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The covering number and the uniformity of the ideal $\mathcal{I}_f$

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1 Introduction

For the ideal $\mathcal{S}\mathcal{N}$ of strongly measure zero subsets of the real line, the cardinal coefficients have been studied[1]. But its cofinality had not been studied. In general, it may be larger than the continuum. Yorioka studied its cofinality(see [2]). One of his results is that the value of $\text{cof}(\mathcal{S}\mathcal{N})$ is equal to the dominating number for $\omega_1^{\omega_1}$ under the continuum hypothesis. In the process, he introduced ideals $\mathcal{I}_f$ for $f \in \omega^\omega$. These ideals were used in the proof. We are interested in the ideals $\mathcal{I}_f$ themselves. These ideals are subideals of the null ideal $\mathcal{N}$ and include $\mathcal{S}\mathcal{N}$. The properties of these ideals depend on $f$.

In this paper, we discuss the following contents. In section 3, we show a characterization of $\text{cov}(\mathcal{I}_f) \geq b$. In section 4, we define a forcing notion which has the countable chain condition. And with the results of section 3 we show that its $\omega_2$-stage finite support iteration by bookkeeping method lifts up $\text{cov}(\mathcal{I}_f)$ from a ground model with the continuum hypothesis. In section 5, we introduce a sufficient condition not to lift up $\text{cov}(\mathcal{I}_f)$ for forcing notions which satisfy axiom $A$.

2 Definitions and notation

Throughout this paper, we use the standard terminology for forcing of set theory and cardinal coefficients (see[1]). We regard the set of all reals as the Cantor set $2^\omega$. We denote by $\mathcal{M}$ and $\mathcal{N}$ the set of all meager subsets of $2^\omega$ and the set of all null subsets of $2^\omega$ respectively.

For functions $f$, $g$ in $\omega^\omega$ we write "$f \leq g$" to mean that $g$ dominates $f$ everywhere, that is, $f(n) \leq g(n)$ for all $n < \omega$. And we let "$f \leq^* g$" mean that $g$ eventually dominates $f$, that is, there exists an $n < \omega$ such that $f(m) \leq g(m)$ holds for all $m < \omega$ larger than $n$. We denote by $S$ the set of all non-decreasing functions $d$ in $\omega^\omega$ which diverges to infinity and $d(0) = 0$. We denote by $\mathcal{C}$ and $\mathcal{D}$ the Cohen forcing notion and the dominating forcing notion respectively[1]. For each ideal (or family if there is not a problem in particular) $\mathcal{I}$ on $2^\omega$ which contains all singletons, we denote by $\text{add}(\mathcal{I})$, $\text{cov}(\mathcal{I})$, $\text{non}(\mathcal{I})$ and $\text{cof}(\mathcal{I})$ the additivity, covering number,
uniformity and cofinality of $\mathcal{I}$ respectively which means that:

1. $\text{add}(\mathcal{I}) = \min\{|A| : A \subset \mathcal{I}, A \not\in \mathcal{I}\}$,
2. $\text{cov}(\mathcal{I}) = \min\{|A| : A \subset \mathcal{I}, A = 2^\omega\}$,
3. $\text{non}(\mathcal{I}) = \min\{|Y| : Y \subset 2^\omega, Y \not\in \mathcal{I}\}$,
4. $\text{cof}(\mathcal{I}) = \min\{|A| : A \subset \mathcal{I}, \forall B \in \mathcal{I}, \exists A \in A (B \subset A)\}$.

We have that $\text{add}(\mathcal{I}) \leq \text{cov}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ and $\text{add}(\mathcal{I}) \leq \text{non}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ for each ideal or family $\mathcal{I}$ on $2^\omega$ which contains all singletons.

We define some notation before we define the ideals $\mathcal{I}_f$ and $\mathcal{K}_f$.

