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The covering number and the uniformity of the ideal $\mathcal{I}_f$

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1 Introduction

For the ideal $\mathcal{S}\mathcal{N}$ of strongly measure zero subsets of the real line, the cardinal coefficients have been studied[1]. But its cofinality had not been studied. In general, it may be larger than the continuum. Yorioka studied its cofinality(see [2]). One of his results is that the value of $\text{cof}(\mathcal{S}\mathcal{N})$ is equal to the dominating number for $\omega_1^{\omega_1}$ under the continuum hypothesis. In the process, he introduced ideals $\mathcal{I}_f$ for $f \in \omega^\omega$. These ideals were used in the proof. We are interested in the ideals $\mathcal{I}_f$ themselves. These ideals are subideals of the null ideal $\mathcal{N}$ and include $\mathcal{S}\mathcal{N}$. The properties of these ideals depend on $f$.

In this paper, we discuss the following contents. In section 3, we show a characterization of $\text{cov}(\mathcal{I}_f) \geq b$. In section 4, we define a forcing notion which has the countable chain condition. And with the results of section 3 we show that its $\omega_2$-stage finite support iteration by bookkeeping method lifts up $\text{cov}(\mathcal{I}_f)$ from a ground model with the continuum hypothesis. In section 5, we introduce a sufficient condition not to lift up $\text{cov}(\mathcal{I}_f)$ for forcing notions which satisfy axiom $\Lambda$.

2 Definitions and notation

Throughout this paper, we use the standard terminology for forcing of set theory and cardinal coefficients (see[1]). We regard the set of all reals as the Cantor set $2^\omega$. We denote by $\mathcal{M}$ and $\mathcal{N}$ the set of all meager subsets of $2^\omega$ and the set of all null subsets of $2^\omega$ respectively.

For functions $f$, $g$ in $\omega^\omega$ we write "$f \leq g$" to mean that $g$ dominates $f$ everywhere, that is, $f(n) \leq g(n)$ for all $n < \omega$. And we let "$f \leq^* g$" mean that $g$ eventually dominates $f$, that is, there exists an $n < \omega$ such that $f(m) \leq g(m)$ holds for all $m < \omega$ larger than $n$. We denote by $\mathcal{S}$ the set of all non-decreasing functions $d$ in $\omega^\omega$ which diverges to infinity and $d(0) = 0$. We denote by $\mathcal{C}$ and $\mathcal{D}$ the Cohen forcing notion and the dominating forcing notion respectively[1]. For each ideal (or family if there is not a problem in particular) $\mathcal{I}$ on $2^\omega$ which contains all singletons, we denote by $\text{add}(\mathcal{I})$, $\text{cov}(\mathcal{I})$, $\text{non}(\mathcal{I})$ and $\text{cof}(\mathcal{I})$ the additivity, covering number,
uniformity and cofinality of $\mathcal{I}$ respectively which means that:

1. $\text{add}(\mathcal{I}) = \min \{ |A| \mid A \subseteq I \cup A \not\in \mathcal{I} \}$,
2. $\text{cov}(\mathcal{I}) = \min \{ |A| \mid A \subseteq I \cup A = 2^\omega \}$,
3. $\text{non}(\mathcal{I}) = \min \{ |Y| \mid Y \subseteq 2^\omega \setminus Y \not\in \mathcal{I} \}$,
4. $\text{cof}(\mathcal{I}) = \min \{ |A| \mid A \subseteq I \forall B \in \mathcal{I} \exists A \in A (B \subseteq A) \}$.

We have that $\text{add}(\mathcal{I}) \leq \text{cov}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ and $\text{add}(\mathcal{I}) \leq \text{non}(\mathcal{I}) \leq \text{cof}(\mathcal{J})$ for each ideal or family $\mathcal{I}$ on $2^\omega$ which contains all singletons.

We define some notation before we define the ideals $\mathcal{I}_f$ and $\mathcal{K}_f$.

**Definition 2.1** Let $f, g$ be functions in $\omega^\omega$.

1. We define the order "$\ll$" on $\omega^\omega$ by $f \ll g$ iff $\forall k < \omega \exists N < \omega \forall n \geq N (f(n^k) \leq g(n))$.
2. We define the order "$\lll$" on $\omega^\omega$ by $g_\sigma(n) = |\sigma(n)|$ for all $n < \omega$.

