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The covering number and the uniformity of the ideal $\mathcal{I}_f$

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1 Introduction

For the ideal $\mathcal{S}\mathcal{N}$ of strongly measure zero subsets of the real line, the cardinal coefficients have been studied[1]. But its cofinality had not been studied. In general, it may be larger than the continuum. Yorioka studied its cofinality(see [2]). One of his results is that the value of $\text{cof}(\mathcal{S}\mathcal{N})$ is equal to the dominating number for $\omega_1^{\omega_1}$ under the continuum hypothesis. In the process, he introduced ideals $\mathcal{I}_f$ for $f \in \omega^{\omega}$. These ideals were used in the proof. We are interested in the ideals $\mathcal{I}_f$ themselves. These ideals are subideals of the null ideal $\mathcal{N}$ and include $\mathcal{S}\mathcal{N}$. The properties of these ideals depend on $f$.

In this paper, we discuss the following contents. In section 3, we show a characterization of $\text{cov}(\mathcal{I}_f) \geq b$. In section 4, we define a forcing notion which has the countable chain condition. And with the results of section 3 we show that its $\omega_2$-stage finite support iteration by bookkeeping method lifts up $\text{cov}(\mathcal{I}_f)$ from a ground model with the continuum hypothesis. In section 5, we introduce a sufficient condition not to lift up $\text{cov}(\mathcal{I}_f)$ for forcing notions which satisfy axiom $\Lambda$.

2 Definitions and notation

Throughout this paper, we use the standard terminology for forcing of set theory and cardinal coefficients (see[1]). We regard the set of all reals as the Cantor set $2^{\omega}$. We denote by $\mathcal{M}$ and $\mathcal{N}$ the set of all meager subsets of $2^{\omega}$ and the set of all null subsets of $2^{\omega}$ respectively.

For functions $f$, $g$ in $\omega^{\omega}$ we write “$f \leq g$” to mean that $g$ dominates $f$ everywhere, that is, $f(n) \leq g(n)$ for all $n < \omega$. And we let “$f \leq^* g$” mean that $g$ eventually dominates $f$, that is, there exists an $n < \omega$ such that $f(m) \leq g(m)$ holds for all $m < \omega$ larger than $n$. We denote by $S$ the set of all non-decreasing functions $d$ in $\omega^{\omega}$ which diverges to infinity and $d(0) = 0$. We denote by $\mathcal{C}$ and $\mathcal{D}$ the Cohen forcing notion and the dominating forcing notion respectively[1]. For each ideal (or family if there is not a problem in particular) $\mathcal{I}$ on $2^{\omega}$ which contains all singletons, we denote by $\text{add}(\mathcal{I})$, $\text{cov}(\mathcal{I})$, $\text{non}(\mathcal{I})$ and $\text{cof}(\mathcal{I})$ the additivity, covering number,
uniformity and cofinality of $\mathcal{I}$ respectively which means that:

1. $\text{add}(\mathcal{I}) = \min \{|A| \mid A \subset \mathcal{I} \cup A \not\in \mathcal{I}\}$,
2. $\text{cov}(\mathcal{I}) = \min \{|A| \mid A \subset \mathcal{I} \cup A = 2^\omega\}$,
3. $\text{non}(\mathcal{I}) = \min \{|Y| \mid Y \subset 2^\omega \cup Y \not\in \mathcal{I}\}$,
4. $\text{cof}(\mathcal{I}) = \min \{|A| \mid A \subset \mathcal{I} \forall B \in \mathcal{I} \exists A \in A (B \subset A)\}$.

We have that $\text{add}(\mathcal{I}) \leq \text{cov}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ and $\text{add}(\mathcal{I}) \leq \text{non}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ for each ideal or family $\mathcal{I}$ on $2^\omega$ which contains all singletons.

We define some notation before we define the ideals $\mathcal{I}_f$ and $\mathcal{K}_f$.

