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Kyoto University
Distributivity numbers of $\mathcal{P}(\omega)/\text{fin}$ and its friends

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Abstract

This brief survey on distributivity numbers is an exposition of the talk which I gave at RIMS in October 2005.

1 Distributivity numbers of Boolean algebras

Let $\mathbb{P}$ be a separative partial order. $D \subseteq \mathbb{P}$ is dense if for all $p \in \mathbb{P}$ there is $q \leq p$ with $q \in D$. $D$ is open if for all $p \in D$, any $q \leq p$ belongs to $D$. The distributivity number (or height) of $\mathbb{P}$, $h(\mathbb{P})$, is the least size of a family $D$ of open dense subsets of $\mathbb{P}$ such that $\bigcap D$ is not dense. Note that $\bigcap D$ necessarily is open. Equivalently, $h(\mathbb{P})$ is the least size of a family $\mathcal{A}$ of maximal antichains of $\mathbb{P}$ which has no common refinement. Here, for maximal antichains $A, B \subseteq \mathbb{P}$, we say that $A$ refines $B$ if for all $p \in A$ there is $q \in B$ with $p \leq q$. If $\mathcal{A}$ is an atomless Boolean algebra, we let $A^+ = A \setminus \{0\}$ denote the partial order of its positive elements and define $h(A) := h(A^+)$. Similarly for other cardinals.

Fact 1. $h(\mathbb{P})$ is a regular cardinal. \Box

$h(\mathbb{P})$ is an invariant of $\mathbb{P}$ as a forcing notion, that is, it does not depend on the particular realization of $\mathbb{P}$. Equivalently, $h(\mathbb{P}) = h(\text{r.o.}(\mathbb{P}))$ where $\text{r.o.}(\mathbb{P})$ is the completion of $\mathbb{P}$, i.e., the unique complete Boolean algebra forcing equivalent with $\mathbb{P}$. For a topological space $X$, $\text{r.o.}(X)$ is the Boolean algebra of regular open subsets of $X$ where $O \subseteq X$ is called regular open if it is open and $\text{Int} (\text{Cl}(O)) = O$. It is well-known that $\text{r.o.}(X)$ is a complete Boolean algebra. If $X = \mathbb{P}$ with the topology introduced above, the mapping $p \mapsto O_p = \{q \in \mathbb{P} : q \leq p\}$ is a dense embedding of $\mathbb{P}$ into $\text{r.o.}(\mathbb{P})$ and thus $\mathbb{P}$ and $\text{r.o.}(\mathbb{P})$ are indeed forcing.

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equivalent. In case $A$ is an atomless Boolean algebra, there is an alternative description of $\text{r.o.}(A) := \text{r.o.}(A^+)$: namely, if $\text{St}(A)$ is the Stone space of $A$, then $\text{r.o.}(A) = \text{r.o.}(\text{St}(A))$.

From the forcing-theoretic point of view, $h(P)$ is the minimal cardinal $\kappa$ such that there are $p \in P$ and a $P$-name $\dot{f}$ for a function from $\kappa$ to the ground model $V$ such that $p \Vdash_P \dot{f} \notin V$. Indeed if $\lambda < h(P)$ and $\dot{f} : \lambda \to V$, then, letting $D_\alpha$, $\alpha < \lambda$, be the open dense subset of $P$ consisting of conditions which decide the value $\dot{f}(\alpha)$, $D = \bigcap_{\alpha < \lambda} D_\alpha$ is dense and for any $p \in D$ there is $f_p \in V$ such that $p \Vdash_P \dot{f} = f_p$. Thus $p \Vdash_P \dot{f} \notin V$. On the other hand, if $D_\alpha$, $\alpha < h(P)$, are open dense such that $D = \bigcap_{\alpha < h(P)} D_\alpha$ is not dense and $A_\alpha = \{p_{\alpha, \gamma} : \gamma < \kappa_\alpha\} \subseteq D_\alpha$ are maximal antichains, then, letting $\dot{f} : h(P) \to V$ be the $P$-name defined by $p_{\alpha, \gamma} \Vdash_P \dot{f}(\alpha) = \gamma$, we see that if $p \in P$ is such that no $q \leq p$ belongs to $D$ then $p \Vdash_P \dot{f} \notin V$.