3. For $\sigma \in (2^\omega)^\omega$, define the subset $Y(\sigma) \subseteq 2^\omega$ by $Y(\sigma) = \bigcap_{n \in \omega} \bigcup_{m \geq n} [\sigma(m)]$, where $[s] = \{ x \in 2^\omega \mid s \subseteq x \}$ for each $s \in 2^{<\omega}$.

Define the subsets $S(f)$, $T(f)$ and $U(f)$ of $2^{<\omega}_d$ by

$S(f) = \{ \sigma \in (2^{<\omega})^\omega \mid g_\sigma \gg f \}$,

$T(f) = \{ \sigma \in (2^{<\omega})^\omega \mid g_\sigma = f \}$.

**Definition 2.2** Let $f \in \omega^\omega$. Define the families $\mathcal{I}_f$, $\mathcal{J}_f$ and $\mathcal{K}_f$ on $2^\omega$ by

$\mathcal{I}_f = \{ X \subseteq 2^\omega \mid \exists \sigma \in S(f) X \subseteq Y(\sigma) \}$,

$\mathcal{J}_f = \{ X \subseteq 2^\omega \mid \exists \sigma \in T(f) X \subseteq Y(\sigma) \}$.

The following definition is not necessary for the definition of ideal $\mathcal{I}_f$. But it is the very useful.

**Definition 2.3** Let $f \in \omega^\omega$. For each $d \in S$, we define the functions $g_d^{(f)}$ and $h_d^{(f)} \in \omega^\omega$ by $g_d^{(f)}(n) = f(n^{k+2})$ if $n \in [d(k), d(k + 1))$ for all $n < \omega$, respectively. If $g \in \omega^\omega$ is $g = g_d^{(f)}$ for some $d \in \omega^\omega$, then we say "$g$ is generated by $d$ (and $f$) for $\ll$".
3 \ cov(I_f) , \ cov(J_f) and bouding number b

In this section, we show that the ideal I_f and the family J_f are related to bounding number b intimately. For each d \in S, g_{d}^{(f)} \gg f holds where g_{d}^{(f)} was introduced in chapter 2. In addition, for each g \gg f there exists a d \in S by the definitions of g_{d}^{(f)} and \ll such that g_{d}^{(f)} \leq^* g. Therefore, the following hold.

Lemma 3.1 For each family F \subset \omega^\omega such that |F| < b and \forall g \in F (g \gg f), there exists d \in S such that \forall g \in F (g_{d}^{(f)} \leq^* g).

Proof of Lemma3.1 Let \mathcal{F} \subset \omega^\omega satisfy \forall g \in \mathcal{F} (g \gg f) and |\mathcal{F}| < b. For each g \in \mathcal{F}, there exists d_{g} \in S such that g_{d_{g}}^{(f)} \leq g. Since |\mathcal{F}| < b, the family \{ d_{g} \mid g \in \mathcal{F} \} is bounded family in \omega^\omega. So there exists d \in S which dominates for all functions in \{ d_{g} \mid g \in \mathcal{F} \}. \blacksquare (Lemma3.1)

Lemma 3.2 There exists a family F \subset \omega^\omega such that |F| = b and \forall g \in F (g \gg f) and \forall h \gg f \exists g \in F (h \not\gg g).

Proof of Lemma3.2 Take a unbounded family B \subset S. Then a family \{ g_{d}^{(f)} \mid d \in B \} is as desired. \blacksquare (Lemma3.2)

For all d \in S, \cov(I_f) \leq \cov(J_{g_{d}}^{(f)}) holds by I_f = \bigcup_{g \gg f} J_{g_{d}}^{(f)} = \bigcup_{d \in S} J_{g_{d}}^{(f)}. By this, if \cov(I_f) is larger than b, then \cov(J_{g_{d}}^{(f)}) is larger than b for all d \in S. The inverse holds.

Theorem 3.1 \cov(I_f) \geq b \iff \cov(J_{g_{d}}^{(f)}) \geq b \ for all d \in S.