**Definition 2.1** Let $f, g$ be functions in $\omega^\omega$.

1. We define the order "$\ll$" on $\omega^\omega$ by $f \ll g$ iff $\forall k < \omega \exists N < \omega \forall n \geq N (f(n^k) \leq g(n))$.
2. We define the order "$\ll_1$" on $\omega^\omega$ by $g_\sigma(n) = \lfloor \sigma(n) \rfloor$ for all $n < \omega$.
3. For $\sigma \in (2^{<\omega})^\omega$, define the subset $\mathcal{Y}(\sigma) \subset 2^\omega$ by $\mathcal{Y}(\sigma) = \bigcap \{[\sigma(n)]_n \mid n < \omega, m \geq n\}$, where $[s] = \{x \in 2^\omega \mid s \subset x\}$ for each $s \in 2^{<\omega}$.

Define the subsets $S(f)$, $T(f)$ and $U(f)$ of $(2^{<\omega})^\omega$ by

\[
S(f) = \{ \sigma \in (2^{<\omega})^\omega \mid g_\sigma \gg f \},
T(f) = \{ \sigma \in (2^{<\omega})^\omega \mid g_\sigma = f \}.
\]

**Definition 2.2** Let $f \in \omega^\omega$. Define the families $\mathcal{I}_f$, $\mathcal{J}_f$ and $\mathcal{K}_f$ on $2^\omega$ by

\[
\mathcal{I}_f = \{ X \subset 2^\omega \mid \exists \sigma \in S(f) X \subset \mathcal{Y}(\sigma) \},
\mathcal{J}_f = \{ X \subset 2^\omega \mid \exists \sigma \in T(f) X \subset \mathcal{Y}(\sigma) \}.
\]

The following definition is not necessary for the definition of ideal $\mathcal{I}_f$. But it is the very useful.

**Definition 2.3** Let $f \in \omega^\omega$. For each $d \in S$, we define the functions $g_d^{(f)}$ and $h_d^{(f)} \in \omega^\omega$ by $g_d^{(f)}(n) = f(n^{d+2})$ if $n \in [d(k), d(k+1))$ for all $n < \omega$, respectively. If $g \in \omega^\omega$ is $g = g_d^{(f)}$ for some $d \in \omega^{<\omega}$, then we say "$g$ is generated by $d$ (and $f$) for $\ll$".
3 \ cov(\mathcal{I}_f) , \ cov(\mathcal{J}_f) and bounding number b

In this section, we show that the ideal \( \mathcal{I}_f \) and the family \( \mathcal{J}_f \) are related to bounding number \( b \) intimately. For each \( d \in \mathcal{S} \), \( g_d^{(f)} \gg f \) holds where \( g_d^{(f)} \) was introduced in chapter 2. In addition, for each \( g \gg f \) there exists a \( d \in \mathcal{S} \) by the definitions of \( g_d^{(f)} \) and \( \ll \) such that \( g_d^{(f)} \leq^* g \). Therefore, the following hold.

Lemma 3.1 For each family \( \mathcal{F} \subset \omega^\omega \) such that \( |\mathcal{F}| < b \) and \( \forall g \in \mathcal{F} \ (g \gg f) \), there exists \( d \in \mathcal{S} \) such that \( \forall g \in \mathcal{F} \ (g_d^{(f)} \leq^* g) \).

Proof of Lemma3.1 Let \( \mathcal{F} \subset \omega^\omega \) satisfy \( \forall g \in \mathcal{F} \ (g \gg f) \) and \( |\mathcal{F}| < b \). For each \( g \in \mathcal{F} \), there exists \( d_g \in \mathcal{S} \) such that \( g_d^{(f)} \leq g \). Since \( |\mathcal{F}| < b \), the family \( \{d_g \mid g \in \mathcal{F}\} \) is bounded family in \( \omega^\omega \). So there exists \( d \in \mathcal{S} \) which dominates for all functions in \( \{d_g \mid g \in \mathcal{F}\} \). \( \Box \) (Lemma3.1)

Lemma 3.2 There exists a family \( \mathcal{F} \subset \omega^\omega \) such that \( |\mathcal{F}| = b \) and \( \forall g \in \mathcal{F} \ (g \gg f) \) and \( \forall h \gg f \exists g \in \mathcal{F} \ (h \leq^* g) \).