If $P$ is homogeneous, that is, if $O_P$ is forcing equivalent with $P$ for all $p \in P$ (equivalently, if $\text{r.o.}(P) = \text{r.o.}(O_P)$ for all $p \in P$), then $h(P)$ is the least size of a family $D$ of open dense subsets of $P$ with $\bigcap D = \emptyset$. Equivalently, $h(P)$ is the least $\kappa$ such that $\forall p \in P \exists f : h(P) \leftrightarrow \kappa \to V$ for some $P$-name $\check{f}$.

We write $P \lhd Q$ if there is a complete embedding $e : P \to Q$.

**Fact 2.** If $P \lhd Q$ then $h(P) \geq h(Q)$. $\Box$

We briefly mention two cardinals which are closely related to $h(P)$. A tower $T \subseteq P$ is a well-ordered decreasing chain without a lower bound. The tower number $t(P)$ of $P$ is the least size of a tower in $P$. $q \in P$ splits $p \in P$ if $p$ and $q$ are compatible and there is $r \leq p$ incompatible with $q$. $S \subseteq P$ is a splitting family if every member of $P$ is split by a member of $S$. The splitting number $s(P)$ of $P$ is the least size of a splitting family. Unlike $h$, $t$ and $s$ are not invariant under forcing equivalence: e.g. $t(A) = \aleph_0$ for every complete atomless Boolean algebra $A$. Also the base-matrix theorem [BPS] (see also [BS, Theorem 3.4]) implies that $s(\text{r.o.}(P(\omega)/\text{fin})) = h(\text{r.o.}(P(\omega)/\text{fin}))$.

**Fact 3.** $t(P)$ is a regular cardinal. $\Box$

**Fact 4.** $t(P) \leq h(P) \leq s(P)$.

**Proof.** Let $D_\alpha$, $\alpha < h(P)$, be open dense such that there is $p \in P$ with $\bigcap_{\alpha < h(P)} D_\alpha \cap O_p = \emptyset$. Recursively construct $p_\alpha \in D_\alpha \cap O_p$ such that $p_\alpha \geq p_\beta$ for $\alpha \leq \beta$. If there is a limit $\lambda < h(P)$ such that $p_\lambda$ cannot be found, $\{p_\alpha : \alpha < \lambda\}$ is a tower and $t(P) \leq cf(\lambda)$. Otherwise $\{p_\alpha : \alpha < h(P)\}$ must be a tower.

Let $\{p_\alpha : \alpha < s(P)\}$ be a splitting family. For each $\alpha$ let $A_\alpha \subseteq P$ be a maximal antichain containing $p_\alpha$. Clearly the $A_\alpha$ have no common refinement. $\Box$

Thus $t(P)$ and $s(P)$ are useful because they give natural lower and upper bounds of $h(P)$, respectively.
2 Products and reduced powers

For separative partial orders $\mathbb{P}$ and $\mathbb{Q}$ consider the product $\mathbb{P} \times \mathbb{Q}$ equipped with the product ordering (that is, $(p', q') \leq_{\mathbb{P} \times \mathbb{Q}} (p, q)$ iff $p' \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{Q}} q$). Since $\mathbb{P} \leq \mathbb{P} \times \mathbb{Q}$ and $\mathbb{Q} \leq \mathbb{P} \times \mathbb{Q}$ we see

Fact 5. $h(\mathbb{P} \times \mathbb{Q}) \leq \min\{h(\mathbb{P}), h(\mathbb{Q})\}$. □

Notice that if $A$ and $B$ are Boolean algebras, then $A \times B$ denotes what is called the free product in Boolean algebra theory, namely, $(A^+ \times B^+) \cup \{0\}$.

For a Boolean algebra $A$ let $A^\omega/\text{fin} := \{[f] : f \in A^\omega\}$ where $[f] = \{g \in A^\omega : \forall^\infty n (f(n) = g(n))\}$, ordered by $[f] \leq [g]$ if $f(n) \leq g(n)$ holds for almost all $n$. The reduced power $A^\omega/\text{fin}$ is again a Boolean algebra.