**Definition 2.1** Let $f, g$ be functions in $\omega^\omega$.

1. We define the order "$\ll$" on $\omega^\omega$ by
   
   $f \ll g$ iff $\forall k < \omega \exists N < \omega \forall n \geq N (f(n^k) \leq g(n))$.
2. We define the order "$\ll'$" on $\omega^\omega$ by
   
   $g_\sigma(n) = |\sigma(n)|$ for all $n < \omega$.
3. For $\sigma \in (2^{<\omega})^\omega$, define the subset $\mathcal{Y}(\sigma) \subset 2^\omega$ by
   
   $\mathcal{Y}(\sigma) = \bigcap_{n < \omega} \bigcup_{m \geq n} \{x \in 2^\omega : s \subset x\}$, where $[s] = \{x \in 2^\omega : s \subset x\}$ for each $s \in 2^{<\omega}$.

Define the subsets $S(f)$, $T(f)$ and $\mathcal{U}(f)$ of $(2^{<\omega})^\omega$ by

- $S(f) = \{\sigma \in (2^{<\omega})^\omega : g_\sigma \gg f\}$,
- $T(f) = \{\sigma \in (2^{<\omega})^\omega : g_\sigma = f\}$.

**Definition 2.2** Let $f \in \omega^\omega$. Define the families $\mathcal{I}_f$, $\mathcal{J}_f$ and $\mathcal{K}_f$ on $2^\omega$ by

- $\mathcal{I}_f = \{X \subset 2^\omega : \exists \sigma \in S(f) X \subset \mathcal{Y}(\sigma)\}$,
- $\mathcal{J}_f = \{X \subset 2^\omega : \exists \sigma \in T(f) X \subset \mathcal{Y}(\sigma)\}$.

The following definition is not necessary for the definition of ideal $\mathcal{I}_f$. But it is the very useful.

**Definition 2.3** Let $f \in \omega^\omega$. For each $d \in S$, we define the functions $g_d^{(f)}$ and $h_d^{(f)} \in \omega^\omega$ by

- $g_d^{(f)}(n) = f(n^{k+2})$ if $n \in [d(k), d(k+1))$

for all $n < \omega$, respectively. If $g \in \omega^\omega$ is $g = g_d^{(f)}$ for some $d \in \omega^\omega$, then we say "$g$ is generated by $d$ (and $f$) for $\ll'$".
3 \( \text{cov} (\mathcal{I}_f) \), \( \text{cov} (\mathcal{J}_f) \) and bounding number \( b \)

In this section, we show that the ideal \( \mathcal{I}_f \) and the family \( \mathcal{J}_f \) are related to bounding number \( b \) intimately. For each \( d \in S \), \( g_{d}^{(f)} \gg f \) holds where \( g_{d}^{(f)} \) was introduced in chapter 2. In addition, for each \( g \gg f \) there exists a \( d \in S \) by the definitions of \( g_{d}^{(f)} \) and \( \ll \) such that \( g_{d}^{(f)} \leq^{*} g \). Therefore, the following hold.

**Lemma 3.1** For each family \( \mathcal{F} \subseteq \omega^\omega \) such that \( |\mathcal{F}| < b \) and \( \forall g \in \mathcal{F} \ (g \gg f) \), there exists \( d \in S \) such that \( \forall g \in \mathcal{F} \ (g_{d}^{(f)} \leq^{*} g) \).

**Proof of Lemma 3.1** Let \( \mathcal{F} \subseteq \omega^\omega \) satisfy \( \forall g \in \mathcal{F} \ (g \gg f) \) and \( |\mathcal{F}| < b \). For each \( g \in \mathcal{F} \), there exists \( d_g \in S \) such that \( g_{d_g}^{(f)} \leq g \). Since \( |\mathcal{F}| < b \), the family \( \{d_g \mid g \in \mathcal{F}\} \) is bounded family in \( \omega^\omega \). So there exists \( d \in S \) which dominates for all functions in \( \{d_g \mid g \in \mathcal{F}\} \). \( \square \) (Lemma 3.1)

**Lemma 3.2** There exists a family \( \mathcal{F} \subseteq \omega^\omega \) such that \( |\mathcal{F}| = b \) and \( \forall g \in \mathcal{F} \ (g \gg f) \) and \( \forall h \gg f \exists g \in \mathcal{F} \ (h \lleq g) \).

**Proof of Lemma 3.2** Take a unbounded family \( B \subseteq S \). Then a family \( \{g_{d}^{(f)} \mid d \in B\} \) is as desired.