Proof of Lemma3.2 Take a unbounded family \( \mathcal{B} \subset \mathcal{S} \). Then a family \( \{g_d^{(f)} \mid d \in \mathcal{B}\} \) is as desired. \( \Box \) (Lemma3.2)

For all \( d \in \mathcal{S} \), \( \cov(\mathcal{I}_f) \leq \cov(\mathcal{J}_{g_d^{(f)}}) \) holds by \( \mathcal{I}_f = \bigcup_{g \gg f} \mathcal{J}_g = \bigcup_{d \in \mathcal{S}} \mathcal{J}_{g_d^{(f)}} \). By this, if \( \cov(\mathcal{I}_f) \) is larger than \( b \), then \( \cov(\mathcal{J}_{g_d^{(f)}}) \) is larger than \( b \) for all \( d \in \mathcal{S} \). The inverse holds.

Theorem 3.1 \( \cov(\mathcal{I}_f) \geq b \) if and only if \( \cov(\mathcal{J}_{g_d^{(f)}}) \geq b \) for all \( d \in \mathcal{S} \).

Proof of Theorem3.1 \( \implies \): As above.
\( \iff \): Assume \( \cov(\mathcal{I}_f) < b \). There exists a family \( \mathcal{F} \) such that \( |\mathcal{F}| = \cov(\mathcal{I}_f) < b \) and \( |\mathcal{F}| = 2^\omega \). For each \( X \in \mathcal{F} \), there exists \( \sigma_X \) such that \( X \subset \sigma_X \). By Lemma3.1, there exists \( d \in \mathcal{S} \) such that \( \forall X \in \mathcal{F} \ g_d^{(f)} \leq^* \sigma_X \). For each \( X \in \mathcal{F} \), define \( \tau_X \in \mathcal{T}(g_d^{(f)}) \) by \( \tau_X(n) = \sigma_X(n) \cdot g_d^{(f)}(n) \). Then a family \( \{\{\mathcal{Y}(\tau_X) \mid X \in \mathcal{F}\} \subset \mathcal{J}_{g_d^{(f)}} \) covers \( 2^\omega \). \( \Box \) (Theorem3.1)

However, it is easily proved that \( \cov(\mathcal{I}_f) \geq b \) is independent from ZFC. \( \cov(\mathcal{I}_f) = \omega_1 \) and \( b = \mathfrak{c} \) hold in a generic model which is obtained by a forcing notion satisfying Laver property from a ground model with the continuum hypothesis. Also \( \cov(\mathcal{I}_f) = b = \omega_1 \) holds in a generic model which is obtained by the Cohen forcing notion of any weight from a ground model with continuum hypothesis.

4 The forcing notion \( \mathbb{P}(d) \) for \( d \in \mathcal{S} \) and \( \cov(\mathcal{I}_f) \) and \( \non(\mathcal{I}_f) \)

In this section, we discuss the covering number and the uniformity of ideal \( \mathcal{I}_f \) in the model obtained by a certain iteration of the forcing notion \( \mathbb{P}(d) \). We define the forcing notion \( \mathbb{P}(d) \) for \( d \in \mathcal{S} \).
Definition 4.1 Let \( d \in S \). Define the forcing notion \( \mathbb{P}(d) \) by
\[
\mathbb{P}(d) = \left\{ (s, F) \in 2^{<\omega} \times \left[ T(g_d^{(f)}) \right]^{<\omega} \mid |s| = f(|F|) \right\},
\]
\[(s, F) \leq (s', F') \iff 1. s \supset s' F \supset F' \]
\[2. \forall \sigma \in F' \forall n \in |F| \setminus |F'| \left[ s \sigma(n+1) \neq (s, F) \right] \sigma(n+1) \neq \sigma(n+1) \left[ f(n), f(n+1) \right].\]

Lemma 4.1 For all \( d \in S \), the forcing notion \( \mathbb{P}(d) \) is \( \sigma \)-linked. So it has the countable chain condition.

Proof of Lemma 4.1 Since \( g_d^{(f)}(n+1) - g_d^{(f)}(n) > n \) for all \( n < \omega \), holds that \( \forall (s, F) \in \mathbb{P}(d) \forall F' \in \left[ T(g_d^{(f)}) \right]^{<\omega} \exists (s, H) \leq (s, F) \left( H = F \cup F' \right) \).

