is subtle, then there exists a sequence $\langle S_x | x \in P_{\kappa}\lambda \rangle$ such that

1. $S_x \subset P_{x\cap\kappa} x$,
2. for any $S \subset P_{\kappa}\lambda \{x : S_x = S \cap P_{x\cap \kappa} x\} \in WNS_{\kappa,\lambda}^{+}$.

We denote the above sequence $\tilde{\diamondsuit}_{\kappa,\lambda}$.

We review some definitions.

**Definition 2.1.** For $X \subset \kappa$ let $[X]^2$ denote the set $\{(\alpha, \beta) \in X \times X : \alpha < \beta\}$. We say $X$ is *subtle* if for any sequence $\langle S_\alpha \subset \alpha | \alpha \in X \rangle$ and club $C \subset \kappa$ there exists $(\beta, \gamma) \in [C \cap X]^2$ such that $S_\beta = S_\gamma \cap \beta$.

For $Y \subset P_{\kappa}\lambda$ let $[Y]^2$ denote the set $\{(x, y) \in Y \times Y : x \in P_{y\cap \kappa} y\}$. We say $Y \subset P_{\kappa}\lambda$ is *strongly subtle* if for any sequence $\langle S_z \subset P_{x\cap \kappa} x | z \in Y \rangle$ and $C \in WNS_{\kappa,\lambda}^{+}$ there exists $(x, y) \in [C \cap Y]^2$ such that $S_x = S_y \cap P_{x\cap \kappa} x$. 

Note that $\kappa$ is subtle if and only if $\mathcal{P}_\kappa\lambda$ is strongly subtle [3]. Compare the above with the following:

**Definition 2.2.** $X \subseteq \kappa$ is *ethereal* if for any sequence $(S_\alpha \subseteq \alpha | \alpha \in X)$ with $|S_\alpha| = |\alpha|$ and club $C \subseteq \kappa$ there exists $(\beta, \gamma) \in [(C \cap X)^2]$ such that $|S_\beta \cap S_\gamma| = |\beta|$.

We say $Y \subseteq \mathcal{P}_\kappa\lambda$ is *weakly subtle* if for any sequence $(S_z \subseteq \mathcal{P}_{\pi^{\kappa}z} | z \in Y)$ with $S_x \in I_{\pi^{\kappa}x, \kappa}^+$ and club $C \subseteq \mathcal{P}_\kappa\lambda$ there exists $(x, y) \in [(C \cap Y)_{<}]^2$ such that $S_x \cap S_y \in I_{\pi^{\kappa}x, \kappa}^+$.

**Fact 2.3.** (1) If $\mathcal{P}_\kappa\lambda$ is weakly subtle, then the corresponding ideal is normal, $\{x : x \cap \kappa \text{ is regular}\}$ is in its dual filter, hence $\kappa$ is weakly Mahlo.

(2) If $f : \mathcal{P}_\kappa\lambda \rightarrow \lambda$ is a bijection and $A = \{x \in \mathcal{P}_\kappa\lambda : f^{-1}\mathcal{P}_{\pi\kappa}x = x\}$, then strongly subtle ideal $= \text{weakly subtle ideal} \upharpoonright A$.

Note that $WNS_{\kappa, \lambda} = NS_{\kappa, \lambda} \upharpoonright A$ in (2). We have several questions:

**Question 2.4.** 1) Is it consistent that there is a non-inaccessible weakly subtle cardinal?

2) Does $\check{\hat{\kappa}, \lambda}$ hold if $\kappa$ is weakly subtle and $2^{<\kappa} = \kappa$?

3) Is $\mathcal{P}_\kappa\lambda$ weakly subtle if $\kappa$ is ethereal?

4) Is the definition of weak subtlety “a right one”?

(3) Weak normalities.

We have some $\mathcal{P}_\kappa\lambda$ generalizations of weakly normal ideals on $\kappa$ defined by Kanamori [8].

**Definition 2.5.** An ideal $I$ on $\kappa$ is said to be *weakly normal* if for any $f : \kappa \rightarrow \kappa$ such that $f(\alpha) < \alpha$ for every $\alpha < \kappa$ there exists $\gamma < \kappa$ with $\{\alpha : f(\alpha) \leq \gamma\} \in I^*$.

