The partition property of $\mathcal{P}_{\kappa} \lambda$

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Abstract

We study a relationship between the partition property of $\mathcal{P}_{\kappa} \lambda$ and the Shelah property.

1 Introduction

The partition property of $\mathcal{P}_{\kappa} \lambda$ was introduced by Jech [5] as a generalization of the classical partition property of cardinal. In this paper we study a relation between the partition property and the Shelah property of $\mathcal{P}_{\kappa} \lambda$, the Shelah property is defined by Carr [2] as a generalization of weakly compactness. It is well-known that there is an essential connection between the partition property of a cardinal and weakly compactness: for a cardinal $\kappa$, $\kappa$ is weakly compact iff $\kappa \rightarrow (\kappa)^2_2$. In Carr [4] observed such connection for various partition property and large cardinal property of $\mathcal{P}_{\kappa} \lambda$, including the Shelah property. We will try more deep analysis. Let $\text{NSh}_{\kappa \lambda}$ is the set of all $X \subseteq \mathcal{P}_{\kappa} \lambda$ such that $X$ is not Shelah.

Main Theorem 1 Let $I = \{ X \subseteq \mathcal{P}_{\kappa} \lambda : X \not\rightarrow (\mathcal{I}^+_{\kappa \lambda})^2_2 \}$. Assume $\lambda \geq \kappa$ is regular, $\lambda^{<\lambda} = \lambda$ but not weakly compact. Then there exists a club $C$ of $\mathcal{P}_{\kappa} \lambda$ such that $I|C = \text{NSh}_{\kappa \lambda}$.

Main Theorem 2 Assume $\lambda \geq \kappa$ is regular and $\lambda^{<\lambda} = \lambda$. Then the following are equivalent:

1. $\kappa$ is $\lambda$-Shelah,

2. $C \not\rightarrow (\mathcal{I}^+_{\kappa \lambda})^2_2$ for every club $C$ of $\mathcal{P}_{\kappa} \lambda$.

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Theorem 2 shows the Shelah property of $\mathcal{P}_\kappa\lambda$ is right analogue of weakly compactness. In this sense, Theorem 2 is not surprising. However Theorem 1 is interesting, since if $\lambda = \kappa$ then it must false; in fact if $\lambda = \kappa$ the partition ideal $I$ in Theorem 1 is just unbounded ideal over $\mathcal{P}_\kappa\lambda$, and $\text{NS}_{\kappa\lambda}$ is just the weakly compact ideal. Further Theorem 2 shows that the partition ideal $I$ can be locally normal, but $I$ itself cannot be normal. These results indicate that the partition ideal $I$ over $\mathcal{P}_\kappa\lambda$ has a strange structure under GCH. Note that, if GCH fails, the partition ideal can have a simple form, unbounded ideal (see Shioya [9]).

we will give a partial answer of a question of 5.5 in Carr [4] with a method which will be used to prove theorems.

2 Preliminaries

We refer to Kanamori [7] for general background and basic notation. Throughout this paper, $\kappa$ denotes an inaccessible cardinal and $\lambda$ a cardinal $\geq \kappa$.

An ideal over $\mathcal{P}_\kappa\lambda$ means that $\kappa$-complete fine ideal over $\mathcal{P}_\kappa\lambda$ in this paper. For an ideal $I$ over $\mathcal{P}_\kappa\lambda$, $I^*$ denotes the dual filter of $I$ and $I^* = \mathcal{P}(\mathcal{P}_\kappa\lambda) \setminus I$. An element of $I^*$ is called $I$-positive set. $\text{NS}_{\kappa\lambda}$ is the set of all $X \subseteq \mathcal{P}_\kappa\lambda$ such that $X$ is non-stationary (not unbounded) in $\mathcal{P}_\kappa\lambda$.

**Definition 2.1** For $x, y \in \mathcal{P}_\kappa\lambda$, we define $x < y$ if $x \subseteq y$ and $|x| < |y \cap \kappa|$. For an ideal $I$ on $\mathcal{P}_\kappa\lambda$, $I$ is strongly normal if for all $X \in I^*$ and $<\text{-regressive } f : X \rightarrow \mathcal{P}_\kappa\lambda$, that is, $f(x) < x$ for all $x \in X$ with $|x \cap \kappa| > 0$, there exists $y \in \mathcal{P}_\kappa\lambda$ such that $\{x \in X : f(x) = y\} \in I^*$. □

For $x \in \mathcal{P}_\kappa\lambda$, we denote $\mathcal{P}_x = \{y \in \mathcal{P}_\kappa\lambda : y < x\}$. If $x \cap \kappa$ is a regular cardinal, then properties of $\mathcal{P}_\kappa\lambda$ can be translated into $\mathcal{P}_x$ naturally. For example, $X \subseteq \mathcal{P}_x$ is stationary if for all $f : x \times x \rightarrow \mathcal{P}_x$ there exists $y \in X$ such that $\bigcup f^{\uparrow}(y \times y) \subseteq y$.

**Definition 2.2** For $X \subseteq \mathcal{P}_\kappa\lambda$, $X$ is Shelah if for all $\langle f_x : x \in X \rangle$ with $f_x : x \rightarrow x$, there exists $f : \lambda \rightarrow \lambda$ such that the set $\{x \in X : f(y) = f_x(y)\}$ is unbounded for all $y \in \mathcal{P}_\kappa\lambda$. We say that $\kappa$ is $\lambda$-Shelah if $\mathcal{P}_\kappa\lambda$ is Shelah.

$\text{NS}_{\kappa\lambda}$ is the set of all $X \subseteq \mathcal{P}_\kappa\lambda$ such that $X$ is not Shelah. □

**Fact 2.3** (Carr [2, 3])

1. $\text{NS}_{\kappa\lambda}$ is a normal ideal over $\mathcal{P}_\kappa\lambda$. Moreover it is strongly normal if $\text{cf}(\lambda) \geq \kappa$.

2. if $\kappa$ is $2^{\lambda^{<\kappa}}$-Shelah then $\kappa$ is $\lambda$-supercompact,

3. if $\kappa$ is $\lambda$-supercompact then $\kappa$ is $\lambda$-Shelah. □

(2) of the above fact shows that the Shelah property of $\mathcal{P}_\kappa\lambda$ is a very strong property.
Fact 2.4 (Abe [1]) \( \{ x \in \mathcal{P}_\kappa \lambda : \forall \alpha \in x (|x \cap \alpha| < |x|) \} \in \text{NSh}_\kappa^* \). \( \square \)

Now we define the partition property of \( \mathcal{P}_\kappa \lambda \).

Definition 2.5 Let \( n \) be a natural number \( > 0 \). For \( X \subseteq \mathcal{P}_\kappa \lambda \),

\[ [X]^\kappa_n = \{ \{ x_1, \ldots, x_n \} \subseteq X : x_1 < \cdots < x_n \} \]

For a function \( f \) on \( [X]^\kappa_n \), \( H \) is homogeneous set for \( f \) if \( H \subseteq X \) and \( |f^{-1}[H]^\kappa_n| = 1 \), and \( H \) is called \( x \)-homogeneous if \( f^{-1}[H]^\kappa_n = \{ x \} \) for some \( x \). \( \square \)

When an element of \( [X]^\kappa_n \) is written as \( \{ x_1, \ldots, x_n \} \), it is assumed that \( x_1 < \cdots < x_n \). For \( \{ x_1, \ldots, x_n \} \in [X]^\kappa_n \) and a function \( f \) on \( [X]^\kappa_n \), we shall write \( f(x_1, \ldots, x_n) \) instead of \( f(\{ x_1, \ldots, x_n \}) \).