Proof of Theorem3.1 \implies: As above.
\iff: Assume \cov(I_f) < b. There exists a family \mathcal{F} such that |\mathcal{F}| = \cov(I_f) < b and |\mathcal{F}| = 2^\omega. For each X \in \mathcal{F}, there exists \sigma_X such that X \subset \sigma_X. By Lemma3.1, there exists d \in S such that \forall X \in \mathcal{F} g_{d}^{(f)} \leq^* \sigma_X. For each X \in \mathcal{F}, define \tau_X \in T(g_{d}^{(f)}) by \tau_X(n) = \sigma_X(n)|g_{d}^{(f)}(n). Then a family \{ \tau_X \mid X \in \mathcal{F} \} covers 2^\omega. \blacksquare (Theorem3.1)

However, it is easily proved that \cov(I_f) \geq b is independent from ZFC. \cov(I_f) = \omega_1 and b = \mathfrak{c} hold in a generic model which is obtained by a forcing notion satisfying Laver property from a ground model with the continuum hypothesis. Also \cov(I_f) = b = \omega_1 holds in a generic model which is obtained by the Cohen forcing notion of any weight from a ground model with continuum hypothesis.

4 The forcing notion \mathbb{P}(d) for d \in S and cov(I_f) and non(I_f)

In this section, we discuss the covering number and the uniformity of ideal I_f in the model obtained by a certain iteration of the forcing notion \mathbb{P}(d). We define the forcing notion \mathbb{P}(d) for d \in S.
Definition 4.1 Let $d \in S$. Define the forcing notion $\mathbb{P}(d)$ by
\[
\mathbb{P}(d) = \left\{ (s, F) \in 2^{<\omega} \times \left[ T(g_d^{(f)}) \right]^{<\omega} \mid |s| = f(|F|) \right\},
\]
\[
(s, F) \leq (s', F') \iff 1. s \supset s' \supset F' \\
2. \forall \sigma \in F' \forall n \in |F| \setminus |F'| \ [s \lceil f(n), f(n+1) \rangle \neq \sigma(n+1)[f(n), f(n+1)].
\]

Lemma 4.1 For all $d \in S$, the forcing notion $\mathbb{P}(d)$ is $\sigma$-linked. So it has the countable chain condition.

Proof of Lemma 4.1 Since $g_d^{(f)}(n+1) - g_d^{(f)}(n) > n$ for all $n < \omega$, holds that $\forall (s, F) \in\mathbb{P}(d) \forall F' \in \left[ T(g_d^{(f)}) \right]^{<\omega} \exists (t, H) \leq (s, F)$ ($H = F \cup F'$).

Let $N < \omega$ and $g = g_d^{(f)}$. For each $t \in 2^{<N}$, $\psi \in \prod_{n \in [N, 2N)} \left[ 2^{g(n+1)-g(n)} \right]^{\leq N}$, define a subset $B_{t, \psi}$ of $\mathbb{P}(d)$ by
\[
B_{t, \psi} = \{ (s, F) \in \mathbb{P}(d) \mid s = t\psi = \langle \{ \sigma(n+1) \mid (g(n), g(n+1)) \mid \sigma \in F \mid n \in ||F|, 2|F|) \rangle \}.
\]
Clearly $\mathbb{P}(d) = \bigcup_{t \in 2^{<N\psi}} \bigcup \{ B_{t, \psi} \mid t \in 2^{<N}\psi \in \prod_{n \in [N, 2N)} \left[ 2^{g(n+1)-g(n)} \right]^{\leq N} \}$. We show that for all $N < \omega$, $t \in 2^{<N}$ and $\psi \in \prod_{n \in [N, 2N)} \left[ 2^{g(n+1)-g(n)} \right]^{\leq N}$, any two distinct conditions in $B_{t, \psi}$ are compatible. Let $(s, F)$, $(s', F')$ be in $B_{t, \psi}$ and $(s, F) \neq (s', F')$. By the definition of $B_{t, \psi}$,
\[
s = s' = t|F| = |F'| = N
\]
\[
\{ \{ \sigma(n+1) \mid (g(n), g(n+1)) \mid \sigma \in F \mid n \in ||F|, 2|F|) \}
\]
\[
= \langle \{ \sigma(n+1) \mid (g(n), g(n+1)) \mid \sigma \in F' \mid n \in ||F|, 2|F'|) \rangle = \psi.
\]
There exists $(u, H) \leq (s, F)$ such that $H = F \cup F'$. Clearly $|F'| < |H| \leq 2N$. To prove $(u, H) \leq (s', F')$, let $\sigma \in F'$ and $n \in |H| \setminus |F'|$. Since $(u, H) \leq (s', F')$, $u \lceil g(n), g(n+1) \rangle \neq \tau(n+1)\lceil (g(n), g(n+1)) \rangle$ for all $\tau \in F$, that is, $u \lceil g(n), g(n+1) \rangle \not\in \psi(n)$. But $\sigma(n+1)\lceil (g(n), g(n+1)) \rangle \in \psi(n)$.

