\[ \mathbb{P}_{\max}^{\aleph_1} \text{ and other variations} \]

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1 Introduction of \( \mathbb{P}_{\max} \) variations

\( \mathbb{P}_{\max} \) has been introduced by W. Hugh Woodin who says that in [11], \( \mathbb{P}_{\max} \) forces the canonical model of the negation of the Continuum Hypothesis CH over \( L(\mathbb{R}) \) with some large cardinal assumptions, e.g. \( \text{AD}^{L(\mathbb{R})} \), or there are infinitely many Woodin cardinals with the measurable cardinal above. Under suitable large cardinal assumptions (in this paper, I abbreviate this to \( \text{LC} \)), \( \mathbb{P}_{\max} \) generically adds, over \( L(\mathbb{R}) \), a directed system of countable transitive models of ZFC (or its fragments) whose limit restricted to \( H(\omega_2) \) (in this extension) is the whole \( H(\omega_2) \), and \( \mathbb{P}_{\max} \) forces that the nonstationary ideal \( NS_{\omega_1} \) on \( \omega_1 \) is saturated. One of the important facts on \( \mathbb{P}_{\max} \) is absoluteness of \( \Pi_2 \)-sentences for the structure

\[ \langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \]

for some set \( R \) of reals in \( L(\mathbb{R}) \) as follows:

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If a $\Pi_2$-sentence for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ is $\Omega_{\text{ZFC}}$-consistent (e.g. forceable by set-forcing over ZFC), then it is true in $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ in the extension with $\mathbb{P}_{\text{max}}$ over $L(\mathbb{R})$ with LC.

(Under LC (e.g. there exist proper class many Woodin cardinals), every set of reals in $L(\mathbb{R})$ is universally Baire, and weakly homogeneously Suslin (see e.g. [5]). $R$ is considered as an interpretation of its universally Baire set of reals in each universe. For more historical and technical remarks on $\mathbb{P}_{\text{max}}$, see [11, 7, 1].)

In [11], Woodin studied not only $\mathbb{P}_{\text{max}}$ but also conditional variations of $\mathbb{P}_{\text{max}}$ for e.g. Suslin trees and the Borel Conjecture. $\mathbb{P}_{\text{max}}$ variations have been studied by several set theorists: Feng–Woodin, Larson, Larson–Todorcević, Shelah–Zapletal and Yorioka [3, 4, 6, 8, 10, 12]. In [10], many variations of $\mathbb{P}_{\text{max}}$ for $\Sigma_2$-statements in the structure $H(\omega_2)$ on cardinal invariants of the reals have been investigated. We should notice that all of them are derived from $\Diamond$. For example, the $\mathbb{P}_{\text{max}}$ variation, say $\mathbb{P}_{\text{max}}^{\exists = \aleph_1}$, for the statement that the dominating number $\delta$ in $\omega^\omega$ is $\aleph_1$ has been studied. It has been proved in [10, §2] that the extension with $\mathbb{P}_{\text{max}}^{\exists = \aleph_1}$ over $L(\mathbb{R})$ under LC satisfies ZFC, the continuum $\mathfrak{c}$ is $\aleph_2$, $NS_{\omega_1}$ is saturated, $\delta = \aleph_1$ holds, and maximality with respect to $\Pi_2$-statements in $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ for some set $R$ of reals in $L(\mathbb{R})$, that is, under LC, the extension with $\mathbb{P}_{\text{max}}^{\exists = \aleph_1}$ over $L(\mathbb{R})$ satisfies the following property, called $\Pi_2$-compactness in [10]:

If $\psi$ is a $\Pi_2$-sentence for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ and the statement $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \models "\delta = \aleph_1 \land \psi"$ is $\Omega_{\text{ZFC}}$-consistent, then it is true in $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$.