\( \square \) (Lemma 3.2)

For all \( d \in S \), \( \text{cov} (\mathcal{I}_f) \leq \text{cov} (\mathcal{J}_{g_{d}^{(f)}}) \) holds by \( \mathcal{I}_f = \bigcup_{g \gg f} \mathcal{J}_g = \bigcup_{d \in S} \mathcal{J}_{g_{d}^{(f)}} \). By this, if \( \text{cov} (\mathcal{I}_f) \) is larger than \( b \), then \( \text{cov} (\mathcal{J}_{g_{d}^{(f)}}) \) is larger than \( b \) for all \( d \in S \). The inverse holds.

**Theorem 3.1** \( \text{cov} (\mathcal{I}_f) \geq b \) iff \( \text{cov} (\mathcal{J}_{g_{d}^{(f)}}) \geq b \) for all \( d \in S \).

**Proof of Theorem 3.1** \( \Rightarrow \): As above.

\( \Leftarrow \): Assume \( \text{cov} (\mathcal{I}_f) < b \). There exists a family \( \mathcal{F} \) such that \( |\mathcal{F}| = \text{cov} (\mathcal{I}_f) < b \) and \( |\mathcal{F}| = 2^\omega \). For each \( X \in \mathcal{F} \), there exists \( \sigma_X \) such that \( X \subseteq \sigma_X \). By Lemma 3.1, there exists \( d \in S \) such that \( \forall X \in \mathcal{F} \ g_{d}^{(f)} \leq^{*} \sigma_X \). For each \( X \in \mathcal{F} \), define \( \tau_X \in T (g_{d}^{(f)}) \) by \( \tau_X (n) = \sigma_X (n) | g_{d}^{(f)} (n) \). Then a family \( \{Y (\tau_X) \mid X \in \mathcal{F} \} \subseteq \mathcal{J}_{g_{d}^{(f)}} \) covers \( 2^\omega \). \( \square \) (Theorem 3.1)

However, it is easily proved that \( \text{cov} (\mathcal{I}_f) \geq b \) is independent from ZFC. \( \text{cov} (\mathcal{I}_f) = \omega_1 \) and \( b = \omega_1 \) hold in a generic model which is obtained by a forcing notion satisfying Laver property from a ground model with the continuum hypothesis. Also \( \text{cov} (\mathcal{I}_f) = b = \omega_1 \) holds in a generic model which is obtained by the Cohen forcing notion of any weight from a ground model with continuum hypothesis.

4 The forcing notion \( \mathbb{P}(d) \) for \( d \in S \) and \( \text{cov} (\mathcal{I}_f) \) and \( \text{non} (\mathcal{I}_f) \)

In this section, we discuss the covering number and the uniformity of ideal \( \mathcal{I}_f \) in the model obtained by a certain iteration of the forcing notion \( \mathbb{P}(d) \). We define the forcing notion \( \mathbb{P}(d) \) for \( d \in S \).
Definition 4.1 Let $d \in S$. Define the forcing notion $\mathbb{P}(d)$ by

$$\mathbb{P}(d) = \left\{ (s, F) \in 2^{<\omega} \times \left[ T(g_{d}^{(f)}) \right]^{<\omega} \mid |s| = f(|F|) \right\},$$

$$(s, F) \leq (s', F') \iff 1. s \supset s' \supset F'$$

$$2. \forall \sigma \in F' \forall n \in |F| \setminus |F'| \ [s \cup \{\sigma(n+1)\} \setminus \{f(n), f(n+1)\} \neq \sigma(n+1) | f(n), f(n+1)] .$$

Lemma 4.1 For all $d \in S$, the forcing notion $\mathbb{P}(d)$ is $\sigma$-linked. So it has the countable chain condition.

Proof of Lemma 4.1 Since $g_{d}^{(f)}(n+1) - g_{d}^{(f)}(n) > n$ for all $n < \omega$, holds that $\forall (s, F) \in \mathbb{P}(d) \forall F' \in \left[ T(g_{d}^{(f)}) \right]^{<\omega} \exists (t, H) \leq (s, F) (H = F \cup F')$.