Let \( N < \omega \) and \( g = g_d^{(f)} \). For each \( t \in 2^{<\omega} \), \( \psi \in \prod_{n \in [N,2N) \left[ 2^{g(n+1) - g(n)} \right]^{\leq N} \), define a subset \( B_{t,\psi} \) of \( \mathbb{P}(d) \) by
\[B_{t,\psi} = \{ (s, F) \in \mathbb{P}(d) \mid s = s' = \langle \{ \sigma(n+1) \mid |g(n), g(n+1)| \sigma \in F \mid n \in |F|, 2|F| \rangle \} \).
\[\text{Clearly } \mathbb{P}(d) = \bigcup_{N \in \mathbb{N}} \bigcup \{ B_{t,\psi} \mid t \in 2^{<\omega}, \psi \in \prod_{n \in [N,2N)} \left[ 2^{g(n+1) - g(n)} \right]^{\leq N} \}.\]
We show that for all \( N < \omega \) and \( t \in 2^{<\omega} \) and \( \psi \in \prod_{n \in [N,2N)} \left[ 2^{g(n+1) - g(n)} \right]^{\leq N} \), any two distinct conditions in \( B_{t,\psi} \) are compatible. Let \( (s, F), (s', F') \) be in \( B_{t,\psi} \) and \( (s, F) \neq (s', F') \). By the definition of \( B_{t,\psi} \),
\[
s = s' = t |F| = |F'| = N \]
\[\langle \{ \sigma(n+1) \mid |g(n), g(n+1)| \sigma \in F \mid n \in |F|, 2|F| \rangle \rangle = \psi.\]

There exists \( (u, H) \leq (s, F) \) such that \( H = F \cup F' \). Clearly \( |F'| < |H| \leq 2N \). To prove \( (u, H) \leq (s', F') \), let \( \sigma \in F' \) and \( n \in |H| \setminus |F'| \). Since \( (u, H) \leq (s, F), u \mid g(n), g(n+1) \neq \sigma(n+1) \mid g(n), g(n+1) \) for all \( \sigma \in F \), that is, \( u \mid g(n), g(n+1) \in \psi \).

But \( \sigma(n+1) \mid g(n), g(n+1) \in \psi(n) \).

Therefore \( u \mid g(n), g(n+1) \neq \sigma(n+1) \mid g(n), g(n+1) \).

\[\Box \text{(Lemma 4.1)}\]

For each \( d \in S \), \( \sigma \in T(g_d^{(f)}) \) and \( n < \omega \), define the subsets \( D_\sigma, E_n \subset \mathbb{P}(d) \) as follows:
\[D_\sigma = \{ (s, F) \in \mathbb{P}(d) \mid \sigma \in F \}, \]
\[E_n = \{ (s, F) \in \mathbb{P}(d) \mid |F| \geq n \} \].

Lemma 4.2 For all \( s \in S \), \( \sigma \in T(g_d^{(f)}) \) and \( n < \omega \), the subsets \( D_\sigma \) and \( E_n \) are dense open sets in \( \mathbb{P}(d) \).

Proof of Lemma 4.2 Let \( \sigma \in T(g_d^{(f)}) \), \( n < \omega \) and \( (s, F) \in \mathbb{P}(d) \). Take \( F' \subset T(g_d^{(f)}) \) such that \( \sigma \in F' \) and \( |F| \geq n \). There exists \( (t, H) \leq (s, F) \) such that \( H = F \cup F' \). Since \( \sigma \in H \) and \( |H| \geq n \), \( t, H \in D_\sigma \) and \( t, H \in E_n \).

\[\Box \text{(Lemma 4.2)}\]
We are interested in the generic model of $\mathbb{P}(d)$. Let $d \in S$ and $\dot{G}$ be the canonical generic $\mathbb{P}(d)$-name. Define $\mathbb{P}(d)$-name $\dot{a}_G$ by

$$\models_{\mathbb{P}(d)} \dot{a}_G = \bigcup \left\{ s \mid \exists F \ (s, F) \in \dot{G} \right\} \in 2^\omega.$$  

**Lemma 4.3** For all $d \in S$, $\models_{\mathbb{P}(d)} \forall \sigma \in T(g_d^{(f)}) \cap \mathcal{V} \ (\dot{a}_G \notin Y(\sigma))$.