So this model can be considered as the canonical model of $\delta = \aleph_1$. In [10], there are many examples and counterexamples of $\Pi_2$-compact statements. One non-$\Pi_2$-compact statement, which does not appear in [10], is that the additivity $\text{add}(\mathcal{M})$ of the meager ideal is $\aleph_1$: By Miller–Truss’s characterization of $\text{add}(\mathcal{M})$, $\text{add}(\mathcal{M})$ is the minimum of the bounding number $\mathfrak{b}$ and the covering number $\text{cov}(\mathcal{M})$ of the meager ideal. However both $\text{"}\aleph_1 = \text{add}(\mathcal{M}) < \mathfrak{b}\text{"}$ and $\text{"}\aleph_1 = \text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})\text{"}$ are consistent with ZFC, and both $\text{"}\text{cov}(\mathcal{M}) > \aleph_1\text{"}$ and $\text{"}\mathfrak{b} > \aleph_1\text{"}$ are $\Pi_2$-statements in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$. (The statement that the additivity of the null ideal is $\aleph_1$ is not $\Pi_2$-compact either. It is known that $\text{add}(\mathcal{N}) = \min\{\text{add}^*(\mathcal{N}), \mathfrak{b}\}$. See [2, Theorem 2.7.13.] or [9].)
In this paper, we work in ZFC except for the definition of $\mathbb{P}_{\text{max}}^\phi$ and the proof of Theorem Schemes because when we force by $\mathbb{P}_{\text{max}}^\phi$, we always consider $L(\mathbb{R})$ as the ground model which never satisfies the Axiom of Choice (by our assumption). $\mathbb{P}_{\text{max}}$ can be defined by various ways. One of them is defined by use of iterable pairs. Suppose a suitable large cardinal property, $M$ is a countable transitive model of ZFC and $I$ is a member of $M$ which is a uniform normal ideal on $\omega_1^M$ in $M$. We can take a direct system $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \omega_1 \rangle$, called an iteration of $(M, I)$ (of length $\omega_1$), such that

- $M_0 = M$,
- $G_\beta$ is an $M_\beta$-generic filter of the forcing notion $(\mathcal{P}(\mathcal{P}(\omega_1^{M_\beta})/j_{0,\beta}(I)))^{M_\beta}$ (or $(\mathcal{P}(\mathcal{P}(\omega_1^{M_\beta}) \setminus j_{0,\beta}(I)))^{M_\beta}$) for every $\beta \in \omega_1$,
- $j_{\gamma,\gamma}$ is the identity on $M_\gamma$ for every $\gamma \in \omega_1 + 1$,
- $M_{\beta+1}$ is (the transitive collapse of) the generic ultrapower of $M_\beta$ by $G_\beta$ (if it is wellfounded, otherwise we stop the construction), and $j_{\gamma,\gamma+1}$ is the ultrapower embedding induced by $G_\gamma$ for every $\gamma \in \omega_1$, and
- if $\alpha \in \omega_1 + 1$ is a limit ordinal, then $M_\alpha$ is (the transitive collapse of) the direct limit of the system $\langle M_\gamma, j_{\gamma,\delta}; \gamma \leq \delta < \alpha \rangle$ and $j_{\gamma,\alpha}$ is the induced embedding for every $\gamma \in \alpha$.

(See [11, Definition 3.5. or Definition 4.1.] or [7, 1.2 Definition].) A pair $(M, I)$ as above is called iterable if all $M_\gamma$, $\gamma \in \omega_1$, are wellfounded regardless of the choice of generic filters $G_\beta$. Woodin proved that if $I$ is precipitous, then $(M, I)$ is iterable (see [11, Lemma 3.10. and Lemma 4.5.]).

In many cases, we define the $\mathbb{P}_{\text{max}}$ variation $\mathbb{P}_{\text{max}}^\phi$ for a $\Sigma_2$-sentence $\phi$ in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$ which is derived from $\diamondsuit$. For example, $\mathfrak{d} = \aleph_1$ holds, and there exists a coherent Suslin tree, etc. In $[10]$, variations of $\mathbb{P}_{\text{max}}$ are defined by use of stationary tower forcing ([5]). In this paper, we adopt a definition in [7, §10.2], however all of proofs in this paper can be applied to any type of $\mathbb{P}_{\text{max}}^\phi$ variations.