Therefore $u \lceil g(n), g(n+1) \rangle \not\in \sigma(n+1)\lceil (g(n), g(n+1)) \rangle$. \hfill $\Box$(Lemma 4.1)

For each $d \in S$, $\sigma \in T(g_d^{(f)})$ and $n < \omega$, define the subsets $D_{\sigma}$, $E_n \subset \mathbb{P}(d)$ as follows:
\[
D_{\sigma} = \{ (s, F) \in \mathbb{P}(d) \mid \sigma \in F \},
\]
\[
E_n = \{ (s, F) \in \mathbb{P}(d) \mid |F| \geq n \}.
\]

Lemma 4.2 For all $s \in S$, $\sigma \in T(g_d^{(f)})$ and $n < \omega$, the subsets $D_{\sigma}$ and $E_n$ are dense open sets in $\mathbb{P}(d)$.

Proof of Lemma 4.2 Let $\sigma \in T(g_d^{(f)})$, $n < \omega$ and $(s, F) \in \mathbb{P}(d)$. Take $F' \subset T(g_d^{(f)})$ such that $\sigma \in F'$ and $|F| \geq n$. There exists $(t, H) \leq (s, F)$ such that $H = F \cup F'$. Since $\sigma \in H$ and $|H| \geq n$, $(t, H) \in D_\sigma$ and $(t, H) \in E_n$. \hfill $\Box$(Lemma 4.2)
We are interested in the generic model of $\mathbb{P}(d)$. Let $d \in S$ and $\dot{G}$ be the canonical generic $\mathbb{P}(d)$-name. Define $\mathbb{P}(d)$-name $\dot{a}_G$ by

$$\Vdash_{\mathbb{P}(d)} \dot{a}_G = \bigcup \left\{ s \mid \exists F \ (s, F) \in \dot{G} \right\} \in 2^\omega.$$ 

**Lemma 4.3** For all $d \in S$, $\Vdash_{\mathbb{P}(d)} \forall \sigma \in T(g_d^{(f)}) \cap V \ (\dot{a}_G \notin Y(\sigma)).$

**Proof of Lemma 4.3** Let $d \in S, \sigma \in T(g_d^{(f)})$ and $(s, F) \in \mathbb{P}(d)$. By Lemma 4.2, there exists $(s', F') \leq (s, F)$ such that $\sigma \in F'$. To prove that $(s', F') \Vdash_{\mathbb{P}(d)} \sigma(n) \notin \dot{a}_G$ for all $n > |F'|$, let $n > |F'|$. By Lemma 4.2, there exists $(s'', F'') \leq (s', F')$ such that $|F''| \geq n$. Then $(s'', F'') \Vdash_{\mathbb{P}(d)} " s'' \subset \dot{a}_G \sigma(n), g_d^{(f)}(n) \neq \sigma(n) |g_d^{(f)}(n-1), g_d^{(f)}(n)) "$. Therefore $(s'', F'') \Vdash_{\mathbb{P}(d)} \sigma(n) \notin \dot{a}_G$. □(Lemma 4.3)

**Lemma 4.4** For all $d \in S$, $\Vdash_{\mathbb{P}(d)} 2^\omega \cap V \in \mathcal{J}_d^{(f)}$.

**Proof of Lemma 4.4** This is directly followed from the fact that $\mathbb{P}(d)$ adds Cohen reals in $
abla_{n<\omega} 2^{g_d^{(f)}(n)}$. □(Lemma 4.4)

To define a finite support iteration of $\mathbb{P}(d)$, let $\kappa$ be an uncountable regular cardinal and $\pi$ be a bijection from $\kappa$ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$ for all $\alpha < \kappa$. Let $\pi_0$ and $\pi_1$ be the first and second coordinate of the value of $\pi$ respectively.