Definition 2.6 Let \( A \subseteq \mathcal{P}(\mathcal{P}_\kappa \lambda) \). For a natural number \( n \), an ordinal \( \alpha \) and \( X \subseteq \mathcal{P}_\kappa \lambda \), we say that \( X \rightarrow (A)^\alpha_n \) holds if for all \( f : [X]^\kappa_n \rightarrow \alpha \) there exists a homogeneous set \( Y \in A \) for \( f \).

For \( B \subseteq \mathcal{P}(\mathcal{P}_\kappa \lambda) \), \( B \rightarrow (A)^\alpha_n \) holds if \( X \rightarrow (A)^\alpha_n \) holds for all \( X \in B \).

We say that \( \text{Part}(\kappa, \lambda)^n_\kappa \) holds if \( \mathcal{P}_\kappa \lambda \rightarrow (\mathcal{I}^+_\kappa \lambda)^n_\kappa \) holds, and \( \text{Part}^*(\kappa, \lambda)^n_\kappa \) holds if \( \mathcal{P}_\kappa \lambda \rightarrow (\text{NS}^+_{\kappa \lambda})^n_\kappa \) holds.

As usual, \( \not \rightarrow \) means the negation of the corresponding partition property.

Remark that Jech’s partition property was defined with the order \( \subseteq \), not \( \rightarrow \). The partition property with \( \subseteq \) is stronger than with \( \rightarrow \), but the author does not know that there is an essential difference between those properties.

Fact 2.7 (Carr [4], Jech [5], Magidor [8])

1. If \( \text{Part}(\kappa, \lambda)^2_\kappa \) holds for some \( \lambda \) then \( \kappa \) is weakly compact,

2. if \( \text{Part}(\kappa, \lambda)^3_\kappa \) holds for all \( \lambda \) then \( \kappa \) is strongly compact,

3. \( \kappa \) is supercompact iff \( \text{Part}^*(\kappa, \lambda)^2_\kappa \) holds for all \( \lambda \). \( \square \)

Fix \( n \) a natural number \( > 0 \) and put \( I = \{ X \subseteq \mathcal{P}_\kappa \lambda : X \rightarrow (\mathcal{I}^+_\kappa \lambda)^n_\kappa \} \). Then it is easy to check that \( I \) forms an ideal over \( \mathcal{P}_\kappa \lambda \). \( I \) is often called the partition ideal over \( \mathcal{P}_\kappa \lambda \).

3 The Shelah property and the partition property

We start proofs of Theorem 1 and 2. First we prove that the Shelah property of \( \mathcal{P}_\kappa \lambda \) implies the partition property.
Lemma 3.1 Assume $\lambda$ is regular and $\lambda^{<\lambda} = \lambda$. For $X \subseteq \mathcal{P}_\kappa \lambda$, if $X$ is Shelah then $X$ satisfies the following property: for any $\langle f_x : x \in X \rangle$ with $f_x : x \to x$ there exists $f : \lambda \to \lambda$ such that for all $\alpha < \lambda \{ x \in X : f|x \cap \alpha = f_x|x \cap \alpha \} \in \text{NSh}^+_{\kappa \lambda}$.

Proof: Fix $\langle f_x : \xi < \lambda \rangle$ an enumeration of $\bigcup \{ \alpha : \alpha < \lambda \}$. Let $Z = \{ x \in \mathcal{P}_\kappa \lambda : \forall \alpha \in x \forall f : x \cap \alpha \to x \exists \xi \in x (f = f_\xi((x \cap \alpha))) \}$. First we claim $Z \in \text{NSh}^+_{\kappa \lambda}$. Assume not. By the normality of $\text{NSh}^+_{\kappa \lambda}$ there exists $\alpha < \lambda$ such that $Y = \{ x \in \mathcal{P}_\kappa \lambda : \exists f_x : x \cap \alpha \to x \forall \xi \in x (f_x \neq f_\xi((x \cap \alpha))) \} \in \text{NSh}^+_{\kappa \lambda}$. For each $x \in Y$, let $g_x : x \to x \cap \alpha$ satisfying $f_x(g_x(\xi)) \neq f_\xi(g_x(\xi))$. Then by the Shelah property of $Y$, there exists $f : \alpha \to \lambda$ and $g : \lambda \to \alpha$ such that $\{ x \in Y : f_x|y = f|y, g_x|y = g|y \}$ is unbounded for any $y \in \mathcal{P}_\kappa \lambda$. Then $f = f_\xi$ for some $\xi < \lambda$. Take $y \in \mathcal{P}_\kappa \lambda$ such that $y$ is closed under $g$ and $\xi \in y$. Then we can take $x \in Y$ such that $y \subseteq x$, $f_x|y = f_\xi|y$ and $g_x|y = g|y$. Then $g(\xi) = g_x(\xi) \in y$, hence $f_x(g(\xi)) = f_\xi(g(\xi))$ holds. But this contradict to the definition of $g_x$, namely $f_x(g_x(\xi)) \neq f_\xi(g_x(\xi))$.

Now let $X \in \text{NSh}^+_{\kappa \lambda}$. We may assume that $X \subseteq Z$. For given $\langle f_x : x \in X \rangle$, define $\langle g_x : x \in X \rangle$ with $g_x : x \to x$ by $f_x|x \cap \xi = f_x(\xi)|x \cap \xi$. By a theorem of Johnson [6], there exists $g : \lambda \to \lambda$ such that for any $y \in \mathcal{P}_\kappa \lambda \{ x \in X : g_x|y = g|y \} \in \text{NSh}^+_{\kappa \lambda}$. Now define $f : \lambda \to \lambda$ by $f(\xi) = f_\eta(\xi)$ for some $\eta > \xi$. It is easy to see that $f$ is well-defined. We see that $f$ has the desired property. Let $\alpha < \lambda$. Take $y \in \mathcal{P}_\kappa \lambda$ such that $\alpha \in y$, $\operatorname{sup}(y) > \alpha$ and closed under $g$. Then $W = \{ x \in X : y \subseteq x, g|y = g_x|y \} \in \text{NSh}^+_{\kappa \lambda}$. Let $x \in W$. Then $f_x|x \cap \alpha = f_x(\alpha)|x \cap \alpha$.