**Definition of $\mathbb{P}_{\text{max}}^\phi$ ([7, §10.2])** Let $\phi$ be a $\Sigma_2$-statement for the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$, and say that $\phi$ forms $\exists u \forall v \phi_0(u,v)$. Conditions of the forcing notion $\mathbb{P}_{\text{max}}^\phi$ are defined by recursion on their ranks as follows. A triple $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{\text{max}}$ if
1. $(M, I)$ is an iterable pair,

2. $a \in H(\omega_2)^M$ and $\langle H(\omega_2), \in, I \rangle^M \models \forall v \phi_0(a, v)$, and

3. $\mathcal{X}$ is a member of $M$ and a set (possibly empty) of pairs $\langle \langle (N, J), b, \mathcal{Y} \rangle, j \rangle$ such that
   - $\langle (N, J), b, \mathcal{Y} \rangle \in \mathbb{P}_{\max}^\phi \cap H(\omega_1)^M$,
   - $j$ is in $M$ and an iteration of $(N, J)$ of length $\omega_1^M$ such that
     $j(J) = I \cap j\left(\mathcal{P}(\omega_1^N)^N\right)$, $j(b) = a$ and $j(\mathcal{Y}) \subseteq \mathcal{X}$, and
   - $\mathcal{X}$ forms a function, i.e. for members $\langle p, j \rangle$ and $\langle p', j' \rangle$ in $\mathcal{X}$, if $p = p'$, then $j = j'$.

For conditions $\langle (M, I), a, \mathcal{X} \rangle$ and $\langle (N, J), b, \mathcal{Y} \rangle$ in $\mathbb{P}_{\max}^\phi$, we define

$$\langle (M, I), a, \mathcal{X} \rangle \prec_{\mathbb{P}_{\max}^\phi} \langle (N, J), b, \mathcal{Y} \rangle$$

if there exists $j$ such that $\langle \langle (N, J), b, \mathcal{Y} \rangle, j \rangle \in \mathcal{X}$.

We have to note that the statement that a pair $(M, I)$ is iterable is $\Pi_2^1$ about a real coding $(M, I)$, so is absolute (see e.g. [7, 1.3 Remark and 1.10 Remark]). Therefore the statement that a triple $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{\max}^\phi$ is also $\Pi_2^1$, and so is absolute. Since $L(\mathbb{R})$ has every real, it also has every countable transitive model. And since a condition of $\mathbb{P}_{\max}^\phi$ can be coded by a real, $(\mathbb{P}_{\max}^\phi)^{L(\mathbb{R})} = \mathbb{P}_{\max}^\phi$. If $\phi$ is trivial (e.g. "0 = 0", or the statement that there exists the empty set), then $\mathbb{P}_{\max}^\phi$ can be considered the standard $\mathbb{P}_{\max}$. (However $\mathbb{P}_{\max}^\phi$ and $\mathbb{P}_{\max}$ are slightly different, see [11, §5.4, in particular Theorem 5.40].)

To analyze the extension by $\mathbb{P}_{\max}^\phi$, we need some game theoretic lemmata. (On definitions of games $G_1^\phi$, $G_\omega^\phi$ and $G_\omega^\phi$, I refer [7, §3 and §10.2].)

We define the game $G_1^\phi$ as follows. Suppose that $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{\max}^\phi$, $J$ is a normal uniform ideal on $\omega_1$. Players $I$ and $\Pi$ collaborate to build an iteration $\langle M_\gamma, G_\beta, j_\gamma, \delta; \beta < \gamma \leq \delta \leq \omega_1 \rangle$ of $(M, I)$ with the following rule: In each round $\alpha$, $\Pi$ chooses a set $A$ in the set $\mathcal{P}(\omega_1^{M_\alpha})^{M_\alpha} \setminus j_{0, \alpha}(I)$, and then $I$ chooses an $(M_\alpha, \mathcal{P}(\omega_1^{M_\alpha})^{M_\alpha})$-generic filter $G_\alpha$ with $A \in G_\alpha$. (To just simplify notation, we force by $\mathcal{P}(\omega_1) \setminus I$ instead of $\mathcal{P}(\omega_1)/I$ in this paper.) After all $\omega_1$ many rounds have been played, $I$ wins if
\[ \langle H(\omega_2), \in, J \rangle \models \forall v \phi_0(j_{0,\omega_1}(a), v) \].

(We should note that player II has a strategy such that after all \(\omega_1\) rounds have been played whenever player II plays according to this strategy,

\[ j_{0,\omega_1}(I) = J \cap M_{\omega_1} \] holds.