Assume the continuum hypothesis. We define $P_\kappa$ by $\kappa$-stage finite support iteration

$$\left\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa \right\rangle$$

as follows:

Assume that $P_\beta$ and the $P_\beta$-names $\dot{d}_\xi^\beta$ for $\xi < \kappa$ with $\Vdash_{\beta} " \left\langle \dot{d}_\xi^\beta \mid \xi < \kappa \right\rangle \text{ is an enumeration of } S''$ are defined for all $\beta \leq \alpha$ in $\alpha$-stage. Define $\Vdash_{\alpha} \dot{Q}_\alpha \simeq P_{\pi(0)}(\dot{d}_{\pi(0)}^\beta) * D$.

**Theorem 4.1** (CH) $\Vdash_{P_\kappa} \exists \xi = b = \kappa \wedge \forall d \in S \text{ cov}(\mathcal{J}_d^{(f)}) = c$.

Therefore, it holds that $\Vdash_{P_\kappa} \text{cov}(\mathcal{I}_f) = c$ by theorem 3.1.

**Proof of Theorem 4.1** Clearly $c = b = \kappa$ in $V[G_\kappa]$. Let $d \in S, \lambda < c$ and a family $\{ X_\delta \mid \delta < \lambda \} \subset \mathcal{J}_d^{(f)}$ in $V[G_\kappa]$. There exists $\alpha < \kappa$ such that $X_\delta$ is coded by $\sigma_\delta \in T(g_d^{(f)})$ for each $\delta < \lambda$ in $V[G_\alpha]$. By Lemma 4.3, $\{ Y(\sigma_\delta) \mid \delta < \lambda \}$ does not cover $2^\omega$ in $V[G_{\alpha+1}]$. Hence $\{ X_\delta \mid \delta < \lambda \}$ does not cover $2^\omega$ in $V[G_\kappa]$. □(Theorem 4.1)

**Theorem 4.2** (CH) $\Vdash_{P_\omega} \text{non}(\mathcal{I}_f) = c$

**Proof of Theorem 4.2** Clearly by Lemma 4.4. □(Theorem 4.2)
5 Property $E$ and $\text{cov}(\mathcal{I}_f) = \omega_1$

In this section, we introduce a certain property for forcing notions which satisfy axiom A. A forcing notion with this property does not add a real which is not covered by all elements of $S(f)$ in ground model. This property is preserved in an iterated forcing. So the countable support iteration of forcing notions with this property does not lift up $\text{cov}(\mathcal{I}_f)$. For example, the infinitely equal forcing notion $EE$ satisfies this property.

**Definition 5.1** Let forcing notion $P$ satisfy axiom A by the fusion orders $(\leq_n \mid n < \omega)$. $P$ has property $E$ if there exists $\varphi \in \omega^{\aleph_0}$ such that

1. for all $p \in P$ and $n < \omega$, if $p \Vdash \dot{a} \in V$ then there exist $q \leq_n p$ and a finite set $B$ such that $|B| \leq \varphi(p, n)$ and $q \Vdash \dot{a} \in B$,
2. for all $p, q \in P$ and $n < \omega$, if $q \leq_n p$ then $\varphi(q, n) = \varphi(p, n)$.

**Lemma 5.1** Suppose that the axiom A forcing notion $P$ has property $E$.

Then $\Vdash_P "2^{\omega} \subset \cup \{Y(\tau) \mid \tau \in T(g) \cap V\}"$ for all strictly increasing function $g \in \omega^\omega$. Therefore, $\Vdash_P "2^{\omega} \subset \cup \{Y(\tau) \mid \tau \in S(f) \cap V\}"$.