We say $I$ on $\mathcal{P}_\kappa\lambda$ is *Kanamori* if for any regressive $f : \mathcal{P}_\kappa\lambda \rightarrow \lambda$ there exists $\gamma < \lambda$ with $\{x : f(x) \leq \gamma\} \in I^*$.

D. Burke[4] and Abe[1] proved:
Fact 2.6. The singular cardinal hypothesis (SCH) holds for $\lambda^\kappa$ if $P_\kappa\lambda$ carries a Kanamori ideal and one of the following holds:

1. $\lambda$ is regular or $\text{cf}(\lambda) \leq \kappa$
2. $\kappa^+ \leq \text{cf}(\lambda) < \lambda$ and there is a measurable cardinal above $\lambda$.

Kanamori ideal may be seen as a weakening of strong compactness and has too strong consequences.

Definition 2.7. We say $I$ is an $AN$-ideal if for any set-regressive function $f$ on $P_\kappa\lambda$ there exists $a \in P_\kappa\lambda$ such that $\{x : f(x) \subset a\} \in I^*$. (For AN-ideals $\kappa$ completeness is not assumed.)

Fact 2.8. Suppose that $I$ is a $\kappa$ complete $AN$-ideal. Then, $I$ is strongly normal, $\kappa$ saturated, and $\{x : S \cap P_{\text{cf}\kappa\lambda} = NS_{\text{cf}\kappa\lambda}^{+}\} \in I^*$ whenever $S \subset P_\kappa\lambda$ is stationary [2].

So $AN$-ideal may be seen as a weakening of supercompactness and is too strong as well.

While Mignon [10] defined a direct weakening of normality:

Definition 2.9. An ideal $I$ on $P_\kappa\lambda$ is weakly normal if for any $X \in I^+$ and regressive $f : X \to \lambda$ there exists $\gamma < \lambda$ with $\{x \in X : f(x) \leq \gamma\} \in I^+$.

3 Definition

We just modify Mignon’s version of weak normality to define a weakening of strong normality.

Definition 3.1. Let $(*)$ denote the following statement:

\[ (*) \text{ for any } X \in I^+ \text{ and set-regressive } f : X \to P_\kappa\lambda \text{ there exists } a \in P_\kappa\lambda \text{ such that } \{x \in X : f(x) \subset a\} \in I^+. \]

Fact 3.2. (1) If $\kappa$ is inaccessible, then $(*)$ is equivalent to strong normality.

(2) If $P_\kappa\lambda$ carries an ideal with $(*)$, then $\kappa$ is weakly inaccessible.

(3) Every normal $\kappa$ saturated ideal on $P_\kappa\lambda$ has the property $(*)$. 
(4) (*) is equivalent to that $I$ is closed under some type of diagonal unions, that is,

$$I = \bigvee \{ X_s : s \in \mathcal{P}_\kappa \lambda \} : X_s \in I, \text{ } X_s \subseteq X_t \text{ whenever } s \subseteq t$$

where $x \in \bigvee \{ X_s : s \in \mathcal{P}_\kappa \lambda \}$ if and only if $x \in X_s$ for some $s < x$.

(5) Suppose that $I$ satisfies (*) in the grand model $V$, $\mathbb{P}$ is a $\delta$-c.c. forcing with $\delta < \kappa$, $G \mod \mathbb{P}$ generic, and $J$ defined in $V[G]$ as $J = \{ X \subseteq \mathcal{P}_\kappa \lambda : X \cap V \subseteq Y \text{ for some } Y \in I \}$. Then the following hold:

(a) $J$ satisfies (*),
(b) $I = \{ X : \forall \mathbb{P} \exists X \in J \}$

(6) Suppose that $\mathbb{P}$ is $\kappa$-c.c., $J$ defined as above satisfies (*) in $V[G]$, and $\mathcal{P}_\kappa \lambda \cap V \notin J$. Then, $I$ satisfies (*).

Remark. The condition underlined in (4) is equivalent to the following:

$$\bigcup \{ X_s : s \subseteq x \} \in J \text{ for every } x \in \mathcal{P}_\kappa \lambda.$$

Concerning the consistency of the existence of a non-strongly normal ideal with (*) we have the following:

Fact 3.3. Let $\kappa$ be Mahlo, $\mathbb{P}$ adding $\kappa$ many Cohen real forcing, and $V[G] \models " J = \{ X \subseteq \mathcal{P}_\kappa \lambda : X \cap V \subseteq Y \text{ for some } Y \in WNS_{\kappa, \lambda}^V \}"$. Then $J$ is the minimal ideal with (*) such that $\mathcal{P}_\kappa \lambda \cap V \in J^*$.