See [11, Lemma 4.36.], [7, 2.8 Lemma], [1, Lemma 1.8].

To show \(\sigma\)-closedness of \(P_{\text{max}}^\phi\) and define the strategic iteration lemma for \(\phi\), we need to define an iterable limit sequence and two games \(G_{\omega}^\phi\) and \(G_{\omega_1}^\phi\). (On this paragraph, see [11, Chapter 4.1 and Lemma 4.43.], [7, \S 3] and [1, \S 2].) Let \(\langle p_i; i \in \omega \rangle\) is a decreasing sequence of \(P_{\text{max}}^\phi\) and write \(p_i := \langle\langle M_i, I_i \rangle, a_i, \mathcal{X}_i\rangle\). Let \(j_{i,i+1} : (M_i, I_i) \to (M_i^*, I_i^*)\) be an iteration witnessing that \(p_{i+1} <_{\text{max}} p_i\) (and if \(p_{i+1} = p_i\), then let \(j_{i,i+1}\) be the identity map) and let \(\{j_{i,i'}; i \leq i' \leq \omega\}\) be the commuting family of embeddings generated by \(\{j_{i,i+1}; i \in \omega\}\). We write \(j_i, w; (M_i, I_i) \to (N_i, J_i)\) for each \(i \in \omega\). Let \(a := \bigcup_{i \in \omega} a_i\) and \(\mathcal{X} := \bigcup_{i \in \omega} \mathcal{X}_i\). In most cases, \(a\) forms a witness of \(\phi\) in every \(N_i\). (At least, every application in any present published paper, including this paper, on \(P_{\text{max}}^\phi\) and its variations is in this case.) Then we can show that

- for each \(i \in \omega\), \((N_i, J_i)\) is an iterable pair,
- for each \(i \in \omega\), \(N_i \in N_{i+1}\) and \(\omega_1^{N_i} = \omega_1^{N_0}\),
- for each \(i \in \omega\), \(J_{i+1} \cap N_i = J_i\),
- \(a \in H(\omega_2)^{N_0}\) and for each \(i \in \omega\), \(\langle H(\omega_2), \in, J_i \rangle^{N_i} \models \forall b \phi_0(a, b) \).

We call \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) a limit sequence if it is constructed as above. For a limit sequence \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\), when an ultrafilter \(G\) on the set

\[ \bigcup_{i \in \omega} P(\omega_1^{N_i}) \setminus J_i \]

satisfies that for every regressive function \(f\) on \(\omega_1^{N_i}\) in \(\bigcup_{i \in \omega} N_i\), \(f\) is constant on some condition in \(G\), we call it a \(\bigcup_{i \in \omega}\)-normal ultrafilter for \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\). Then we form the ultrapower of \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) formed from \(G\) and all functions \(f : \omega_1^{N_0} \to N_i\) in \(\bigcup_{i \in \omega} N_i\). (More precisely, see [11, Definition 4.15.].) Using this ultrapower, we define the iteration of the sequence \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\), and the iterability of the sequence \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) as in the iterable pair. We note that for a limit sequence \(\langle\langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) constructed as above,
\(\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) is iterable.

We define the game \(G^\phi_\omega\) as follows. Suppose that \(\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) is a limit sequence, \(J\) is a normal uniform ideal on \(\omega_1\). Players I and II collaborate to build an iteration of \(\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle\) consisting of limit sequences \(\langle \langle (N^\alpha_i, J^\alpha_i); i \in \omega \rangle, a^\alpha, \mathcal{X}^\alpha \rangle, \cup \{N^\alpha_i; i \in \omega\}\)-normal ultrafilters \(G^\alpha\) for \(\langle \langle (N^\alpha_i, J^\alpha_i); i \in \omega \rangle, a^\alpha, \mathcal{X}^\alpha \rangle\) and a commuting family of embeddings \(j_{\alpha,\beta}\) for \(\alpha \leq \beta \leq \omega_1\) with the following rule: In each round \(\alpha\), II chooses a set \(A\) in the set \(\cup \{P(\omega_1 N^\alpha_i)^N_i \setminus J^\alpha_i; i \in \omega\}\), and then I chooses a \(\cup \{N^\alpha_i; i \in \omega\}\)-normal ultrafilter \(G^\alpha\) for \(\langle \langle (N^\alpha_i, J^\alpha_i); i \in \omega \rangle, a^\alpha, \mathcal{X}^\alpha \rangle\) with \(A \in G^\alpha\). After all \(\omega_1\) many rounds have been played, I wins if

- \(\langle H(\omega_2), \in, J \rangle \models \forall v \phi_0(j_{0,\omega_1}(a), v)\).