**Proof of Lemma 4.3** Let $d \in S$, $\sigma \in T(g_d^{(f)})$ and $(s, F) \in \mathbb{P}(d)$. By Lemma 4.4, there exists $(s', F') \leq (s, F)$ such that $\sigma \in F'$. To prove that $(s', F') \models_{\mathbb{P}(d)} \sigma(n) \notin \dot{a}_G$ for all $n > |F'|$, let $n > |F'|$. By Lemma 4.2, there exists $(s', F') \leq (s', F')$ such that $|F'| \geq n$. Then $(s'', F'') \models_{\mathbb{P}(d)} " s'' \in \dot{a}_G \sigma''[g_d^{(f)}(n-1), \ g_d^{(f)}(n)] \neq \sigma(n)[g_d^{(f)}(n-1), \ g_d^{(f)}(n)] "$. Therefore $(s'', F'') \models_{\mathbb{P}(d)} \sigma(n) \notin \dot{a}_G$. □(Lemma 4.3)

**Lemma 4.4** For all $d \in S$, $\models_{\mathbb{P}(d)} \exists \tau \in J_{d^{(f)}}$.

**Proof of Lemma 4.4** This is directly followed from the fact that $\mathbb{P}(d)$ adds Cohen reals in

$$\prod_{n<\omega} 2^{g_d^{(f)}(n)}.$$  

To define a finite support iteration of $\mathbb{P}(d)$, let $\kappa$ be an uncountable regular cardinal and $\pi$ be a bijection from $\kappa$ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta < \alpha$ for all $\alpha < \kappa$. Let $\pi_0$ and $\pi_1$ be the first and second coordinate of the value of $\pi$ respectively.

Assume the continuum hypothesis. We define $\mathbb{P}_{\kappa}$ by $\kappa$-stage finite support iteration

$$\langle P_\alpha, Q_\alpha \mid \alpha < \kappa \rangle$$

as follows:

Assume that $P_\beta$ and the $P_\beta$-names $d_\beta^\Xi$ for $\Xi < \kappa$ with $\models_{P_\beta} \langle d_\beta^\Xi \mid \Xi < \kappa \rangle$ be an enumeration of $\mathcal{S}^n$ are defined for all $\beta < \alpha$ in $\alpha$-stage. Define $\models_{\alpha} \dot{Q}_\alpha \simeq P(d_{\pi_0(\alpha)} \ast \Pi).$

**Theorem 4.1** (CH) $\models_{\mathbb{P}_{\kappa}} \exists \tau = b = \kappa \land \forall d \in S \ \text{cov}(J_{d^{(f)}}) = \tau.$ Therefore, it holds that $\models_{\mathbb{P}_{\kappa}} \text{cov}(J_f) = \tau$ by Theorem 3.1.

**Proof of Theorem 4.1** Clearly $\tau = \kappa$ in $V[G_\kappa]$. Let $d \in S$, $\lambda < \tau$ and a family

$$\{ X_\delta \mid \delta < \lambda \} \subset J_{d^{(f)}} \text{ in } V[G_\kappa].$$

There exists $\alpha < \kappa$ such that $X_\delta$ is coded by $\sigma_\delta \in T(g_d^{(f)})$ for each $\delta < \lambda$ in $V[G_\alpha]$. By Lemma 4.3, $\{ Y(\sigma_\delta) \mid \delta < \lambda \}$ does not cover $2^\omega$ in $V[G_{\alpha+1}]$. Hence

$$\{ X_\delta \mid \delta < \lambda \} \text{ does not cover } 2^\omega \text{ in } V[G_\kappa].$$ □(Theorem 4.1)

**Theorem 4.2** (CH) $\models_{\mathbb{P}_{\omega_2}} \text{non}(J_f) = \tau$

**Proof of Theorem 4.2** Clearly by Lemma 4.4. □(Theorem 4.2)
5 Property $E$ and $\text{cov}(I_f) = \omega_1$

In this section, we introduce a certain property for forcing notions which satisfy axiom $A$. A forcing notion with this property does not add a real which is not covered by all elements of $\mathcal{S}(f)$ in ground model. This property is preserved in an iterated forcing. So the countable support iteration of forcing notions with this property does not lift up $\text{cov}(I_f)$. For example, the infinitely equal forcing notion $\mathcal{E}$ satisfies this property.