Hence by the definition of $f$, $f(\xi) = f_x(\xi)$ holds for any $\xi \in x \cap \alpha$. \square

Assume $\lambda^{<\lambda} = \lambda$. Let $\langle x_\xi : \xi < \lambda \rangle$ be an enumeration of $\mathcal{P}_\kappa \lambda$. Then by the strong normality of $\text{NSh}^+_{\kappa \lambda}$, we have $\{ x \in \mathcal{P}_\kappa \lambda : \mathcal{P}_x = \{ x_\xi : \xi \in x \} \} \in \text{NSh}^+_{\kappa \lambda}$. Hence we have the following:

Cor. 3.2 Assume $\lambda$ is regular and $\lambda^{<\lambda} = \lambda$. Let $X \in \text{NSh}^+_{\kappa \lambda}$. Then $X$ has the following property: for any $\langle f_x : x \in X \rangle$ with $f_x : x \to \mathcal{P}_x$ there exists $f : \lambda \to \mathcal{P}_\kappa \lambda$ such that for all $\alpha < \lambda \{ x \in X : f_x|x \cap \alpha = f|x \cap \alpha \} \in \text{NSh}^+_{\kappa \lambda}$. \square

Now we shall prove more strong partition property from the Shelah property. For $X \subseteq \mathcal{P}_\kappa \lambda$ and $A, B \subseteq \mathcal{P}(\mathcal{P}_\kappa \lambda)$, we say that $X \overset{\leq}{\rightarrow} (A, B)^n$ holds if for any $f : [X]^n \to 2$, either there exists a 0-homogeneous set $H$ for $f$ with $H \in A$ or 1-homogeneous set $H$ for $f$ with $H \in B$.

Lemma 3.3 Assume $\lambda$ is regular and $\lambda^{<\lambda} = \lambda$. For $X \subseteq \mathcal{P}_\kappa \lambda$, if $X$ is Shelah then $X \overset{\leq}{\rightarrow} (\text{NSh}^+_{\kappa \lambda}, \mathcal{I}^+_{\kappa \lambda})^2$ holds.

Proof:Fix an enumeration $\langle x_\xi : \xi < \lambda \rangle$ of $\mathcal{P}_\kappa \lambda$. For each $x \in X$, we may assume that $\mathcal{P}_x = \{ x_\xi : \xi \in x \}$. Let $f : [X]^n \to 2$. For $x \in X$, we define $g_x : x \cap \alpha_x \to X \cap \mathcal{P}_x$ and $\alpha_x \leq \operatorname{sup}(x)$ by the induction on $\xi \in x$. Let $\xi \in x$ and assume $g_x|x \cap \xi$ is defined. If there exists $z \in \mathcal{P}_x \cap X$ such that $x_\xi \subseteq z$,
\( \forall \eta \in x \cap \xi \left( z \not\subseteq g_z(\eta) \right) \) and \( \forall \eta \in x \cap \xi \left( g_z(\eta) < z \Rightarrow f(g_z(\eta), z) = f(z, x) = 1 \right) \), then set \( g_z(\xi) = z \). If there is no such \( z \in X \cap \mathcal{P}_x \), then we set \( \alpha_x = \xi \). Assume \( g_z(\xi) \) is defined for any \( \xi \in x \), then we set \( \alpha_x = \sup(x) \).

Note that \( \{g_z(\xi) : \xi \in x \cap \alpha_x \} \cup \{x\} \) is 1-homogeneous for \( f \) and if \( \alpha_x = \sup(x) \) then \( \{g_z(\xi) : \xi \in x \cap \alpha_x \} \) is unbounded in \( \mathcal{P}_x \).

Now we consider the following two cases.

**Case 1.** \( \{x \in X : \alpha_x < \sup(x)\} \in \text{NSh}_{\kappa \lambda}^{+} \). By the normality of \( \text{NSh}_{\kappa \lambda}^{+} \), there exists \( \alpha \in \lambda \) such that \( \{x \in X : \alpha_x = \alpha\} \in \text{NSh}_{\kappa \lambda}^{+} \). Then by Cor. 3,2, there exists \( g : \alpha \rightarrow X \) such that \( Y = \{x \in X : g_x[x\cap\alpha = g[x\cap\alpha] \in \text{NSh}_{\kappa \lambda}^{+}\} \). Let \( H = \{x \in Y : x_{\alpha} < x, \forall \xi < \alpha \left( g(\xi) < x \Rightarrow \xi \in x \right) \} \). Then it is easy to see that \( H \in \text{NSh}_{\kappa \lambda}^{+} \). We claim that \( H \) is 0-homogeneous set. Let \( x, y \in H \) with \( x < y \). Assume \( f(x, y) = 1 \).

If \( f(g_y(\eta), x) = 1 \) for all \( \eta \in y\cap\alpha \) with \( g_y(\eta) < x \), then \( x \) witness that \( \alpha \in \text{dom}(g_y) \).

Hence there must exist \( \eta \in y\cap\alpha \) such that \( x \) witness that \( g_y(\eta) < x \) and \( f(g_y(\eta), x) = 0 \). Since \( g_y(\eta) = g(\eta) < x \), we have \( \eta \in x \). Thus \( g_x(\eta) = g_y(\eta) = g(\eta) \) holds. However \( f(g_x(\eta), x) = 1 \) by the definition of \( g_x \), a contradiction.

**Case 2.** \( \{x \in X : \alpha_x = \sup(x)\} \in \text{NSh}_{\kappa \lambda}^{+} \). Let \( Y = \{x \in X : \alpha_x = \sup(x)\} \).

Then for \( x \in Y \), \( \{g_x(\xi) : \xi \in x\} \) is a 1-homogeneous set for \( f \) and unbounded in \( \mathcal{P}_x \).

By Cor. 3,2, there exists \( g : \lambda \rightarrow X \) such that \( \{x \in Y : g_x[x\cap\alpha = g[x\cap\alpha] \in \text{NSh}_{\kappa \lambda}^{+}\} \) for all \( \alpha \in \lambda \). Let \( H = g^\alpha \lambda \). Then it is easy to see that \( H \) is an unbounded 1-homogeneous set for \( f \).

Next we will show that if \( \text{NSh}_{\kappa \lambda}^{*} \rightarrow (\text{I}_{\kappa \lambda}^{+})^2 \) then \( \kappa \) is \( \lambda \)-Shelah. To see this, we need some lemmata.

**Lemma 3.4** Let \( \mu \) be a cardinal with \( \kappa \leq \mu \leq \lambda \). Assume \( \lambda^{<\mu} = \lambda \). Then there exists a club \( C \) of \( \mathcal{P}_{\kappa \lambda} \) such that for every unbounded subset \( X \subseteq C \), \( \alpha < \mu \) and \( f : \alpha \rightarrow \mathcal{P}_{\kappa \lambda} \), \( X \setminus \{x \in X : \forall \xi \in x \cap \alpha \left( f(\xi) < x \right) \} \) is not unbounded.