Fact 6. If $A \leq B$ then $A^\omega/\text{fin} \leq B^\omega/\text{fin}$ (and thus $h(A^\omega/\text{fin}) \geq h(B^\omega/\text{fin})$). □

If $A$ is the trivial algebra $\{0, 1\}$, we see $A^\omega/\text{fin} \cong \mathcal{P}(\omega)/\text{fin}$ where $\mathcal{P}(\omega)/\text{fin} := \{[A] : A \subseteq \omega\}$ with $[A] = \{B \subseteq \omega : |A \Delta B| < \aleph_0\}$, ordered by $[A] \leq [B]$ if $|A \setminus B| < \aleph_0$. In particular $h(B^\omega/\text{fin}) \leq h$ for any Boolean algebra $B$ where $h := h(\mathcal{P}(\omega)/\text{fin})$.

Stone-Čech remainders. Much of the interest in Boolean algebras of the form $A^\omega/\text{fin}$ stems from the fact their completion is isomorphic to the regular open algebra $\text{r.o.}(X^*)$ of the Stone-Čech remainder $X^*$ of some natural space $X$. Briefly recall the construction of the Stone-Čech compactification $\beta X$ of a normal space $X$ [En, Section 3.6]. Let $\beta X$ be the family of all ultrafilters of closed subsets of $X$. Identify $x \in X$ with $U(x) = \{A \subseteq X : \text{closed, } x \in A\} \in \beta X$. Clearly $\bigcap U(x) = \{x\}$. In fact, the maximality of any $U \in \beta X$ entails that either $\bigcap U = \{x\}$ for some $x$ and then $U = U(x)$ or $\bigcap U = \emptyset$ and $U$ is a free ultrafilter. Thus $X^* = \beta X \setminus X$ is the space of free ultrafilters of closed sets. For $O \subseteq X$ open let $O^* = \{U \in \beta X : \exists A \in U \text{ with } A \subseteq O\}$. Clearly $O^* \cap X = O$. The sets $O^*$, $O \subseteq X$ open, form a basis of the topology of $\beta X$ and thus the $O^* \cap X^*$ are a basis of the topology of $X^*$.

We come to specific examples. First let $X = \omega$, equipped with the discrete topology. $\beta \omega$ is the space of all ultrafilters on $\omega$ and $\omega^*$ is the space of free ultrafilters. Basic open sets are of the form $O^* = \{U \in \beta \omega : O \in U\}$ for $O \subseteq \omega$ and, in fact, every regular open set is of this form so that $\text{r.o.}(\beta \omega) = \mathcal{P}(\omega)$. Basic non-empty open sets of $\omega^*$ are of the form $O^* \cap \omega^*$ where $O \subseteq \omega$ is infinite. If $|O_0 \Delta O_1| < \aleph_0$, then clearly $O_0^* \cap \omega^* = O_1^* \cap \omega^*$. Thus a dense subset of $\text{r.o.}(\omega^*)^+$ is isomorphic to $\mathcal{P}(\omega)/\text{fin}^+$ and we obtain

Fact 7. $\text{r.o.}(\omega^*) = \text{r.o.}(\mathcal{P}(\omega)/\text{fin})$. □

Similarly, we get $\text{r.o.}(\beta \omega \times \beta \omega) = \text{r.o.}(\mathcal{P}(\omega) \times \mathcal{P}(\omega))$ and $\text{r.o.}(\omega^* \times \omega^*) = \text{r.o.}(\mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin})$ etc.
Next, let $X = \mathbb{R}$, equipped with the standard topology. For $s \in 2^{<\omega}$, $n \in \mathbb{Z}$ and $\epsilon > 0$, let

$$O_{s,n,\epsilon} = (n + \sum \left\{ \frac{1}{2^{i+1}} : i < |s| \text{ and } s(i) = 1 \right\} - \epsilon, \n + \sum \left\{ \frac{1}{2^{i+1}} : i < |s| \text{ and } s(i) = 1 \right\} + \frac{1}{2^{|s|}} + \epsilon)$$

Clearly the $O_{s,n,\epsilon}$ are a basis of the topology of $\mathbb{R}$ and so the $O_{s,n,\epsilon}^*$ are a basis of the topology of $\beta\mathbb{R}$. For every infinite partial function $f : \mathbb{Z} \rightarrow 2^{<\omega}$ let