**Definition 5.1** Let forcing notion $P$ satisfy axiom $A$ by the fusion orders $\langle \leq_n | n < \omega \rangle$. $P$ has property $E$ if there exists $\varphi \in \omega^{\mathfrak{P} \times \omega}$ such that

1. for all $p \in P$ and $n < \omega$, if $p \Vdash \dot{a} \in \mathcal{V}$ then there exist $q \leq_n p$ and a finite set $B$ such that $|B| \leq \varphi(p, n)$ and $q \Vdash \dot{a} \in B$.
2. for all $p, q \in P$ and $n < \omega$, if $q \leq_n p$ then $\varphi(q, n) = \varphi(p, n)$.

**Lemma 5.1** Suppose that the axiom $A$ forcing notion $P$ has property $E$.

Then $\Vdash P \ "2^\omega \subset \bigcup \{ \mathcal{Y}(\tau) | \tau \in \mathcal{T}(g) \cap \mathcal{V} \} "$ for all strictly increasing function $g \in \omega^\omega$. Therefore, $\Vdash P \ "2^\omega \subset \bigcup \{ \mathcal{Y}(\tau) | \tau \in \mathcal{S}(f) \cap \mathcal{V} \} "$.

**Proof of Lemma 5.1** Let $p \in P$ satisfy $p \Vdash \dot{a} \in 2^\omega$ and $g \in \omega^\omega$ be strictly increasing. By induction on $j < \omega$, define three sequences $\langle p_j \in P | j < \omega \rangle$, $\langle m_j \in \omega | j < \omega \rangle$ and $\langle A_j | j < \omega \rangle$ as follows:

(i) $p_0 = p$,

(ii) $p_{j+1} \leq_{j} p_j$,

(iii) $m_j = \sum_{i < j} \varphi(p_i, i)$,

(iv) $A_j \in 2^{\varphi(m_j + \varphi(p_j, j))}$,

(v) $|A_j| \leq \varphi(p_j, j)$,

(vi) $p_{j+1} \Vdash \dot{a} \in g(m_j + \varphi(p_j, j)) \in A_j$.

for all $j < \omega$. For each $j < \omega$, let $\{ s^j_l | l < \varphi(p_j, j) \}$ be an enumeration of $A_j$. There exists $q \in P$ such that $\forall j < \omega q \leq_{j} p_j$.

We define $\sigma \in (2^{\omega})^\omega$ by for each $n < \omega$, $\sigma(n) = s^j_l \upharpoonright g(n)$ where $n = m_j + l$. To prove that $q \Vdash \dot{a} \in \mathcal{Y}(\sigma)$, let $n < \omega$. There exists $j < \omega$ such that $m_j \geq n$. Since $q \Vdash \dot{a} \in g(m_j + \varphi(p_j, j)) \in A_j$, there exist $l < q$ and $l < \varphi(p_j, j)$ such that $q' \Vdash \dot{a} \in g(m_j + \varphi(p_j, j)) = s^j_l \supset \sigma(m_j + l)$.

$\Box$(Lemma 5.1)

Let $\delta \leq \omega_2$. Let $P_\delta = \langle P_\alpha, \dot{Q}_\alpha | \alpha < \delta \rangle$ be a $\delta$-stage countable support iteration such that $\dot{Q}_\alpha$ is defined by the forcing notion with property $E$ for all $\alpha < \delta$. For $n < \omega$ and $F \in [\delta]^\omega$, $p \in P_\delta$ is $(n, F)$-good if there exists $h \in \omega^F$ such that $p \Vdash \varphi_\gamma \varphi_\gamma(p(\gamma), n) \leq h(\gamma)$ for all $\gamma \in F$ where $\varphi_\gamma$ is $P_\gamma$-name for the function $\varphi$ appeared in the definition of property $E$ for $\dot{Q}_\gamma$. 

Lemma 5.2 Let $\delta \leq \omega_2$. For all $n < \omega$ and $F \in [\delta]^{<\omega}$, the set $\{ p \in P_\delta \mid p \text{ is } (n, F)\text{-good} \}$ is $(n, F)$-dense open in $P_\delta$.

Proof of Lemma 5.2 Since the property $E$ implies the strongly $\omega^\alpha$-bounding, we can prove easily by induction on $\delta \leq \omega_2$.