**Proof:** Let \( \vec{h} = \langle h_\xi : \xi < \lambda \rangle \) be an enumeration of \( \bigcup_{\eta<\mu}^\eta \lambda \) and \( \vec{x} = \langle x_\xi : \xi < \lambda \rangle \) an enumeration of \( \mathcal{P}_{\kappa \lambda} \). We can enumerate with \( \lambda \)-length by our cardinal arithmetic assumption. Let \( \theta \) be a sufficiently large regular cardinal and \( M = \langle \mathcal{H}_\theta, \epsilon, \kappa, \lambda, \vec{h}, \vec{x} \rangle \). Let \( C = \{N \cap \lambda : N < M, \ |N| = \kappa, N \cap \kappa \in \kappa \} \). Then \( C \) forms a club. Note that if \( N \cap \kappa \in C \) and \( x \in N \cap \mathcal{P}_{\kappa \lambda} \) then \( x < N \cap \lambda \). We shall check that \( C \) satisfies the conclusion of lemma. Fix \( X \) an unbounded subset of \( C \). Let \( \alpha < \mu \) and \( f : \alpha \rightarrow \mathcal{P}_{\kappa \lambda} \). For \( f \), define \( h : \alpha \rightarrow \lambda \) by \( f(\xi) = x_h(\xi) \). Then there exists \( \xi < \lambda \) such that \( h = h_\xi \). By the definition of \( C \), for each \( x \in X \) if \( \xi \in x \) then \( h^*(x \cap \alpha) \subseteq x \). Further if \( N < M \), \( N \cap \lambda \in X \) and \( h(\xi) \in N \), then \( f(\xi) = x_{y(\xi)} \in N \), hence \( f(\xi) \in N \). Therefore for each \( N \cap \lambda \in X \) if \( \xi \in N \cap \lambda \) then \( \forall \xi \in N \cap \alpha (f(\xi) < N \cap \lambda) \). Since \( X \setminus \{x \in X : \xi \in X \} \) is not unbounded, we have done. \( \square \)
Lemma 3.5 Let $\mu$ be a cardinal with $\kappa \leq \mu \leq \lambda$ and assume $\lambda^{\lt \mu} = \lambda$. Then there exists some club $C$ of $\mathcal{P}_\kappa \lambda$ such that for any $X \subseteq C$ if $X \not\rightarrow (\mathcal{I}_\kappa^{\lt \lambda})^{n+1}$ holds then $X$ has the following property: whenever $(a_t : t \in [X]^{n}_\kappa)$ with $a_t \subseteq \text{min}(t) \cap \mu$ there exists an unbounded subset $H \subseteq X$ and $A \subseteq \mu$ such that

$$\forall \xi < \mu \exists z_\xi \in \mathcal{P}_\kappa \lambda \forall t \in [H]^{n}_\kappa (z_\xi < \text{min}(t) \Rightarrow A \cap \text{min}(t) \cap \xi = a_t \cap \xi)).$$

Here $\text{min}(t)$ is the minimal element of $t$ with respect to $\lt$.

Proof: Let $C$ be a club shown in Lemma 3.4. Let $X \subseteq C$ be such that $X \not\rightarrow (\mathcal{I}_\kappa^{\lt \lambda})^{n+1}$. We will see that $X$ has the desired property. Let $(a_t : t \in [X]^{n}_\kappa)$ with $a_t \subseteq \text{min}(t) \cap \mu$. We define $f : [X]^{n}_\kappa \rightarrow 2$ as: for $\{x_1, \ldots, x_{n+1}\} \in [X]^{n+1}_\kappa$, if $a_{x_1 \cdots x_n} = a_{x_2 \cdots x_{n+1}} \cap x_1$, then let $f(x_1, \ldots, x_{n+1}) = 0$. Assume $a_{x_1 \cdots x_n} \neq a_{x_2 \cdots x_{n+1}} \cap x_1$ and let $\alpha$ be the minimal element of $a_{x_1 \cdots x_n} \triangle(a_{x_2 \cdots x_{n+1}} \cap x_1)$. If $\alpha \in a_{x_2 \cdots x_{n+1}}$, then $f(x_1, \ldots, x_{n+1}) = 0$. If $\alpha \in a_{x_1 \cdots x_n}$, then $f(x_1, \ldots, x_{n+1}) = 1$.

By $X \not\rightarrow (\mathcal{I}_\kappa^{\lt \lambda})^{n+1}$, we can take an unbounded homogeneous set $H$ for $f$. Now we will construct $A \subseteq \mu$ and $(z_\xi : \xi < \mu)$ by the induction on $\xi < \mu$. Assume $A \cap \eta$ and $z_\eta$ is defined for any $\eta < \xi$ and satisfies the following:

1. $z_\eta \in \mathcal{P}_\kappa \lambda$,
2. for any $t \in [H]^{n}_\kappa$, if $z_\xi < \text{min}(t)$ then $A \cap \eta \cap \text{min}(t) = a_t \cap \eta$.

We define $z_\xi$ and decide whether $\xi \in A$ or not. First assume that $H$ is 0-homogeneous. Let $H' = \{x \in H : \exists \eta \in x \cap \xi (z_\eta \not\in x)\}$. By Lemma 3.4, $H'$ is not unbounded. Fix $z \in H$ such that $\xi \in z$ and $z \not\in x$ for all $x \in H'$. Note that if $x \in H$ and $z < x$ then $\forall \eta \in x \cap \xi (z_\eta < x)$.

**Case 1.** If there exists $\{y_1, \ldots, y_n\} \in [H]^{n}_\kappa$ such that $z < y_1$ and $\xi \in a_{y_1 \cdots y_n}$, then set $z_\xi = y_n$ and $\xi \in A$. We check that $A \cap \xi + 1$ and $z_\xi$ satisfies the induction hypotheses. Let $\{x_1, \ldots, x_n\} \in [H]^{n}_\kappa$ such that $z_\xi < x_1$. Then since $z < y_1 < \cdots < y_n = z < x_1 < \cdots < x_n$, $\forall \eta \in x_i \cap \xi (z_\eta < x_i)$ and $\forall \eta \in y_\xi \cap \xi (z_\eta < y_\xi)$ hold for any $i \leq n$. Hence by the induction hypotheses, for any $\eta \in y_1 \cap \xi$, $A \cap y_1 \cap \eta = a_{y_1 \cdots y_n} \cap \eta$. This means that $A \cap y_1 \cap \xi = a_{y_1 \cdots y_n} \cap \xi$. By the same reason we have $A \cap y_2 \cap \xi = a_{y_2 \cdots y_n, x_1} \cap \xi$. In particular $a_{y_1 \cdots y_n} \cap \xi = a_{y_2 \cdots y_n, x_1} \cap y_1 \cap \xi$. Further $H$ is 0-homogeneous and $\xi \in a_{y_1 \cdots y_n, x_1} \cap \xi$ must be an element of $a_{y_2 \cdots y_n, x_1}$. Repeating this argument $n$-times, we have $\xi \in a_{x_1 \cdots x_n}$ and $A \cap (\xi + 1) = a_{x_1 \cdots x_n} \cap (\xi + 1)$. If $H$ is 1-homogeneous, then we consider the following two cases: there exists $\{y_1, \ldots, y_n\} \in [H]^{n}_\kappa$ such that $z < y_1$ and $\xi \not\in a_{y_1 \cdots y_n}$, and otherwise. The rest follows from a similar argument. □
Now we will prove Theorem 1 and 2 using the above lemma.

**Lemma 3.6** Assume $\lambda$ is regular, $\lambda^{<\lambda} = \lambda$ and $\lambda$ is not strong limit. Let $I = \{X \subseteq P_{\kappa}\lambda : X \not\subseteq (I_{\kappa\lambda}^{+})^{2}\}$. Then there exists a club $D$ of $P_{\kappa}\lambda$ such that $\text{NSh}_{\kappa\lambda} = I|D$ holds.