$$O_f = \bigcup_{n \in \text{dom}(f)} O_{f(n),n,\epsilon_n}$$

where $\epsilon_n = \min\{\frac{1}{2^{|s| + k}} : n - 1 \leq i \leq n + 1\}$ and notice that the $O_f^* \cap \mathbb{R}^*$ form a basis of regular open sets of the topology of $\mathbb{R}^*$ (indeed every $\mathcal{U} \in \mathbb{R}^*$ contains only unbounded closed sets; otherwise $\bigcap \mathcal{U} \neq \emptyset$ by compactness; thus for bounded $O \subseteq \mathbb{R}$, $O^* \cap \mathbb{R}^* = \emptyset$, and it is easy to see every unbounded $O \subseteq \mathbb{R}$ contains a set of the form $O_f$). If $\text{dom}(f) =^* \text{dom}(g)$ and $f(n) = g(n)$ for almost all $n \in \text{dom}(f)$ then $O_f^* \cap \mathbb{R}^* = O_g^* \cap \mathbb{R}^*$. Otherwise they are distinct (by choice of the $\epsilon_n$). This means that $\text{r.o.}(\mathbb{R}^*)^+$ has a dense subset isomorphic to $F/\text{fin} := \{[f] : f \in F\}$ where $F = \{f : \mathbb{Z} \rightarrow 2^{<\omega} : \text{dom}(f) \text{ is infinite}\}$ and $[f] = \{g \in F : \text{dom}(g) =^* \text{dom}(f) \text{ and } \forall^\infty n \in \text{dom}(g) (g(n) = f(n))\}$, ordered by $[f] \leq [g]$ if $\text{dom}(f) \subseteq^* \text{dom}(g)$ and $f(n) \supseteq g(n)$ holds for almost all $n \in \text{dom}(f)$.

Let $\mathcal{C}$ be Cohen forcing, that is, the algebra of clopen subsets of the Cantor space $2^\omega$, ordered by inclusion. Since $\{[s] : s \in 2^{<\omega}\}$ is a dense subset of $\mathcal{C}^+$, $\mathcal{C}^+$ has a dense subset isomorphic to $2^{<\omega}$ ordered by reverse inclusion. Thus $\mathcal{C}^\omega/\text{fin}^+$ has a dense subset isomorphic to $F/\text{fin}$. This shows

**Fact 8.** $\text{r.o.}(\mathbb{R}^*) = \text{r.o.}(\mathcal{C}^\omega/\text{fin})$.

The above discussion motivates the investigation of cardinal numbers like $\mathfrak{h} = \mathfrak{h}(\text{r.o.}(\omega^*))$, $\mathfrak{b}_2 := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin}) = \mathfrak{h}(\text{r.o.}(\omega^* \times \omega^*))$, $\mathfrak{h}(\mathcal{C}^\omega/\text{fin}) = \mathfrak{h}(\text{r.o.}(\mathbb{R}^*))$ etc.

**Distributivity numbers of products and reduced powers.** We know already $\mathfrak{b}_2 \leq \mathfrak{h}$. The following is easy to see

**Fact 9.** $t \leq \mathfrak{b}_2$.

On the other hand $\mathfrak{b}_2$ may be less than $\mathfrak{h}$.

**Theorem 1.** (Shelah-Spinas [SS1]) $\text{CON}(\mathfrak{b}_2 < \mathfrak{h})$.

In fact $\mathfrak{b}_2 < \mathfrak{h}$ holds in the iterated Mathias model (the $\omega_2$-stage countable support iteration of Mathias forcing over a model of CH).

In fact Shelah and Spinas also obtained the consistency of $\mathfrak{b}_{n+1} < \mathfrak{b}_n$ for any $n$ [SS2] where $\mathfrak{b}_n := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin}_n)$. Since $t \leq \mathfrak{b}_n$ for all $n$, the consistency of $t < \mathfrak{h}$ follows immediately.

We know already $\mathfrak{h}(\mathcal{C}^\omega/\text{fin}) \leq \mathfrak{h}$. Again, the inequality is consistently strict.
Theorem 2. (Dow [Do]) $\text{CON}(\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h})$.
In fact $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h}$ holds in the iterated Mathias model. □

The similarity to the Shelah-Spina result lead to the following

Question 1. (Dow [Do]) Is $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) = \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$? Is $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}_2$?

Note that $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}_2$ because $\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin} < \mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin}$. Upper and lower bounds for $\mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ are given by

Theorem 3. (Balcar-Hrušák [BH]) $t \leq \mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \text{add}(\mathcal{M})$. □

Here $\text{add}(\mathcal{M})$ denotes the additivity of the meager ideal, that is, the least size of a family of meager sets whose union is not meager.