Let $N < \omega$ and $g = g_{d}^{(f)}$. For each $t \in 2^{g(N)}$, $\psi \in \prod_{n \in |N, 2N|} [2^{g(n+1)-g(n)}]^{\leq N}$, define a subset $B_{t, \psi}$ of $\mathbb{P}(d)$ by

$$B_{t, \psi} = \{(s, F) \in \mathbb{P}(d) \mid s = t \psi = \{ \sigma(n+1) \mid g(n), g(n+1) \mid \sigma \in F \mid n \in |F|, 2|F|\} \}.$$

Clearly $\mathbb{P}(d) = \bigcup_{N<\omega} \bigcup \{ B_{t, \psi} \mid t \in 2^{g(N)}\psi \in \prod_{n \in |N, 2N|} [2^{g(n+1)-g(n)}]^{\leq N} \}$. We show that for all $N < \omega$, $t \in 2^{g(N)}$ and $\psi \in \prod_{n \in |N, 2N|} [2^{g(n+1)-g(n)}]^{\leq N}$, any two distinct conditions in $B_{t, \psi}$ are compatible. Let $(s, F)$, $(s', F')$ be in $B_{t, \psi}$ and $(s, F) \neq (s', F')$. By the definition of $B_{t, \psi}$,

$$s = s' = t |F| = |F'| = N$$

$$(\{ \sigma(n+1) \mid g(n), g(n+1) \mid \sigma \in F \mid n \in |F|, 2|F|\})$$

$$= (\{ \sigma(n+1) \mid g(n), g(n+1) \mid \sigma \in F' \mid n \in |F'|, 2|F'|\}).$$

There exists $(u, H) \leq (s, F)$ such that $H = F \cup F'$. Clearly $|F'| < |H| \leq 2N$. To prove $(u, H) \leq (s', F')$, let $\sigma \in F'$ and $n \in |H| \setminus |F'|$. Since $(u, H) \leq (s', F')$, $u \cup \{g(n), g(n+1)\} \neq \tau(n+1) | g(n), g(n+1) \}$ for all $\tau \in F$, that is, $u | g(n), g(n+1) \notin \psi(n)$.

But $\sigma(n+1) | g(n), g(n+1) \notin \psi(n)$.

Therefore $u | g(n), g(n+1) \notin \sigma(n+1) | g(n), g(n+1))$. \hfill \Box (Lemma 4.1)

For each $d \in S$, $\sigma \in T(g_{d}^{(f)})$ and $n < \omega$, define the subsets $D_{\sigma}$, $E_{n} \subset \mathbb{P}(d)$ as follows:

$$D_{\sigma} = \{ (s, F) \in \mathbb{P}(d) \mid \sigma \in F \},$$

$$E_{n} = \{ (s, F) \in \mathbb{P}(d) \mid |F| \geq n \}.$$

Lemma 4.2 For all $s \in S$, $\sigma \in T(g_{d}^{(f)})$ and $n < \omega$, the subsets $D_{\sigma}$ and $E_{n}$ are dense open sets in $\mathbb{P}(d)$.

Proof of Lemma 4.2 Let $\sigma \in T(g_{d}^{(f)})$, $n < \omega$ and $(s, F) \in \mathbb{P}(d)$. Take $F' \subset T(g_{d}^{(f)})$ such that $\sigma \in F'$ and $|F| \geq n$. There exists $(t, H) \leq (s, F)$ such that $H = F \cup F'$. Since $\sigma \in H$ and $|H| \geq n$, $(t, H) \in D_{\sigma}$ and $(t, H) \in E_{n}$. \hfill \Box (Lemma 4.2)
We are interested in the generic model of $P(d)$. Let $d \in S$ and $\dot{G}$ be the canonical generic $P(d)$-name. Define $P(d)$-name $\dot{a}_{\dot{G}}$ by

$$\models_{P(d)} \dot{a}_{\dot{G}} = \bigcup \left\{ s \mid \exists F (s, F) \in \dot{G} \right\} \in 2^\omega.$$  

Lemma 4.3 For all $d \in S$, $\models_{P(d)} \forall \sigma \in T(g_d^{(f)}) \cap V (\dot{a}_{\dot{G}} \notin Y(\sigma))$.