**Proof:** Since $\lambda^{<\lambda} = \lambda$ and $\lambda$ is not strong limit, there exists $\nu < \lambda$ such that $2^{\nu} = \lambda$. Fix such a $\nu$. Fix $\langle B_{\xi} : \xi < \lambda \rangle$ a bijective enumeration of $\mathcal{P}(\nu)$. Fix $\pi : \lambda \times \nu \to \lambda$ a bijection. Now let $C$ be a club in Lemma 3.5 with the case $\mu = \lambda$. Let $\theta$ be a sufficiently large regular cardinal and $M = \langle H_{\theta}, \in, \kappa, \lambda, \pi, (B_{\xi} : \xi < \lambda), \ldots \rangle$. Now let $D = \{N \cap \lambda \in C : N \prec M, |N| < \kappa, N \cap \kappa \in \kappa \}$. Then $D$ is a club subset of $P_{\kappa}\lambda$. We will show that $D$ works. Note that for any $x \in D$, $\pi''((x \times (x \cap \nu)) = x$ and for all $\xi, \eta \in x$, if $\xi \neq \eta$ then $B_{\xi} \cap x \neq B_{\eta} \cap x$.

Since $\text{NSh}_{\kappa\lambda}$ is normal, if $X \in \text{NSh}_{\kappa\lambda}^{+}$ then $X \cap D \in \text{NSh}_{\kappa\lambda}^{+}$. Hence by Lemma 3.3 $X \in (I|D)^{+}$ holds.

To see the converse, let $X \in (I|D)^{+}$. We may assume that $X \subseteq D$. Let $\langle f_{x} : x \in X \rangle$ with $f_{x} : x \to x$. For $x \in X$, let $a_{x} = \pi''\langle \eta, \zeta \rangle : \eta \in x, \zeta \in B_{f_{\eta}(\zeta)} \cap x \rangle \subseteq x$. Then by Lemma 3.5, there exists an unbounded $H \subseteq X$, $A \subseteq \lambda$ and $\langle z_{\xi} : \xi < \lambda \rangle$ such that $\forall \xi < \lambda \forall x \in H (z_{\xi} < x \Rightarrow A \cap x \cap \xi = a_{x} \cap \xi)$. For each $\eta < \lambda$, define $A_{\eta} \subseteq \nu$ by $\zeta \in A_{\eta}$ iff $\pi\langle \eta, \zeta \rangle \in A$. Then define $f : \lambda \to \lambda$ by $f_{x} = B_{f_{\xi}(\eta)}$. We claim for any $y \in P_{\kappa}\lambda$ there exists $x \in H$ such that $y \subseteq x$ and $f|y = f_{x}|y$, this completes a proof. Let $y \in P_{\kappa}\lambda$. If necessary we may assume that $y$ is closed under $f$. Take a large $\xi < \lambda$ such that $\sup(y) < \xi$ and $\pi''(\xi \times \nu) = \xi$. Then we can take $x \in H$ such that $z_{\xi} < x$, $y < x$ and $A \cap x \cap \xi = a_{x} \cap \xi$. We check that $f|y = f_{x}|y$. Note that $\pi''((x \cap \xi) \times (x \cap \mu)) = x \cap \xi$. Let $\eta \in y$. Since $f(\eta), f_{x}(\eta) \in x$, it suffices to show that $B_{f_{\eta}(\xi)} \cap x = B_{f_{\xi}(\eta)} \cap x$. Let $\xi \in B_{f_{\xi}(\eta)} \cap x$. Then $\pi\langle \eta, \zeta \rangle \in a_{\eta}$. Since $\eta < \xi$, $\pi\langle \eta, \zeta \rangle \in a_{x} \cap \xi = A \cap x \cap \xi$. Then by the definition of $A$, we have $\zeta \in B_{f_{\eta}(\xi)}$. The converse can be verified by the same argument.

**Lemma 3.7** Assume $\lambda$ is regular, $\lambda^{<\lambda} = \lambda$ and there exists a $\lambda$-Aronszajn tree. Let $I = \{X \subseteq P_{\kappa}\lambda : X \not\subseteq (I_{\kappa\lambda}^{+})^{2}\}$. Then there exists a club $D$ of $P_{\kappa}\lambda$ such that $\text{NSh}_{\kappa\lambda} = I|D$ holds.

**Proof:** Fix $T = \langle T, \leq_{T} \rangle$ a $\lambda$-Aronszajn tree. We may assume that $T = \lambda$. For $\alpha < \lambda$, $T_{\alpha}$ denotes the $\alpha$-th level of $T$. Fix $\pi : \lambda \times \lambda \to \lambda$ a bijection. Let $\theta$ be a sufficiently large regular cardinal. Let $M = \langle H_{\theta}, \in, \kappa, \lambda, T, \pi, \ldots \rangle$. Let $C$ be a club in Lemma 3.5 with the case $\mu = \lambda$. Let $D = \{N \cap \lambda \in C : N \prec M, |N| < \kappa, N \cap \kappa \in \kappa \}$. We will show that $D$ works.

$I|D \subseteq \text{NSh}_{\kappa\lambda}$ is Lemma 3.3. Let $X \in (I|D)^{+}$ be such that $X \subseteq D$. For $\langle f_{x} : x \in X \rangle$ with $f_{x} : x \to x$, define $a_{x}$ for $x \in X$ as follows: for $\eta \in x$, take $a_{x}^{\eta} \in T_{f_{\eta}(\eta)} \cap x$. Note that such an $a_{x}^{\eta}$ exists since $x = N \cap \lambda$ for some $N \prec M$. Let
\( b_\eta^x = \{ \beta \in T : \beta \leq_T \alpha_\eta^x \} \cap x \). Hence \( \alpha_\eta^x \) is the max element of \( b_\eta^x \) with respect to the order \( \leq_T \). Now let \( a_x = \pi^{\mathrm{a}}(\langle \eta, \zeta \rangle : \eta \in x, \zeta \in b_\eta^x) \subseteq x \).

We take an unbounded \( H \subseteq X, A \subseteq \lambda \langle z_\xi : \xi < \lambda \rangle \) by Lemma 3.5. For each \( \eta < \lambda \), define \( B_\eta \subseteq \lambda \) by \( \bar{\eta} \in B_\eta \iff \bar{\eta} \in A \).

Fix \( \eta < \lambda \). We check \( B_\eta \) forms a chain of \( T \). Let \( \zeta_1, \zeta_2 \in B_\eta \). Take \( \xi < \lambda \) such that \( \pi(\langle \eta, \zeta_1 \rangle), \pi(\langle \eta, \zeta_2 \rangle) < \xi \). Then we can take \( x \in H \) such that \( \pi(\langle \eta, \zeta_1 \rangle), \pi(\langle \eta, \zeta_2 \rangle) \subseteq x \) and \( a_x \cap \xi = A \cap \xi \cap \eta \). Thus \( \pi(\langle \eta, \zeta_1 \rangle), \pi(\langle \eta, \zeta_2 \rangle) \subseteq A \cap \xi \cap \eta \). By the definition of \( a_x \), both \( \zeta_1 \) and \( \zeta_2 \) belong \( b_\eta^x \), hence \( \zeta_1, \zeta_2 \) are compatible. As the above argument, we can show that if \( \zeta_1 \in B_\eta \) and \( \zeta_2 \leq_T \zeta_1 \) then \( \zeta_2 \in B_\eta \).