Balcar and Hrušák also observed [BH] that $t < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ is consistent. Moreover, Dow's Theorem 2 is a Corollary of Theorem 3. Namely, it is well-known (and much easier to prove than Dow's argument for $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) = \aleph_1$) that $\text{add}(\mathcal{M}) = \aleph_1$ in the iterated Mathias model (see, e.g., [BJ]). On the other hand, since $\mathfrak{h} < \text{add}(\mathcal{M})$ in the Hechler model, the consistency of $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \text{add}(\mathcal{M})$ follows. This naturally leads to

Question 2. (Balcar-Hrušák [BH]) Is $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \min\{\mathfrak{h}, \text{add}(\mathcal{M})\}$ consistent?

Both questions can be answered with basically the same method.

Theorem 4. ([Br3]) $\text{CON}(\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \min\{\mathfrak{h}, \text{add}(\mathcal{M})\})$.

Theorem 5. ([Br3]) $\text{CON}(\mathfrak{h}_2 < \mathfrak{h}(\mathbb{C}^\omega/\text{fin}))$.
Thus $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ is consistent as well.

Notice that the converse, namely, the consistency of $\mathfrak{h}_2 > \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$, follows from the consistency of $\mathfrak{h}_2 > \text{add}(\mathcal{M})$ established by Shelah and Spina [SS2] and from Theorem 3.

Sketch of proof. We briefly sketch the proof of Theorem 4. Unlike earlier results on the independence of distributivity numbers ([Do], [SS1], [SS2]), we use a finite support iteration $\langle \mathcal{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ of ccc forcing over a model of CH. This is natural because we have to add Cohen reals anyway (in Theorem 4 we want to add increase $\text{add}(\mathcal{M})$, and in Theorem 5 we have to increase $\text{add}(\mathcal{M})$ by Theorem 3).

Roughly speaking, the iteration adds dominating reals (via Hechler forcing $\mathbb{D}$, see [BJ]) in successor stages and limit stages of cofinality $\omega$ while we use Laver forcing $\mathbb{L}_U$ with a Ramsey ultrafilter $U$ at limit stages of cofinality $\omega_1$ (see below for the definition). The Hechler reals (as well as the Laver reals) guarantee that $b = \aleph_2$ while the Cohen reals give $\text{cov}(\mathcal{M}) = \aleph_2$. So $\text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\} = \aleph_2$ holds.

Recall that an ultrafilter $U$ on $\omega$ is Ramsey if for all partitions $\langle X_n : n \in \omega \rangle$ of $\omega$ either $X_n \in U$ for some $n \in \omega$ or there is $Y \in U$ with $|Y \cap X_n| \leq 1$ for all $n$. $\mathbb{L}_U$ consists of all trees $T \subseteq \omega^{<\omega}$ such that for all $s \in T$ below the
stem (i.e. \( s \supseteq \text{stem}(T) \)) the set of successor nodes \( \{ n : n \in T \} \) belongs to \( \mathcal{U} \). \( \mathbb{L}_\mathcal{U} \) is ordered by inclusion. It is easy to see that the generic Laver real \( \ell_\mathcal{U} := \bigcap \{ [T] : T \in G \} \in \omega^\omega \) dominates the ground model reals and that \( \text{ran}(\ell_\mathcal{U}) \subseteq \omega \) diagonalizes \( \mathcal{U} \) (i.e. \( \ell_\mathcal{U} \subseteq^* U \) for all \( U \in \mathcal{U} \)). Here \( G \) denotes the generic filter. Thus iterating \( \mathbb{L}_\mathcal{U} \) naturally increases \( b \) and \( s \). The effect of \( \mathbb{L}_\mathcal{U} \) on \( \mathfrak{h} \), however, is a more subtle issue and depends very much on the choice of the ultrafilter \( \mathcal{U} \). In some situations \( \mathfrak{h} \) (and its relatives) stay small (see [Br2, Section 2] for such a construction) and we obtain a natural model for \( \mathfrak{h} < \min \{ b, s \} \), the consistency of which was originally obtained by Shelah [Sh1] (see also [Sh2, Theorem VI.8.2]).