Proof of Lemma 4.3 Let $d \in S$, $\sigma \in T(g_d^{(f)})$ and $(s, F) \in P(d)$. By Lemma 4.2, there exists $(s', F') \leq (s, F)$ such that $\sigma \in F'$. To prove that $(s', F') \models_{P(d)} \sigma(n) \notin \dot{a}_{\dot{G}}$ for all $n > |F'|$, let $n > |F'|$. By Lemma 4.2, there exists $(s'', F'') \leq (s', F')$ such that $|F''| \geq n$. Then $(s'', F'') \models_{P(d)} s'' \in \dot{a}_{\dot{G}} \sigma^{\alpha}$, $g_d^{(f)}(n-1), g_d^{(f)}(n)) \neq \sigma(n) |(g_d^{(f)}(n-1), g_d^{(f)}(n))$. Therefore $(s'', F'') \models_{P(d)} \sigma(n) \notin \dot{a}_{\dot{G}}$. □(Lemma 4.3)

Lemma 4.4 For all $d \in S$, $\models_{P(d)} 2^\omega \cap V \in \mathcal{J}_{g_d^{(f)}}$.

Proof of Lemma 4.4 This is directly followed from the fact that $P(d)$ adds Cohen reals in

$$\prod_{n < \omega} 2^{g_d^{(f)}(n)}.$$  

□(Lemma 4.4)

To define a finite support iteration of $P(d)$, let $\kappa$ be an uncountable regular cardinal and $\pi$ be a bijection from $\kappa$ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta < \alpha$ for all $\alpha < \kappa$. Let $\pi_0$ and $\pi_1$ be the first and second coordinate of the value of $\pi$ respectively.

Assume the continuum hypothesis. We define $P_\kappa$ by $\kappa$-stage finite support iteration

$$\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa \rangle$$  

as follows:

Assume that $P_\beta$ and the $P_\beta$-names $\dot{d}_\xi^\beta$ for $\xi < \kappa$ with $\models_{P_\beta} \langle \dot{d}_\xi^\beta \mid \xi < \kappa \rangle$ be an enumeration of $\mathbb{S}^n$ are defined for all $\beta < \alpha$ in $\alpha$-stage. Define $\models_{P_\alpha} \dot{Q}_\alpha \simeq \mathbb{P}(\langle \dot{d}_{\pi\alpha(\beta)} \rangle)^* \mathcal{D}$.

Theorem 4.1 (CH) $\models_{P_\kappa} \kappa = b = \kappa \land \forall d \in S \text{ cov}(\mathcal{J}_{g_d^{(f)}}) = c$.

Therefore, it holds that $\models_{P_\kappa} \text{ cov}(\mathcal{I}_f) = c$ by theorem 3.1.

Proof of Theorem 4.1 Clearly $c = b = \kappa$ in $V[G_\kappa]$. Let $d \in S$, $\lambda < \kappa$ and a family $\left\{ X_\delta \mid \delta < \lambda \right\} \subset \mathcal{J}_{g_d^{(f)}}$ in $V[G_\kappa]$. There exists $\alpha < \kappa$ such that $X_\delta$ is coded by $\sigma_\delta \in T(g_d^{(f)})$ for each $\delta < \lambda$ in $V[G_\alpha]$. By Lemma 4.3, $\left\{ Y(\alpha) \mid \delta < \lambda \right\}$ does not cover $2^\omega$ in $V[G_{\alpha+1}]$. Hence $\left\{ X_\delta \mid \delta < \lambda \right\}$ does not cover $2^\omega$ in $V[G_\kappa]$. □(Theorem 4.1)

Theorem 4.2 (CH) $\models_{P_{\omega_2}} \text{non}(\mathcal{I}_f) = c$

Proof of Theorem 4.2 Clearly by Lemma 4.4. □(Theorem 4.2)
Property $E$ and $\text{cov}(I_f) = \omega_1$

In this section, we introduce a certain property for forcing notions which satisfy axiom A. A forcing notion with this property does not add a real which is not covered by all elements of $S(f)$ in ground model. This property is preserved in an iterated forcing. So the countable support iteration of forcing notions with this property does not lift up $\text{cov}(I_f)$. For example, the infinitely equal forcing notion $\mathbb{E}$ satisfies this property.