4 Combinatorial characterization of the minimal ideal with (*)

$NS_{\kappa, \lambda}$ and $WNS_{\kappa, \lambda}$ are characterized as follows:

Fact 4.1. Let $X \subseteq \mathcal{P}_\kappa \lambda$.

(1) $X \in NS_{\kappa, \lambda}$ if and only if there exists $f : \lambda^2 \rightarrow \mathcal{P}_\kappa \lambda$ such that $C_f \cap X = \emptyset$, where $C_f = \{ x : f \subseteq x \subseteq \mathcal{P}(x) \}$.

(2) $X \in WNS_{\kappa, \lambda}$ if and only if there exists $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$ such that $C_f \cap X = \emptyset$, where $C_f = \{ x : f \subseteq \mathcal{P}_{x \cap \kappa} x \subseteq \mathcal{P}(x) \}$.

If $\kappa$ is inaccessible or $\kappa = \nu^+$ with $\nu^{< \nu} = \nu$, then $\bigcup f \mathcal{P}_{x \cap \kappa} x \subseteq \mathcal{P}_\kappa \lambda$ for every $f \in \mathcal{P}_\kappa \lambda \mathcal{P}_\kappa \lambda$ and $x \in \mathcal{P}_\kappa \lambda$. 
Definition 4.2. Let $\mathcal{F} = \{f \in P_{\kappa}\lambda : \cup f^w P_{\kappa\cap} x \in P_{\kappa}\lambda \text{ for every } x \in P_{\kappa}\lambda\}$, and $\tilde{C}_f = \{x : f^w P_{\kappa\cap} x \subset P(x)\}$ for $f \in \mathcal{F}$.
Set $I_0 = \{X \subset P_{\kappa}\lambda : \tilde{C}_f \cap X = \emptyset \text{ for some } f \in \mathcal{F}\}$.

Fact 4.3. Let $\kappa$ be weakly Mahlo. Then,
(1) For any $f \in \mathcal{F}$, $\tilde{C}_f \in I_{\kappa,\lambda}^+$. 
(2) $I_0$ satisfies $(*)$.

Recall that $WNS_{\kappa,\lambda}$ has another characterization:

Fact 4.4. For any $X \subset P_{\kappa}\lambda$, $X \in WNS_{\kappa,\lambda}$ if and only if there exists a set-regressive $f : X \to P_{\kappa}\lambda$ such that $f^{-1}\{a\} \in I_{\kappa,\lambda}$ for any $a \in P_{\kappa}\lambda$.

We now define another ideal.

Definition 4.5. Define $J_0$ by:

$X \in J_0$ if $X \subset P_{\kappa}\lambda$ and there exists a set regressive $f : X \to P_{\kappa}\lambda$ such that for any $a \in P_{\kappa}\lambda \{x \in X : f(x) \subset a\} \in I_{\kappa,\lambda}$.

We easily have:

Fact 4.6. $NS_{\kappa,\lambda} \subset J_0 = \tilde{\nabla}_{\prec} I_{\kappa,\lambda}$.

We know $\nabla \nabla I = \nabla I$ and $\nabla \nabla \nabla I = \nabla I$ for every ideal $I$. (If $NS_{\kappa,\lambda} \subset I$, then $\nabla \nabla I = \nabla I$.) The author does not know how about for the operation $\tilde{\nabla}_{\prec}$.

Question 4.7. (1) Is $J_0$ normal? 
(2) $\tilde{\nabla}_{\prec} I = \tilde{\nabla}_{\prec} \tilde{\nabla}_{\prec} I$ for every ideal $I$?

Fact 4.6 suggests a different ideal.

Definition 4.8. Define $J_1$ by:

$X \in J_1$ if $X \subset P_{\kappa}\lambda$ and there exists a set regressive $f : X \to P_{\kappa}\lambda$ such that for any $a \in P_{\kappa}\lambda \{x \in X : f(x) \subset a\} \in NS_{\kappa,\lambda}$.

Clearly $J_1$ is normal.

Question 4.9. $J_1 = I_0$?
参考文献