We assume \( \diamondsuit_2 \) holds in the ground model. This means there is a sequence \( \langle Z_\alpha : cf(\alpha) = \omega_1, \alpha < \omega_2 \rangle \) such that for all \( Z \subseteq \omega_2 \), the set \( \{ \alpha < \omega_2 : cf(\alpha) = \omega_1 \) and \( Z \cap \alpha = Z_\alpha \} \) is stationary. \( \diamondsuit_2 \) is used for guessing (initial segments of) names for potential witnesses for \( \mathfrak{h} = \aleph_1 \). Notice that if \( \dot{A} \) is a \( \mathcal{P}_{\omega_2} \)-name for such a witness, then by \( CH \) and ccc, \( \dot{A} \) can be thought of as an object of size \( \omega_2 \) and can be coded into a subset of \( \omega_2 \). Similarly if \( \alpha < \omega_2, |\alpha| = \aleph_1 \), and \( \dot{A} \) is a \( \mathcal{P}_\alpha \)-name for a witness of \( \mathfrak{h} = \aleph_1 \), \( \dot{A} \) can be coded into a subset of \( \alpha \). Thus, if at stage \( \alpha \) where \( cf(\alpha) = \omega_1 \), \( Z_\alpha \) codes such a \( \mathcal{P}_\alpha \)-name \( \dot{A} = \{ \dot{A}_\beta : \beta < \omega_1 \} \), then we construct a Ramsey ultrafilter \( \mathbb{U}_\alpha \) such that \( \forall_\alpha \mathbb{U}_\alpha \cap \dot{A}_\beta \neq \emptyset \) for all \( \beta < \omega_1 \) and force with \( \mathbb{Q}_\alpha = \mathbb{L}_\mathcal{U}_\alpha \). This destroys the witness \( \dot{A} \). Now, if \( \dot{A} \) is a \( \mathcal{P}_{\omega_2} \)-name for a witness for \( \mathfrak{h} = \aleph_1 \) coded by \( Z \subseteq \omega_2 \), then \( C = \{ \alpha < \omega_2 : cf(\alpha) = \omega_1 \) and \( \dot{A} \lceil \alpha \) is a \( \mathcal{P}_\alpha \)-name for a witness for \( \mathfrak{h} = \aleph_1 \} \) is \( \omega_1 \)-club. By \( \diamondsuit_2 \) there is \( \alpha \in C \) with \( Z \cap \alpha = Z_\alpha \). So \( Z_\alpha \) codes \( \dot{A} \lceil \alpha \) and \( \mathbb{L}_\mathcal{U}_\alpha \) destroys \( \dot{A} \lceil \alpha \) and also \( \dot{A} \). This shows \( \mathfrak{h} = \aleph_2 \).

The most difficult part of the argument is the proof of \( \mathfrak{h}(\mathbb{C}^{\omega}/\text{fin}) = \aleph_1 \). We build a witness \( \mathcal{F} = \{ F_\beta \subseteq \mathbb{C}^{\omega} : \beta < \omega_1 \} \) along the iteration. The main point is that if the Ramsey ultrafilter \( \mathbb{U}_\alpha \) is carefully chosen, then \( \mathbb{Q}_\alpha = \mathbb{L}_\mathcal{U}_\alpha \) does not destroy (the initial segment of) this witness \( \mathcal{F} \). This is a technical argument which relies heavily on a rank analysis of \( \mathbb{L}_\mathcal{U}_\alpha \)-names. See [Br3] for details. To be able to build the required Ramsey ultrafilter in limit stages of cofinality \( \omega_1 \), we use the Hechler reals which we added in successor stages. For Hechler forcing it is much easier to see that it preserves (the initial segment of) the witness \( \mathcal{F} \). Thus \( \mathfrak{h}(\mathbb{C}^{\omega}/\text{fin}) = \aleph_1 \) follows. \( \Box \)

We close this section with some comments and questions on related cardinals. Balcar and Hrušák [BH] proved that \( t(\mathbb{C}^{\omega}/\text{fin}) = t \) (and thus \( t(\mathbb{C}^{\omega}/\text{fin}) < \mathfrak{h}(\mathbb{C}^{\omega}/\text{fin}) \) is consistent as well, see above). But little seems to be known about \( s(\mathbb{C}^{\omega}/\text{fin}) \) except for the trivial \( s(\mathbb{C}^{\omega}/\text{fin}) \leq s \).

**Problem 1.** Investigate \( s(\mathbb{C}^{\omega}/\text{fin}) \)! Investigate \( \tau(\mathbb{C}^{\omega}/\text{fin}) \) for other cardinal invariants \( \tau \)!