**Definition 5.1** Let forcing notion $P$ satisfy axiom A by the fusion orders $\langle \leq_n \mid n < \omega \rangle$. $P$ has property $E$ if there exists $\phi \in \omega^{\mathbb{P} \times \omega}$ such that

1. for all $p \in P$ and $n < \omega$, if $p \Vdash \dot{a} \in \mathbb{V}$ then
   
   there exist $q \leq_n p$ and a finite set $B$ such that $|B| \leq \phi(p, n)$ and $q \Vdash \dot{a} \in B$,

2. for all $p, q \in P$ and $n < \omega$, if $q \leq_n p$ then $\phi(q, n) = \phi(p, n)$.

**Lemma 5.1** Suppose that the axiom A forcing notion $P$ has property E. Then $P \Vdash \omega^\mathbb{A} \subset \bigcup \{Y(\tau) \mid \tau \in T(g) \cap \mathbb{V}\}$ for all strictly increasing function $g \in \omega^\mathbb{A}$. Therefore, $P \Vdash \omega^\mathbb{A} \subset \bigcup \{Y(\tau) \mid \tau \in S(f) \cap \mathbb{V}\}$.

**Proof of Lemma 5.1** Let $p \in P$ satisfy $p \Vdash \dot{x} \in 2^\omega$ and $g \in \omega^\omega$ be strictly increasing. By induction on $j < \omega$, define three sequences $\langle p_j \in P \mid j < \omega \rangle$, $\langle m_j < \omega \mid j < \omega \rangle$ and $\langle A_j \mid j < \omega \rangle$ as follows:

1. $p_0 = p$,
2. $p_{j+1} \leq_j p_j$,
3. $m_j = \sum_{i < j} \phi(p_i, i)$,
4. $A_j \subset 2^{\phi(m_j + \phi(p_j, j))}$,
5. $|A_j| \leq \phi(p_j, j)$,
6. $p_{j+1} \Vdash \dot{x} \upharpoonright \phi(m_j + \phi(p_j, j)) \in A_j$,

for all $j < \omega$. For each $j < \omega$, let $\{s_j^f \mid l < \phi(p_j, j)\}$ be a enumeration of $A_j$. There exists $q \in P$ such that $\forall j < \omega$ $q \leq_j p_j$.

We define $\sigma \in \langle 2^{\omega^\omega} \rangle^\omega$ by for each $n < \omega$, $\sigma(n) = s_j^f \upharpoonright \phi(n)$ where $n = m_j + l$. To prove that $q \Vdash \dot{x} \in Y(\sigma)$, let $n < \omega$. There exists $j < \omega$ such that $m_j \geq n$. Since $q \Vdash \dot{x} \upharpoonright \phi(m_j + \phi(p_j, j)) \in A_j$, there exist $q' \leq q$ and $l < \phi(p_j, j)$ such that $q' \Vdash \dot{x} \upharpoonright \phi(m_j + \phi(p_j, j)) = s_j^f \supset \sigma(m_j + l)$. □ (Lemma 5.1)

Let $\delta \leq \omega_2$. Let $P_\delta = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \delta \rangle$ be a $\delta$-stage countable support iteration such that $\dot{Q}_\alpha$ is defined by the forcing notion with property E for all $\alpha < \delta$. For $n < \omega$ and $F \in [\delta]^{\omega}$, $p \in P_\delta$ is $(n, F)$-good if there exists $h \in \omega^F$ such that $p \Vdash \phi \in \phi_n(p(\gamma), n) \leq h(\gamma)$ for all $\gamma \in F$ where $\phi_\gamma$ is $P_\gamma$-name for the function $\phi$ appeared in the definition of property E for $\dot{Q}_\gamma$. 

Lemma 5.2 Let $\delta \leq \omega_2$. For all $n < \omega$ and $F \in [\delta]^{<\omega}$, the set $\{ p \in P_\delta \mid p \text{ is } (n, F)\text{-good} \}$ is $(n, F)$-dense open in $P_\delta$.

Proof of Lemma 5.2 Since the property $E$ implies the strongly $\omega^\omega$-bounding, we can prove easily by induction on $\delta \leq \omega_2$. \(\square\) (Lemma 5.2)

By the lemma above, we may suppose only the condition that is $(n, F)$-good. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, define $h_{p,n,F} \in \omega^F$ by

(a) $p|\gamma \models_{\gamma} \varphi_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$,

(b) if $q \leq_{n,F} p$ then $h_{q,n,F}(\gamma) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$.