Since \( T \) is an Aronsjazn tree, \( B_\eta \) is not cofinal in \( T \). Take \( \delta_\eta < \lambda \) such that \( B_\eta \subseteq \bigcup_{\beta \leq \delta_\eta} T_\beta \) but \( B_\eta \cap T_{\delta_\eta} = \emptyset \). Now we claim that \( \delta_\eta \) is a successor ordinal, hence \( B_\eta \) has the max element. Assume not. Take \( \xi < \lambda \) such that \( \eta, \delta_\eta < \xi \), \( \bigcup_{\beta < \delta_\eta} T_\beta \subseteq \xi \) and \( \pi(\xi \times \xi) \subseteq \xi \). Take \( x \in H \) such that \( \delta_\eta \in x \) and \( A \cap \xi = a_x \cap \xi \). By Lemma 3.4 and the fact \( H \subseteq C \), we may assume that for each \( \beta \in x \cap \delta_\eta \), the \( \beta \)-th element of \( B_\eta \) (with respect to \( \leq_T \)) is in \( x \). If \( f_x(\eta) \geq \delta_\eta \), there exists \( \gamma \in b_\eta \cap x \cap \xi \).

But since \( \bigcup_{\beta < \delta_\eta} T_\beta \cap x \subseteq \xi \), we have \( \pi(\langle \eta, \gamma \rangle) \subseteq x \cap \xi \subseteq x \cap \xi \). Hence \( \gamma \in B_\eta \cap T_{\delta_\eta} \neq \emptyset \), a contradiction. Thus \( f_x(\eta) < \delta_\eta \). If \( f_x(\eta) + 1 < \delta_\eta \), then \( f_x(\eta) + 1 \in \xi \). Hence we can take \( x \cap T_{f_x(\eta)+1} \cap B_\eta \). Then \( \gamma < \xi \). Thus \( \pi(\langle \eta, \gamma \rangle) \subseteq A \cap \xi \cap \eta \). However then \( \gamma \in b_\eta \cap T_{f_x(\eta)+1} \), a contradiction. Therefore we have \( \delta_\eta = f_x(\eta) + 1 \). Further notice that this arguments indicates the max element of \( b_\eta^x \) is equal to of \( B_\eta \).

For \( \eta < \lambda \), let \( \alpha_\eta \) be the max element of \( B_\eta \). Now define \( f : \lambda \to \lambda \) by \( f(\eta) = \) the height of \( \alpha_\eta \). We will see that for any \( y \in P_\lambda \), there exists \( x \in H \) such that \( y \subseteq x \) and \( f|y = f_x|y \). Let \( y \in P_\lambda \). If necessary we may assume that \( y \) is closed under \( f \). Take a large \( \xi < \lambda \) such that \( \sup(y) < \xi \) and \( f^\xi \subseteq \xi \). Then we can take \( x \in H \) such that \( y \subseteq x \) and \( A \cap \xi = a_x \cap \xi \). As the above argument, we may assume that for any \( \eta \in y \), the max element of \( b_\eta^x \) is equal to of \( B_\eta \). Then by the definition of \( b_\eta^x \) and \( f \), we have \( f_x(\eta) = f(\eta) \) holds for all \( \eta \in y \). \( \square \)

This completes the proof of Main Theorem 1.

**Cor. 3.8** Assume \( \lambda \) is regular, \( \lambda^{< \lambda} = \lambda \) but not weakly compact. Then for \( X \subseteq P_\lambda \), the following are equivalent:

1. \( X \) is Shelah,
2. \( (\text{NS}_{\lambda})^X \bullet \xrightarrow{\leq} (I^{+}_{\lambda})^2 \) holds,
3. \( X \xrightarrow{\leq} (\text{NSh}^+_{\lambda})^2 \) holds,
4. \( X \xrightarrow{\leq} (\text{NSh}^+_{\lambda})^2 \) holds. \( \square \)
Proof: (4) ⇒ (3) is trivial. (1) ⇒ (4) is Lemma 3.3. (2) ⇒ (1) follows from Lemma 3.6 and 3.7.

(3) ⇒ (2). Assume $X \cap C \not\rightarrow (I^+_{\kappa\lambda})^2$ for some club $C$ of $\mathcal{P}_\kappa\lambda$. Since (3) holds, it must hold that $X \setminus C \not\rightarrow (\text{NS}^*_{\kappa\lambda}, I^+_{\kappa\lambda})^2$. However this is impossible; consider the constant function $f : [X \setminus C]^2_\kappa \rightarrow \{0\}$. □

In the next section, we will prove that we cannot delete the assumption "$\lambda$ is not weakly compact" of the above Lemma.

For a proof of Theorem 2, we must prove the case that $\lambda$ is weakly compact. To see this, we need the following lemma.

Lemma 3.9 Let $\nu$ be a cardinal with $\kappa \leq \nu < \lambda$ and assume $\lambda^\nu = \lambda$. If $\text{NS}^*_{\kappa\lambda} \not\rightarrow (I^+_{\kappa\nu})^2$ holds then $\mathcal{P}_\kappa\nu$ satisfies the following property: whenever $\langle a_x : x \in \mathcal{P}_\kappa\nu \rangle$ with $a_x \subseteq x$, there exists $A \subseteq \nu$ such that $\{x \in \mathcal{P}_\kappa\nu : a_x = A \cap x\}$ is unbounded in $\mathcal{P}_\kappa\nu$.

Remark that the above property of $\mathcal{P}_\kappa\nu$ is known as almost ineffability (see Carr [3]). Almost ineffability of $\mathcal{P}_\kappa\nu$ is stronger than the Shelah property, so the above lemma also shows that if $\lambda^\nu = \lambda$ and $\text{NS}^*_{\kappa\lambda} \not\rightarrow (I^+_{\kappa\lambda})^2$ holds then $\kappa$ is $\nu$-Shelah.

Proof: Let $\langle a_x : x \in \mathcal{P}_\kappa\nu \rangle$ be a sequence such that $a_x \subseteq x$. For each $x \in \mathcal{P}_\kappa\lambda$, let $b_x = a_{x\cap\nu} \subseteq x \cap \nu$. Then by Lemma 3.5 with the case $\nu^+ = \mu$, there exists unbounded $H \subseteq \mathcal{P}_\kappa\lambda$, $B \subseteq \nu$ and $z \in \mathcal{P}_\kappa\lambda$ such that for any $x \in H$ if $z < x$ then $b_z = B \cap x$. Let $H^* = \{x \cap \nu : x \in H, z < x\}$. Then it is easy to see that $H^*$ is unbounded in $\mathcal{P}_\kappa\nu$ and for all $x \in H^*$, $a_x = B \cap x$. □

Lemma 3.10 Assume $\lambda$ is weakly compact. Then the followings are equivalent:

(1) $\kappa$ is $\lambda$-Shelah,

(2) $\text{NS}^*_{\kappa\lambda} \not\rightarrow (I^+_{\kappa\lambda})^2$ holds.