By the lemma above, we may suppose only the condition that is $(n, F)$-good. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, define $h_{p,n,F} \in \omega^F$ by

(a) $p\upharpoonright \gamma \models \varphi_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$,

(b) if $q \leq_{n,F} p$ then $h_{q,n,F}(\gamma) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$.

Lemma 5.3 Let $\delta \leq \omega_2$. There exists $\hat{\varphi}_\delta \in \omega^{P_\delta \times [\delta]^{<\omega}}$ such that

(1) for all $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, if $p \upharpoonright \delta \hat{\epsilon} \in V$ then there exist $q \leq_{n,F} p$ and a finite set $B$ such that $|B| \leq \varphi_{\alpha}(p,n,F)$ and $q \upharpoonright \delta \hat{\epsilon} \in B$,

(2) for all $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$, if $q \leq_{n,F} p$ then $\varphi_\delta(q,n,F) \leq \varphi_\delta(p,n,F)$.

Proof of Lemma 5.3 We prove by induction on $\delta \leq \omega_2$. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good, we define $\hat{\varphi}_\delta(p,n,F)$ as follows:

Case 1: $\delta$ is limit ordinal.

Let $\alpha = \max(F) + 1$. Then $F \subset \alpha$. By induction hypothesis, there exists $\varphi_\alpha \in \omega^{P_\delta \times [\alpha]^{<\omega}}$ such that (1) and (2). So we define $\hat{\varphi}_\delta(p,n,F)$ by $\varphi_\alpha(p, \alpha, n, F)$.

We show that (1) and (2). (1): Let $p \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $p \upharpoonright \delta \hat{\epsilon} \in V$. Suppose $\alpha = \max(F) + 1$. Since $p \upharpoonright \alpha \models b \in V \lor \hat{f} \in P_\delta \forall \hat{\alpha} \models \hat{a} = b$ for some $P_\delta$-name $\hat{b}$ and $\hat{f}$, there exist $r \leq_{n,F} p \upharpoonright \alpha$, finite set $B$ and $g \in P_\delta$ such that $|B| \leq \varphi_\alpha(p, \alpha, n, F) = \varphi_\alpha(p, n, F)$ and $r \upharpoonright \alpha \models \hat{b} \in B \land \hat{f} = g$. Let $q = r \cup g$. Then $q \leq_{n,F} p$ and $q \upharpoonright \delta \hat{\epsilon} \in B$.

(2): Let $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $q \leq_{n,F} p$. Suppose $\alpha = \max(F) + 1$. Then since $q \upharpoonright \alpha \leq_{n,F} p \upharpoonright \alpha$,

$\varphi_\delta(q,n,F) \leq \varphi_\alpha(q,\alpha, n, F) \leq \varphi_\alpha(p, \alpha, n, F) = \varphi_\delta(p, n, F)$

Case 2: $\delta = \gamma + 1$.

In the case of $F \subset \gamma$, we define in the same way as the case of that $\delta$ is limit ordinal.

Suppose $\gamma \in F$.

By induction hypothesis, there exists $\varphi_\gamma$ such that for all $p' \in P_\gamma$, $n' < \omega$ and $F' \in [\gamma]^{<\omega}$, if $p \upharpoonright \gamma \models \hat{\alpha} \in V$, there exist $r \leq_{n,F \cap \gamma} p'$ and $B$ such that $|B| \leq \varphi_\gamma(p, \gamma, n, F \cap \gamma)$ and $r \upharpoonright \gamma \models \hat{\alpha} \in B$.

So we define $\hat{\varphi}_\delta(p, n, F)$ by $\overline{\varphi}(p, \gamma, n, F \cap \gamma) \cdot \varphi_{p,n,F}(\gamma)$.