For a topological space \( X \) without isolated points, the Baire number of \( X \) (also called Novák number), \( n(X) \), is the least size of a family of nowhere dense sets covering \( X \). Let \( n := n(\omega^*) \).

**Theorem 6.** (Balcar-Pelant-Simon [BPS], see also [BS, Theorem 3.10])
(i) If $h < c$, then $h \leq n \leq h^+$. 

(ii) If $h = c$, then $c \leq n \leq 2^c$. 

The analogous result holds for $h(\mathbb{R}^*)$ and $n(\mathbb{R}^*)$. Also $n(\mathbb{R}^*) \leq n$ is easy to see, but the following is still open.

**Question 3.** (van Douwen, see [Do]) Is $n(\mathbb{R}^*) < n$ consistent?

### 3 Further friends of $\mathcal{P}(\omega)/\text{fin}$

We briefly discuss the distributivity number of other structures related to $\mathcal{P}(\omega)/\text{fin}$.

**Dense (Q) / nwd.** Let Dense($\mathbb{Q}$) denote the family of dense subsets of the rationals $\mathbb{Q}$, and let nwd stand for the nowhere dense sets of rationals. Let $\text{Dense}(\mathbb{Q})/\text{nwd} = \{[A] : A \in \text{Dense}(\mathbb{Q})\}$ where $[A] = \{B : A \Delta B \in \text{nwd}\}$ for $A \in \text{Dense}(\mathbb{Q})$, ordered by $[A] \leq [B]$ if $A \setminus B \in \text{nwd}$. Let $h_\mathbb{Q} = h(\text{Dense}(\mathbb{Q})/\text{nwd})$. $s_\mathbb{Q}$ and $t_\mathbb{Q}$ are defined similarly. The investigation of Dense($\mathbb{Q}$)/nwd has been started by Balcar, Hernández and Hrušák [BHH].

**Theorem 7.** (Balcar-Hernández-Hrušák [BHH], Brendle [Br2])

(i) $t_\mathbb{Q} = t$. 

(ii) $s_\mathbb{Q} \leq \min\{s, \text{add}(\mathcal{M})\}$. 

Balcar, Hernández and Hrušák [BHH] also proved the consistency of $t_\mathbb{Q} < h_\mathbb{Q}$ and of $h_\mathbb{Q} < h$. In fact, by (ii) of Theorem 7, $s_\mathbb{Q} < h$ holds in the iterated Mathias model. Furthermore:

**Theorem 8.** [Br2]

(i) $\text{CON}(h_\mathbb{Q} < s_\mathbb{Q})$.

(ii) $\text{CON}(h < h_\mathbb{Q})$. 

The argument for the proof of (ii) is similar to the argument for Theorems 4 and 5, see above. The following is still open.

**Question 4.** [Br2] Is $s_\mathbb{Q} < \min\{s, \text{add}(\mathcal{M})\}$ consistent?

**Partitions of $\omega$.** Let $(\omega)$ denote the collection of partitions of $\omega$. $(\omega)^\omega$ is the *infinite partitions* of $\omega$ (i.e. the partitions into infinitely many blocks), and $(\omega)^c$ is the *non-trivial partitions* of $\omega$. Here, we say $A \in (\omega)$ is trivial if $\{n\} \in A$ for almost all $n$ (equivalently, $A$ has no infinite block and almost all blocks are singletons). Write $A \leq B$ if $A$ is coarser than $B$ iff all blocks of $A$ are unions of blocks of $B$. Say $X$ is a *finite coarsening* of $A$ if $X$ is gotten from $A$ by merging finitely many blocks of $A$. Write $A \leq^* B$ if there is a finite coarsening $X$ of $A$ such that $X \leq B$. Say $A =^* B$ if $A \leq^* B$ and $B \leq^* A$ iff there is
X which is a finite coarsening of both A and B. Let \([A] = \{B : A =^* B\}\) and set \([A] \leq [B]\) if \(A \leq^* B\). \(((\omega)^\omega/ =^*, \leq)\) is the separative quotient of \(((\omega)^\omega, \leq)\). It is called the dual structure and we let \(h_d = h((\omega)^\omega/ =^*)\). As usual, we work with \(((\omega)^\omega, \leq)\) instead of \(((\omega)^\omega/ =^*, \leq)\). It is easy to see that \(P(\omega)/\text{fin} <^{<)} (\omega)^\omega/ =^*\); namely, \(h : (\omega)^\omega \to [\omega]^\omega\) given by \(h(A) = \{\min(b) : b \in A\}\) induces the projection mapping giving rise to the complete embedding. Thus \(h_d \leq h\). The investigation of cardinal invariants of \(((\omega)^\omega, \leq^*)\) has been started by Cichoń, Krawczyk, Majcher-Iwanow and Węglorz [CKMW].