Proof: The case that $\lambda = \kappa$ is well-known. Thus we may assume that $\lambda > \kappa$. (1) ⇒ (2) is Lemma 3.3. We see (2) ⇒ (1). By Lemma 3.9, $\kappa$ is $\mu$-Shelah for any $\mu < \lambda$. Now assume that $\kappa$ is not $\lambda$-Shelah. Let $\vec{f} = \langle f_x : x \in \mathcal{P}_\kappa\lambda \rangle$ be a counterexample of the Shelah property of $\mathcal{P}_\kappa\lambda$. Consider the structure $(V_{\lambda}, \in, \vec{f}, \mathcal{P}_\kappa\lambda)$. The assertion that "$\vec{f}$ is a counterexample of the Shelah property of $\mathcal{P}_\kappa\lambda$" can be describable as $\Pi^1_1$-sentence over $(V_{\lambda}, \in, \vec{f}, \mathcal{P}_\kappa\lambda)$. Since weakly compact cardinal is $\Pi^1_1$-indescribable, this assertion is reflected to $\mu$ for some inaccessible $\mu < \lambda$. However this means that $\kappa$ is not $\mu$-Shelah, a contradiction. □

Therefore we conclude the following:

Cor. 3.11 Assume $\lambda$ is regular and $\lambda^{<\lambda} = \lambda$. Then the following are equivalent:
(1) $\kappa$ is $\lambda$-Shelah,

(2) $\text{NS}_{\kappa\lambda}^* \leq (\text{I}_{\kappa\lambda}^+)_{2}^{2}$ holds,

(3) $\mathcal{P}_{\kappa}\lambda \leq (\text{NS}_{\kappa\lambda}^+, \text{I}_{\kappa\lambda}^+)_{2}^{2}$ holds,

(4) $\mathcal{P}_{\kappa}\lambda \leq (\text{NSh}_{\kappa\lambda}^+, \text{I}_{\kappa\lambda}^+)_{2}^{2}$ holds. □

This and corollary 3.8 are partial answers of a question of 5.5 in Carr [4].

Using Lemma 3.9, we have a slight improvement of a Magidor's theorem((3) of Fact 2.7). Notice that Part"$(\kappa, \lambda)_{2}^{2}$ implies $\text{NS}_{\kappa\lambda}^* \leq (\text{I}_{\kappa\lambda}^+)_{2}^{2}$, but the converse does not hold in general.

**Cor. 3.12 The followings are equivalent:**

(1) $\kappa$ is supercompact,

(2) $\text{NS}_{\kappa\lambda}^* \leq (\text{I}_{\kappa\lambda}^+)_{2}^{2}$ holds for any $\lambda$,

(3) for any countable language structure $M$ with $\kappa \subseteq M$ and $f : [\{N \in \mathcal{P}_{\kappa}M : N \prec M, N \cap \kappa \in \kappa\}]_{<}^{2} \rightarrow 2$ there exists an $H$ such that $H$ is unbounded in $\mathcal{P}_{\kappa}M$ and homogeneous for $f$. Where for $X \subseteq \mathcal{P}_{\kappa}M$, $[X]_{<}^{2} = \{\{N, N'\} \subseteq X : N \subseteq N', |N| < |N' \cap \kappa\}$.

4 Some related results

In Theorem 1 and Cor. 3.8, it was assumed that $\lambda$ is not weakly compact. Now we show that this assumption is needed.

**Fact 4.1** Let $\theta$ be a sufficiently large regular cardinal and $\mu < \theta$ a cardinal. Let $\Delta$ be a well-order of $\mathcal{H}_\theta$. Let $M = \langle \mathcal{H}_\theta, \in, \Delta, \mu, \ldots \rangle$. For $N \prec M$ and $\alpha < \mu$, let $N[\alpha] = \{f(\alpha) : f \in {}^\mu N \cap N\}$. Then $N \subseteq N[\alpha], \alpha \in N[\alpha]$ and $N[\alpha] \prec M$. □

In fact $N[\alpha]$ is just the Skolem hull of $N \cup \{\alpha\}$ under $M$.

**Lemma 4.2** Assume $\lambda$ is weakly compact $> \kappa$ and $\kappa$ is $\lambda$-Shelah. Let $W = \{x \in \mathcal{P}_{\kappa}\lambda : \exists \alpha \in x(|x| = |x \cap \alpha|)\}$. Then for any club $C$ of $\mathcal{P}_{\kappa}\lambda$, $(C \cap W) \leq (\text{I}_{\kappa\lambda}^+)_{2}^{2}$ holds.

**Proof:** Let $C$ be an arbitrary club and $g : \lambda \times \lambda \rightarrow \lambda$ generating $C$, that is, if $x \cap \kappa \in \kappa$ and $x$ is closed under $g$ then $x \in C$. Fix a sufficiently large regular cardinal $\theta$ and a well-order $\Delta$ on $\mathcal{H}_\theta$. Let $M = \langle \mathcal{H}_\theta, \in, \Delta, \kappa, \lambda, g \rangle$. Let $M^* = \text{Skull}^M(\lambda)$.
Then by Carr [2], there exists a \( \lambda \)-complete proper \( M^* \)-normal ultra filter \( F \) over \( \lambda \), here \( M^* \)-normal ultra mean that for all \( A \in M^* \cap \mathcal{P}(\lambda) \), either \( A \in F \) or \( \lambda \setminus A \in F \), and for any regressive \( f \in \lambda \lambda \cap M^* \) there exists \( \beta < \lambda \) such that \( \{\alpha < \lambda : f(\alpha) = \beta\} \in F \).

By Abe [1], we can take \( Y \in \text{NSh}_{\lambda}^* \) such that

1. for each \( x \in Y \), \( x \cap \kappa < \kappa \) and \( \text{Skull}^M(x) \cap \lambda = x \), here \( \text{Skull}^M(x) \) is the Skolem hull of \( x \) under \( M \),

2. for \( x, y \in Y \), if \( x \neq y \) then \( \text{sup}(x) \neq \text{sup}(y) \).

For \( x \in Y \), let \( M_x = \text{Skull}^M(x) \). Note that \( |M_x| = |x| \). Now define \( s_x : x \in Y \) by the induction on \( \text{sup}(x) < \lambda \). Let \( x \in Y \) and assume \( s_y < \lambda \) is defined for any \( y \in Y \) with \( \text{sup}(y) < \text{sup}(x) \). Consider \( A = \bigcap \{ B : B \in M_x \cap \mathcal{P}(\lambda) \} \). Since \( F \) is \( \lambda \)-complete, \( A \in F \). Hence we can take \( s_x \in A \) such that \( s_x > \text{sup}(M_y[s_y] \cap \lambda) \) for any \( y \in Y \) with \( \text{sup}(y) < \text{sup}(x) \).

Now we claim the following:

**Claim 4.3** \( \{x \in Y : M_x[s_x] \cap s_x \neq x\} \) is non-stationary.

**Proof:** Assume not. Let \( \pi : \lambda \to M^* \) be a bijection. Then \( \{x \in \mathcal{P}_\lambda : M_x \cap \lambda = x, \pi''x = M_x\} \) is club, so \( Z = \{ x \in Y : \pi''x = M_x, M_x[s_x] \cap s_x \neq x \} \) is stationary. Let \( x \in Z \). Then by the definition of \( M_x[s_x] \), there exists \( f_x \in M_x \) such that \( f_x(s_x) \in (M[s_x] \cap s_x) \setminus x \). Then we may assume that \( f_x \in \lambda \lambda \) and \( f_x \) is regressive. By Fodor’s lemma, there exists \( f \in M^* \cap \lambda \lambda \) such that \( \{ x \in Z : f_x = f \} \) is stationary. Since \( f \in M^* \) and \( f \) is regressive, there exists \( \beta < \lambda \) such that \( \{\alpha < \lambda : f(\alpha) = \beta\} \in F \). Then we can take \( x \in Z \) such that \( f = f_x \) and \( \beta \in x \). Since \( f, \beta \in x \), we have \( \{\alpha < \lambda : f(\alpha) = \beta\} \in F \cap M_x \). Then \( s_x \in \{\alpha < \lambda : f(\alpha) = \beta\} \) thus \( f_x(s_x) = \beta \), a contradiction. \( \square \)