Lemma 5.3 Let $\delta \leq \omega_2$. There exists $\tilde{\varphi}_\delta \in \omega^{P_\delta \times \omega \times [\delta]^{<\omega}}$ such that

1. for all $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, if $p \models_\delta \dot{a} \in \mathbb{V}$ then there exists $q \leq_{n,F} p$ and a finite set $B$ such that $|B| \leq \tilde{\varphi}_\delta(p, n, F)$ and $q \models_\delta \dot{a} \in B$,

2. for all $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$, if $q \leq_{n,F} p$ then $\tilde{\varphi}_\delta(q, n, F) \leq \tilde{\varphi}_\delta(p, n, F)$.

Proof of Lemma 5.3 We prove by induction on $\delta \leq \omega_2$. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, we define $\tilde{\varphi}_\delta(p, n, F)$ as follows:

Case 1: $\delta$ is limit ordinal.

Let $\alpha = \max(F) + 1$. Then $F \subset \alpha$. By induction hypothesis, there exists $\tilde{\varphi}_\alpha \in \omega^{P_\alpha \times \omega \times [\alpha]^{<\omega}}$ such that (1) and (2). So we define $\tilde{\varphi}_\delta(p, n, F)$ by $\tilde{\varphi}_\alpha(p|\alpha, n, F)$.

We show that (1) and (2). (1): Let $p \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $p \models_\delta \dot{a} \in \mathbb{V}$. Suppose $\alpha = \max(F) + 1$. Since $p|\alpha \models_\alpha \dot{b} \in \mathbb{V}$, $f \in P_\alpha$ if $p \models_\delta \dot{a} = \dot{b}^\alpha$ for some $P_\alpha$-name $\dot{b}$ and $\tilde{f}$, there exist $r \leq_{n,F} p|\alpha$, finite set $B$ and $g \in P_\alpha$ such that $|B| \leq \tilde{\varphi}_\alpha(p|\alpha, n, F)$ and $r \models_{\alpha} \dot{b} \in B \land \tilde{f} = g^\alpha$. Let $q = r \cup g$. Then $q \leq_{n,F} p$ and $q \models_\delta \dot{a} \in B$.

(2): Let $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $q \leq_{n,F} p$. Suppose $\alpha = \max(F) + 1$. Then since $q \models_{n,F} p|\alpha$,

$\tilde{\varphi}_\delta(q, n, F) = \tilde{\varphi}_\alpha(q|\alpha, n, F)$

$\leq \tilde{\varphi}_\alpha(p|\alpha, n, F)$

$= \tilde{\varphi}_\delta(p, n, F)$

Case 2: $\delta = \gamma + 1$.

In the case of $F \subset \gamma$, we define in the same way as the case of that $\delta$ is limit ordinal.

Suppose $\gamma \in F$.

By induction hypothesis, there exists $\tilde{\varphi}_\gamma$ such that for all $p' \in P_\gamma$, $n' < \omega$ and $F' \in [\gamma]^{<\omega}$, if $p|\gamma \models_{\gamma} \dot{a} \in \mathbb{V}$, there exist $r \leq_{n,F,F_\gamma} p'$ and $B$ such that $|B| \leq \tilde{\varphi}_\gamma(p|\gamma, n, F \cap \gamma)$ and $r \models_{\gamma} \dot{a} \in B$.

So we define $\tilde{\varphi}_\delta(p, n, F)$ by $\tilde{\varphi}_\gamma(p|\gamma, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma)$.