**Theorem 9.** (Carlson [Mat]) \(t_d = \aleph_1\). □

**Theorem 10.** (i) (Halbeisen [Ha]) \(\text{CON}(h_d > \aleph_1)\).

Namely, \(c = h_d = \aleph_1\) holds in the iterated dual Mathias model.

(ii) (Spinas [Sp]) \(\text{CON}(h_d < h)\).

In fact \(h_d < h\) holds in the iterated Mathias model.

(iii) [Br1] \(\text{CON}(h_d = \aleph_1 + MA + \neg CH)\). □

Note that (iii) strengthens (ii) because \(MA\) implies \(t = c\) and, thus, \(h = c\) and \(h(P) = c\) where \(P\) is any of the partial orders considered in Section 2 or \(P = \text{Dense}(\mathbb{Q})/\text{ndw}\). (The main distinction seems to be that for all \(P\) considered earlier in this paper, \(t(P) = t\) in ZFC while \(t_d = \aleph_1\).) On the other hand, Cichoń et al. [CKMW] already observed that \(MA\) implies \(s_d = c\) so that \(h_d < s_d\) is consistent as well.

There is another natural structure associated with \((\omega)\), which is obtained by turning the order upside down and looking at refining instead of coarsening. Say \(A \leq c B\) if \(A\) is finer than \(B\) iff \(B \leq A\). \(X\) is a finite refinement of \(A\) if for some finite \(x \subseteq \omega, X = \{b \setminus x : b \in A\} \cup \{\{n\} : n \in x\}\). Write \(A =^* c B\) if there is a finite refinement \(X\) of \(A\) such that \(X \leq c B\). Say \(A =^* c B\) if \(A \leq^* c B\) and \(B \leq^* c A\) if there is \(X\) which is a finite refinement of both \(A\) and \(B\). Notice that \(A \leq^* c B\) implies \(B \leq^* c A\) (and equivalence holds for partitions which contain only finite blocks). As usual let \([A] = \{B : A =^* c B\}\), \([A] \leq [B]\) if \(A \leq^* c B\) and consider the converse dual structure \(((\omega)^c/ =^*, \leq)\) which may be identified with \(((\omega)^c, \leq^*)\). The reason for considering \(\leq^* c\) instead of \(\geq^*\) is that the former gives indeed rise to the separative quotient of \(((\omega)^c, \leq c)\) while \(((\omega)^c, \geq^*)\) does not. This structure has been investigated by Majcher-Iwanow [Maj]. Again \(P(\omega)/\text{fin} <^{<)} (\omega)^c/ =^*\); but more is true: \(((\omega)^c, \leq^*)\) is locally isomorphic to \(((\omega)^\omega, \leq^*)\) [Maj] so that \(\text{r.o.}(P(\omega)/\text{fin}) = \text{r.o.}(\omega)^c/ =^*)\). Thus \(h_c = h\) where \(h_c = h((\omega)^c/ =^*)\). In fact, equality also holds for several other cardinal invariants of the continuum; e.g., \(t_c = t\) and \(s_c = s\), see [BZ] for details.

**The General Philosophy** behind the results obtained so far is that distributivity numbers are independent unless there is an order relationship for trivial reasons, namely, unless there is a complete embedding between the partial orderings. Indeed, in all cases investigated so far, either \(P <^{<)} Q\) or \(\text{CON}(h(P) < h(Q))\) has been established.
References


[BH] B. Balcar and M. Hrušák, Distributivity of the algebra of regular open subsets of $\beta\mathbb{R}\setminus\mathbb{R}$, Top. Appl.


[Do] A. Dow, The regular open algebra of $\beta\mathbb{R}\setminus\mathbb{R}$ is not equal to the completion of $\mathcal{P}(\omega)/\text{fin}$, Fund. Math. 157 (1998), 33-41.