Let \( X = \{ x \in Y : M_x[s_x] \cap s_x = x \} \). By the above claim \( X \in \text{NSh}_{\lambda}^* \). Note that for \( x \in X \), \( M_x[s_x] \cap \kappa = x \cap \kappa \in \kappa \), and \( M_x[s_x] \) is closed under \( g \). Thus we have \( M_x[s_x] \cap \lambda \subseteq C \). For \( x \in X \), \(|x| = |M_x[s_x] \cap s_x| = |M_x[s_x] \cap \lambda| \). Therefore \( \{M_x[s_x] \cap \lambda : x \in X\} \subseteq C \cap \{ x \in \mathcal{P}_\lambda : \exists \alpha \in x (|x| = |x \cap \alpha|)\} \). We will see that \( \{M_x[s_x] \cap \lambda : x \in X\} \overset{\leq}{\to} (1^*_\lambda)^\| \). To see this, we claim the following: for any \( x, y \in X \), if \( M_x[s_x] \cap \lambda < M_y[s_y] \cap \lambda \) then \( x < y \). Since \(|x| = |M_x[s_x] \cap \lambda| \) and \(|y| = |M_y[s_y] \cap \lambda| \), we have \(|x| < |y \cap \kappa| \). We check that \( x \subseteq y \). We consider three cases.

1. If \( \text{sup}(x) = \text{sup}(y) \), then \( x = y \) by the definition of \( Y \), a contradiction.

2. If \( \text{sup}(x) > \text{sup}(y) \). Then \( s_x > \sup(M_y[s_y] \cap \lambda) \) by the choice of \( s_x \). Hence \( s_x \not\in M_y[s_y] \cap \lambda \), but this contradict to \( M_x[s_x] \cap \lambda \subseteq M_y[s_y] \cap \lambda \).

3. If \( \text{sup}(x) < \text{sup}(y) \). Note that then \( s_x < s_y \). Hence \( x = M_x[s_x] \cap s_x \subseteq M_y[s_y] \cap s_y = y \) and we have done.
For given $f : \{M_x[x] \cap \lambda : x \in X\}^\lambda \rightarrow 2$, define $f' : [X]^\lambda \rightarrow 2$ by $f'(x, y) = f(M_x[x] \cap \lambda, M_y[s_y] \cap \lambda)$ if $M_x[s_x] \cap \lambda < M_y[s_y] \cap \lambda$. Since $X \in \text{NSh}^\alpha$, there exists an unbounded homogeneous set $H$ for $f'$. Then it is easy to see that $\{M_x[x] \cap \lambda : x \in H\}$ is unbounded homogeneous set for $f$. \$
$
Combining the above lemma and Fact 2.4, we have the following.

**Cor. 4.4** Assume $\lambda$ is weakly compact $> \kappa$ and $\kappa$ is $\lambda$-Shelah. Then $I|C \subseteq \text{NSh}^\kappa\lambda$ holds for any club $C$ of $\mathcal{P}\kappa\lambda$, here $I = \{X \subseteq \mathcal{P}\kappa\lambda : X \not\rightarrow \langle 1^+\rangle^\kappa\lambda\}$. \$
$
Next we argue more possibility of the local normality of the partition ideal. The local normality of the 2-array partition ideal was shown. We see the case $n \geq 2$ with a bit weak assumption.

**Lemma 4.5** Let $n$ be a natural number $> 0$ and $I = \{X \subseteq \mathcal{P}\kappa\lambda : X \not\rightarrow \langle 1^+\rangle^n\kappa\lambda\}$. Assume $\lambda = 2^\nu$ for some $\nu < \lambda$. Then there exists a club $D$ of $\mathcal{P}\kappa\lambda$ such that $I|D$ is normal.

**Proof:** Fix $\nu < \lambda$ with $2^\nu = \lambda$. Note that $\kappa \leq \nu < \text{cf}(\lambda)$ and $\lambda^\nu = \lambda$ holds.

Fix $\vec{A} = \langle A_\xi : \xi < \lambda\rangle$ a bijective enumeration of $\mathcal{P}(\nu)$. Take a club $C$ shown in Lemma 3.5 with the case $\mu = \nu^+$. Fix a sufficiently large regular cardinal $\theta$ and let $M = \langle H_\theta, \in, \kappa, \lambda, \vec{A}\rangle$. Let $D = \{N \cap \lambda \in C : N \prec M, |N| < \kappa, N \cap \kappa \in \kappa\}$. We will prove $D$ is a desired club.

Let $X \in (I|D)^+$ with $X \subseteq D$. Let $g : X \rightarrow \lambda$ be a regressive function. Assume $X_\alpha = \{x \in X : g(x) = \alpha\} \in I$ for all $\alpha < \lambda$. Let $f_\alpha : [X_\alpha]^{n+1} \rightarrow 2$ be a counterexample of $X_\alpha \not\rightarrow \langle 1^+\rangle^n\kappa\lambda$. For $t \in [X]_\xi$, set $a_t = A_{g(\min(t))} \cap \min(t) \subseteq \min(t) \cap \nu$. Now define $f : [X]^{n+1}_\xi \rightarrow 2$ as: for $\{x_1, \ldots, x_{n+1}\} \subseteq [X]^{n+1}_\xi$, if $g(x_1) = \cdots = g(x_{n+1}) = \alpha$, then $f(x_1, \ldots, x_{n+1}) = f_\alpha(x_1, \ldots, x_{n+1})$. Suppose not. Assume $a_{x_1} = \cdots = a_{x_{n+1}} \cap x_1$ and let $\xi = \min(a_{x_1} = \cdots = a_{x_{n+1}} \cap x_1)$. If $\xi \in a_{x_1} = \cdots = a_{x_{n+1}} \cap x_1$, then set $f(x_1, \ldots, x_{n+1}) = 0$. If $\xi \in a_{x_1} = \cdots = a_{x_{n+1}} \cap x_1$, then set $f(x_1, \ldots, x_{n+1}) = 1$.

Then, by a similar argument of Lemma 3.5, there exists an unbounded homogeneous set $H \subseteq X$ for $f$, $A \subseteq \nu$ and $z \in \mathcal{P}\kappa\lambda$ such that for any $t \in [H]_z^{n+1}$ if $z < \min(t)$ then $A \cap \min(t) = a_t$. Take $\alpha < \lambda$ such that $A = A_\alpha$ and put $H^* = \{x \in H : z < x, \alpha \in x\}$. It is easy to check that $H^* \subseteq X_\alpha$. Then by the definition of $f$, $H^*$ is an unbounded homogeneous set for $f_\alpha$, a contradiction. \$
$
Note that the above lemma shows that the partition ideal over $\mathcal{P}\kappa\lambda$ can be locally normal even if $\lambda$ is singular.

Combining arguments of Lemma 3.7 with Lemma 4.5, we have the following.

**Lemma 4.6** Let $n$ be a natural number $> 0$ and $I = \{X \subseteq \mathcal{P}\kappa\lambda : X \not\rightarrow \langle 1^+\rangle^n\kappa\lambda\}$. Assume $\lambda$ is inaccessible but not weakly compact. Then there exists a club $D$ of $\mathcal{P}\kappa\lambda$ such that $I|D$ is normal. \$

References