We show that (1) and (2). (1): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and $p$ with $(n, F)$-good. In the case of $F \subset \gamma$, we can show in the same way as the case of that $\delta$ is limit ordinal.
Assume $\gamma \in F$. Also there exist $P_\gamma$-names $\dot{q}$ and $\dot{B}$ such that $p \models_{\gamma} \varphi(p(\gamma), n) \land \dot{B} \in V \land |\dot{B}| \leq \varphi(p(\gamma), n) \land \dot{q} \Vdash_{\gamma} \dot{a} \in \dot{B}$. By $p$ is $(n, F)$-good, $p \models_{\gamma}$ is $(n, F \cap \gamma)$-good and $p \models_{\gamma} |\dot{B}| \leq \varphi(p(\gamma), n) \leq h_{p, n, F}(\gamma)$. Let $\langle b_j \mid j < h_{p, n, F}(\gamma) \rangle$ be a sequence of $P_\gamma$-names for an enumeration of $\dot{B}$. That is $p \models_{\gamma} \{ b_j \mid j < h_{p, n, F}(\gamma) \} = \dot{B} \in V$. By induction on $j < h_{p, n, F}$, we construct two sequences $\langle r_j \mid j < h_{p, n, F} \rangle$ and $\langle B_j \mid j < h_{p, n, F} \rangle$ such that (let $r_{-1} = p \models_{\gamma}$)

(a) $r_j \leq_{n, F \cap \gamma} r_{j-1}$ for all $j < h_{p, n, F}(\gamma)$,
(b) $|B_j| \leq \varphi(p(r_{j-1}, n, F \cap \gamma)) \leq \varphi(p(\gamma, n, F \cap \gamma))$ for $j < h_{p, n, F}$,
(c) $r_j \models_{\gamma} b_j \in B_j$ for all $j < h_{p, n, F}$.

Let $q = r_{h_{p, n, F}(\gamma)-1} \cup \{(\gamma, \dot{q})\}$ and $B = \bigcup \{ B_j \mid j < h_{p, n, F}(\gamma) \}$. Clearly $q \models_{\dot{a}} \dot{a} \in B$ and

$|B| \leq \sum_{j < h_{p, n, F}} |B_j| \leq \sum_{j < h_{p, n, F}} \varphi(p(q, n, F \cap \gamma)) = \varphi(p(q, n, F \cap \gamma)) \cdot h_{p, n, F}(\gamma) = \varphi(p, n, F).

(2): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and $p, q$ satisfy $(n, F)$-good and $q \leq_{n, F} p$. In the case of $F \subset \gamma$, we can show in the same way as the case of that $\delta$ is limit ordinal. Suppose $\gamma \in F$. Then

$\varphi(q, n, F) = \varphi(q(\gamma, n, F \cap \gamma)) \cdot h_{q, n, F}(\gamma) \leq \varphi(p(\gamma, n, F \cap \gamma)) \cdot h_{p, n, F}(\gamma) = \varphi(p, n, F).

Theorem 5.1 $\models_{\mathcal{P}_2} 2^{\omega} \subset \bigcup \{ Y(\tau) \mid \tau \in T(g) \cap V \}$, for all strictly increasing function $g \in \omega^{\omega}$. therefore, $\models_{\mathcal{P}_2} 2^{\omega} \subset \bigcup \{ Y(\tau) \mid \tau \in S(f) \cap V \}$.

Proof of Theorem5.1 By Lemma5.3, we can show in the same way as Lemma5.1.

Corollary 5.4 (CH) $\models_{\mathcal{P}_2} \text{"cov}(I_f) = \text{cov}(J_g) = \omega_1$ for all strictly increasing function $g \in \omega^{\omega}$.

6 The diagram of cardinal coefficients of $I_f$

In this section, we give the results for the cardinal coefficients of ideal $I_f$ of the forcing notions that we studied. Let $\kappa$ be an uncountable regular cardinal. We express the parts which we do not yet understand in '?'.

□(Lemma5.3)

□(Theorem5.1)
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<th>cov</th>
<th>non</th>
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<tr>
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</table>

$\mathbb{O}(f)_{\kappa}$: the $\kappa$-stage finite support iteration of the forcing notion $\mathbb{O}(f)$ introduced by T. Yorioka,

$P_{\kappa}$: the $\kappa$-stage finite support iteration of the forcing notion $P(d)$ by bookkeeping method,

$C_{\kappa}$: the Cohen forcing notion which adds $\kappa$ many Cohen reals,

$EE_{\omega_2}$: the $\omega_2$-stage countable support iteration of the infinitely equal forcing notion,

$S_{\omega_2}$: the $\omega_2$-stage countable support iteration of the Sacks forcing notion.

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References