**Proof of Lemma 5.1** Let $p \in P$ satisfy $p \Vdash \dot{x} \in 2^\omega$ and $g \in \omega^\omega$ be strictly increasing. By induction on $j < \omega$, define three sequences $(p_j \in P \mid j < \omega)$, $(m_j \in \omega \mid j < \omega)$ and $(A_j \mid j < \omega)$ as follows:

(i) $p_0 = p$,
(ii) $p_{j+1} \leq_j p_j$,
(iii) $m_j = \sum_{i < j} \varphi(p_i, i)$,
(iv) $A_j \subseteq 2^{g(m_j + \varphi(p_j, j))}$,
(v) $|A_j| \leq \varphi(p_j, j)$,
(vi) $p_{j+1} \Vdash \dot{x} \in g(m_j + \varphi(p_j, j)) \in A_j$,

for all $j < \omega$. For each $j < \omega$, let $\{s^j_l \mid l < \varphi(p_j, j)\}$ be a enumeration of $A_j$. There exists $q \in P$ such that $\forall j < \omega$ $q \leq_j p_j$.

We define $\sigma \in (2^{\omega})^\omega$ by for each $n < \omega$, $\sigma(n) = s^j_l \in g(n)$ where $m_j = m_j + l$. To prove that $q \Vdash \dot{x} \in Y(\sigma)$, let $n < \omega$. There exists $j < \omega$ such that $m_j \geq n$. Since $q \Vdash \dot{x} \in g(m_j + \varphi(p_j, j)) \in A_j$, there exist $q' \leq q$ and $l < \varphi(p_j, j)$ such that $q' \Vdash \dot{x} \in (m_j + \varphi(p_j, j)) = s^j_l \supseteq \sigma(m_j + l)$. □(Lemma 5.1)

Let $\delta \leq \omega_2$. Let $P_\delta = \langle P_\delta, \dot{Q}_\alpha \mid \alpha < \delta \rangle$ be a $\delta$-stage countable support iteration such that $\dot{Q}_\alpha$ is defined by the forcing notion with property $E$ for all $\alpha < \delta$. For $n < \omega$ and $F \in [\delta]^{\omega}$, $p \in P_\delta$ is $(n, F)$-good if there exists $h \in \omega^F$ such that $p|\gamma \Vdash \dot{\varphi}_\gamma (\varphi(\gamma), n) \leq h(\gamma)$ for all $\gamma \in F$ where $\varphi_\gamma$ is $P_\gamma$-name for the function $\varphi$ appeared in the definition of property $E$ for $\dot{Q}_\gamma$. 


Lemma 5.2 Let $\delta \leq \omega_2$. For all $n < \omega$ and $F \in [\delta]^{<\omega}$, the set $\{ p \in P_3 \mid p \text{ is } (n, F)\text{-good} \}$ is $(n, F)$-dense open in $P_3$.

Proof of Lemma 5.2 Since the property $E$ implies the strongly $\omega^\omega$-bounding, we can prove easily by induction on $\delta \leq \omega_2$. □(Lemma 5.2)

By the lemma above, we may suppose only the condition that is $(n, F)$-good. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, define $h_{p,n,F} \in \omega^F$ by

(a) $p \upharpoonright \gamma \vdash_{\gamma} \varphi_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$,

(b) if $q \leq_{n,F} p$ then $h_{q,n,F}(\gamma) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$.

Lemma 5.3 Let $\delta \leq \omega_2$. There exists $\tilde{\varphi}_\delta \in \omega^{\omega \times \omega \times [\delta]^{<\omega}}$ such that

(1) for all $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, if $p \vdash_{\delta} \dot{a} \in V$ then there exists $q \leq_{n,F} p$ and a finite set $B$ such that $|B| \leq \tilde{\varphi}_\delta(p,n,F)$ and $q \vdash_{\delta} \dot{a} \in B$, (2) for all $p, q \in P_3$, $n < \omega$ and $F \in [\delta]^{<\omega}$, if $q \leq_{n,F} p$ then $\tilde{\varphi}_\delta(q,n,F) \leq \tilde{\varphi}_\delta(p,n,F)$.

Proof of Lemma 5.3 We prove by induction on $\delta \leq \omega_2$. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, we define $\tilde{\varphi}_\delta(p,n,F)$ as follows:

Case 1 : $\delta$ is limit ordinal.

Let $\alpha = \max(F) + 1$. Then $F \subset \alpha$. By induction hypothesis, there exists $\varphi_\alpha \in \omega^{P_\alpha \times \omega \times [\alpha]^{<\omega}}$ such that (1) and (2). So we define $\tilde{\varphi}_\delta(p,n,F)$ by $\varphi_\alpha(p\upharpoonright \alpha, n, F)$.