We show that (1) and (2). (1): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good. In this case of $F \subset \gamma$, we can show in the same way as the case of that $\delta$ is limit ordinal.
Assume $\gamma \in F$. Also there exist $P_\gamma$-names $\dot{q}$ and $\dot{\hat{B}}$ such that $p\gamma \models "\dot{q} \leq p(\gamma) \land \dot{\hat{B}} \subseteq V \land \dot{\hat{B}} \leq \varphi_\gamma(p(\gamma), n) \land \dot{q} \models \dot{\alpha} \in \dot{\hat{B}}"$. By $p$ is $(n,F)$-good, $p\gamma$ is $(n,F \cap \gamma)$-good and $p\gamma \models "\dot{B} \leq \varphi_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)"$. Let $\{b_j \mid j < h_{p,n,F}(\gamma)\}$ be a sequence of $P_\gamma$-names for an enumeration of $\dot{B}$. That is $p\gamma \models "\dot{b_j} \in \dot{B} \subseteq V\lceil \gamma"$. By induction on $j < h_{p,n,F}$, we construct two sequences $\langle r_j \mid j < h_{p,n,F} \rangle$ and $\langle B_j \mid j < h_{p,n,F} \rangle$ such that

(a) $r_j \leq_{n,F \cap \gamma} r_{j-1}$ for all $j < h_{p,n,F}(\gamma)$,
(b) $|B_j| \leq \varphi'_\gamma(r_{j-1}, n, F \cap \gamma) \leq \varphi'_\gamma(p(\gamma), n, F \cap \gamma)$ for $j < h_{p,n,F}$,
(c) $r_j \models \gamma \models \dot{b_j} \in B_j$ for all $j < h_{p,n,F}$

Let $q = r_{h_{p,n,F}(\gamma)} \cup \{(\gamma, \dot{q})\}$ and $B = \cup \{B_j \mid j < h_{p,n,F}(\gamma)\}$. Clearly $q \models \dot{\alpha} \in B$ and

$|B| \leq \sum_{j < h_{p,n,F}} |B_j| \leq \sum_{j < h_{p,n,F}} \varphi'_\gamma(p(\gamma), n, F \cap \gamma) = \varphi'_\gamma(p(\gamma), n, F \cap \gamma) \cdot h_{p,n,F}(\gamma) = \varphi'_\delta(p, n, F)$.

(2): Let $n < \omega$, $F \subseteq [\delta]^{<\omega}$ and $p, q$ satisfy $(n,F)$-good and $q \leq_{n,F} p$. In the case of $F \subseteq \gamma$, we can show in the same way as the case of that $\delta$ is limit ordinal. Suppose $\gamma \in F$. Then

$\varphi'_\delta(q, n, F) = \varphi'_\gamma(q, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma) \leq \varphi'_\gamma(p, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma) = \varphi'_\delta(p, n, F)$.

$\square$(Lemma 5.3)

Theorem 5.1 $p \models P_\omega \subset \cup \{Y(\tau) \mid \tau \in T(g) \cap V\}$, for all strictly increasing function $g \in \omega^\omega$. Therefore, $p \models P_\omega \subset \cup \{Y(\tau) \mid \tau \in S(f) \cap V\}$.

Proof of Theorem 5.1 By Lemma 5.3, we can show in the same way as Lemma 5.1. $\square$(Theorem 5.1)

Corollary 5.4 (CH) $p \models "\text{cov}(I_f) = \text{cov}(J_g) = \omega_1"$ for all strictly increasing function $g \in \omega^\omega$.

6 The diagram of cardinal coefficients of $I_f$

In this section, we give the results for the cardinal coefficients of ideal $I_f$ of the forcing notions that we studied. Let $\kappa$ be an uncountable regular cardinal. We express the parts which we do not yet understand in '?'.
<table>
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<tr>
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<td>$\omega_1$</td>
<td>$\omega_2$</td>
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</tbody>
</table>

$\mathbb{O}(f)_\kappa$: the $\kappa$-stage finite support iteration of the forcing notion $\mathbb{O}(f)$ introduced by T. Yorioka,

$\mathbb{P}_\kappa$: the $\kappa$-stage finite support iteration of the forcing notion $\mathbb{P}(d)$ by bookkeeping method,

$\mathbb{C}_\kappa$: the Cohen forcing notion which adds $\kappa$ many Cohen reals,

$\mathbb{E}\mathbb{E}_{\omega_2}$: the $\omega_2$-stage countable support iteration of the infinitely equal forcing notion,

(\text{the infinitely equal forcing notion has property E}),

$\mathbb{S}_{\omega_2}$: the $\omega_2$-stage countable support iteration of the Sacks forcing notion.

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References