We show that (1) and (2). (1): Let $p \in P_3$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $p \vdash_{\delta} \dot{a} \in V$. Suppose $\alpha = \max(F) + 1$. Since $p\upharpoonright \alpha \vdash_{\alpha} \dot{b} \in V$ $\dot{f} \in P_\alpha \cdot f \vdash_{\alpha} \dot{a} = b^\dot{b}$ for some $P_\alpha$-name $\dot{b}$ and $\dot{f}$, there exist $r \leq_{n,F} p\upharpoonright \alpha$, finite set $B$ and $g \in P_\alpha$ such that $|B| \leq \varphi_\alpha(p\upharpoonright \alpha, n,F) = \tilde{\varphi}_\delta(p,n,F)$ and $r \vdash_{\alpha} \dot{a} \in F \cdot \dot{f} = g^\alpha$. Let $q = r \cup g$. Then $q \leq_{n,F} p$ and $q \vdash_{\delta} \dot{a} \in B$.

(2): Let $p, q \in P_3$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $q \leq_{n,F} p$ $q$. Suppose $\alpha = \max(F) + 1$. Then since $q \upharpoonright \alpha \leq_{n,F} p\upharpoonright \alpha$,

$$\tilde{\varphi}_\delta(q,n,F) = \varphi_\alpha(q\upharpoonright \alpha, n,F) \leq \varphi_\alpha(p\upharpoonright \alpha, n,F) = \tilde{\varphi}_\delta(p,n,F)$$

Case 2 : $\delta = \gamma + 1$.

In the case of $F \subset \gamma$, we define in the same way as the case of that $\delta$ is limit ordinal.

Suppose $\gamma \in F$.

By induction hypothesis, there exists $\tilde{\varphi}_\gamma$ such that for all $p' \in P_\gamma$, $n' < \omega$ and $F' \in [\gamma]^{<\omega}$, if $p\upharpoonright \gamma \vdash_{\gamma} \dot{a} \in V$, there exist $r \leq_{n,F} \gamma$, $p'$ and $B$ such that $|B| \leq \varphi_\gamma(p'\upharpoonright \gamma, n,F \cap \gamma)$ and $r \vdash_{\gamma} \dot{a} \in B$.

So we define $\tilde{\varphi}_\delta(p,n,F)$ by $\varphi_\gamma(p\upharpoonright \gamma, n,F \cap \gamma) \cdot h_{p,n,F}(\gamma)$.

We show that (1) and (2). (1): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good. In the case of $F \subset \gamma$, we can show in the same way as the case of that $\delta$ is limit ordinal.
Assume $\gamma \in F$. Also there exist $P_{\gamma}$-names $\dot{q}$ and $\dot{B}$ such that $p \models_{\gamma} "\dot{q} \leq_{n} p(\gamma) \wedge \dot{B} \subseteq V \wedge |\dot{B}| \leq \check{\varphi}_{\gamma}(p(\gamma), n) \wedge \dot{b} \in \dot{B}"$. By $p$ is $(n, F)$-good, $p \models_{\gamma}$ is $(n, F \cap \gamma)$-good and $p \models_{\gamma} "\dot{B} \subseteq \check{\varphi}_{\gamma}(p(\gamma), n) \leq h_{p, n, F}(\gamma)\). Let $\{b_{j} \mid j < h_{p, n, F}(\gamma)\}$ be a sequence of $P_{\gamma}$-names for an enumeration of $\dot{B}$. That is $p \models_{\gamma} \{b_{j} \mid j < h_{p, n, F}(\gamma)\} = \dot{B} \subseteq V$. By induction on $j < h_{p, n, F}$, we construct two sequences $\langle r_{j} \mid j < h_{p, n, F}\rangle$ and $\langle B_{j} \mid j < h_{p, n, F}\rangle$ such that

(a) $r_{j} \leq_{n, F \cap \gamma} r_{j-1}$ for all $j < h_{p, n, F}(\gamma)$,
(b) $|B_{j}| \leq \check{\varphi}_{\gamma}(r_{j-1}, n, F \cap \gamma) \leq \check{\varphi}_{\gamma}(p(\gamma), n, F \cap \gamma)$ for $j < h_{p, n, F}$,
(c) $r_{j} \models_{\gamma} b_{j} \in B_{j}$ for all $j < h_{p, n, F}$.

Let $q = r_{h_{p, n, F}(\gamma)-1} \cup \{b_{j} \mid j < h_{p, n, F}(\gamma)\}$ and $B = \bigcup \{B_{j} \mid j < h_{p, n, F}(\gamma)\}$. Clearly $q \models \dot{b} \in B$ and

$$|B| \leq \sum_{j < h_{p, n, F}} |B_{j}| \leq \sum_{j < h_{p, n, F}} \check{\varphi}_{\gamma}(p(\gamma), n, F \cap \gamma) \leq \check{\varphi}_{\gamma}(p(\gamma), n, F \cap \gamma) \cdot h_{p, n, F}(\gamma) = \check{\varphi}_{\delta}(r_{p}, n, F).$$

(2): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and $p, q$ satisfy $(n, F)$-good and $q \leq_{n, F} p$. In the case of $F \subseteq \gamma$, we can show in the same way as the case of that $\delta$ is limit ordinal. Suppose $\gamma \in F$. Then

$$\check{\varphi}_{\delta}(q, n, F) = \check{\varphi}_{\gamma}(q \upharpoonright \gamma, n, F \cap \gamma) \cdot h_{p, n, F}(\gamma) \leq \check{\varphi}_{\gamma}(p \upharpoonright \gamma, n, F \cap \gamma) \cdot h_{p, n, F}(\gamma) = \check{\varphi}_{\delta}(p, n, F).$$

\(\square\) (Lemma 5.3)

**Theorem 5.1** $\models_{P_{\omega_{2}}} 2^{\omega} \subseteq \bigcup \{Y(\tau) \mid \tau \in T(g) \cap V\}$, for all strictly increasing function $g \in \omega^{\omega}$. therefore, $\models_{P_{\omega_{2}}} 2^{\omega} \subseteq \bigcup \{Y(\tau) \mid \tau \in S(f) \cap V\}$.

**Proof of Theorem 5.1** By Lemma 5.3, we can show in the same way as Lemma 5.1.

\(\square\) (Theorem 5.1)

**Corollary 5.4** (CH) $\models_{P_{\omega_{2}}} "\text{cov}(I_{f}) = \text{cov}(J_{g}) = \omega_{1}"$ for all strictly increasing function $g \in \omega^{\omega}$.

6 The diagram of cardinal coefficients of $I_{f}$

In this section, we give the results for the cardinal coefficients of ideal $I_{f}$ of the forcing notions that we studied. Let $\kappa$ be an uncountable regular cardinal. We express the parts which we do not yet understand in '?'.
forcing notions | add | cov | non | cof | b | d | c
---|---|---|---|---|---|---|---
$\mathbb{O}(f)_\kappa$ | c | c | c | c | c | c | $\kappa$
$\mathbb{P}_\kappa$ | ? | c | c | c | ? | ? | $\kappa$
$\mathbb{C}_\kappa$ | $\omega_1$ | $\omega_1$ | c | c | $\omega_1$ | c | $\kappa$
$\mathbb{E}\mathbb{E}_{\omega_2}$ | $\omega_1$ | $\omega_1$ | c | c | $\omega_1$ | $\omega_1$ | $\omega_2$
$\mathbb{S}_{\omega_2}$ | $\omega_1$ | $\omega_1$ | $\omega_1$ | $\omega_1$ | $\omega_1$ | $\omega_1$ | $\omega_2$

$\mathbb{O}(f)_\kappa$: the $\kappa$-stage finite support iteration of the forcing notion $\mathbb{O}(f)$ introduced by T. Yorioka,
$\mathbb{P}_\kappa$: the $\kappa$-stage finite support iteration of the forcing notion $\mathbb{P}(d)$ by bookkeeping method,
$\mathbb{C}_\kappa$: the Cohen forcing notion which adds $\kappa$ many Cohen reals,
$\mathbb{E}\mathbb{E}_{\omega_2}$: the $\omega_2$-stage countable support iteration of the infinitely equal forcing notion,
$\mathbb{S}_{\omega_2}$: the $\omega_2$-stage countable support iteration of the Sacks forcing notion.

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References