(We should note that II has a strategy such that after all \(\omega_1\) rounds have been played whenever player II plays according to this strategy,

- \(J^\omega_i = J \cap N^\omega_i\) holds for every \(i \in \omega\).

We can prove \(\sigma\)-closedness of \(\mathbb{P}^\phi_{\max}\) using strategies for both players I and II. See [11, Lemma 4.43.], [7, 3.4 Lemma and 3.5 Lemma], [1, Lemma 2.5].)

We define the game \(G^\phi_{\omega_1}\) as follows. Let \(p_0\) is a condition of \(\mathbb{P}^\phi_{\max}\). Players I and II collaborate to build a decreasing \(\omega_1\)-chain \(\langle p_\alpha; \alpha \in \omega_1 \rangle\) of conditions with the following rule: In each round \(\alpha\), if \(\alpha\) is a successor ordinal, II chooses a condition \(p_\alpha\) below \(p_{\alpha-1}\). If \(\alpha\) is a limit ordinal, then II chooses a cofinal \(\omega\)-sequence of \(\alpha\) and, letting \(\langle \langle (N^\alpha_i, J^\alpha_i); i \in \omega \rangle, a^*_\alpha, \mathcal{X}^*_\alpha \rangle\) be the induced limit sequence, II chooses a set \(A_\alpha\) in the set \(\cup \{P(\omega_1 N_i^\alpha)^N_i \setminus J^\alpha_i; i \in \omega\}\), and then I chooses a condition \(p_\alpha = ((M_\alpha, I_\alpha), a^*_\alpha, \mathcal{X}^*_\alpha)\) below every \(p_\beta\) such that for some iteration \(k\) of \(\langle \langle (N^\alpha_i, J^\alpha_i); i \in \omega \rangle, a^*_\alpha, \mathcal{X}^*_\alpha \rangle, k[\mathcal{X}^*_\alpha] \subseteq \mathcal{X}_\alpha\) and \(\omega_1 N_\alpha^\alpha \in k(A_\alpha)\). After all \(\omega_1\) rounds have been played, I wins if, letting \(j_{\alpha,\beta}\) (\(\alpha < \beta \leq \omega_1\) be the induced commuting family of embeddings on the sequence \(\langle p_\alpha; \alpha \in \omega_1 \rangle\),

- \(\langle H(\omega_2), \in, j_{0,\omega_1}(I_0) \rangle \models \forall v \phi_0(j_{0,\omega_1}(a), v)\).

(In [10], the strategic iteration lemma for \(\phi\) is the following lemma scheme:

\((\text{ZFC} + \diamond)\) Player I has a winning strategy in \(G^\phi_{\omega_1}\).

This is related to [7, 5.2 Theorem].)

The following theorem is a basic theorem of \(\mathbb{P}^\phi_{\max}\).
Theorem Scheme 1 ([11, Chapter 4], [1, §§3-5], [7, §§5-7], [10, §1]) \(\text{ZFC + LC}\)

Let \(\phi\) be a \(\Sigma_2\)-sentence in the structure \(\langle H(\omega_2), \in, NS_{\omega_1}\rangle\). Assume that the following three statements

1. player \(I\) has a winning strategy in \(G_1^\phi\),
2. \((\omega)\) player \(I\) has a winning strategy in \(G_\omega^\phi\),
3. \((\omega_1)\) player \(I\) has a winning strategy in \(G_{\omega_1}^\phi\),

are all \(\Omega_{\text{ZFC}}\)-consistent. Let \(G\) be a \((L(\mathbb{R}), \mathbb{P}_{\text{max}}^\phi)\)-generic filter. Then in \(L(\mathbb{R})[G]\), \(\text{ZFC}\) holds, \(c = \aleph_2\), \(NS_{\omega_1}\) is saturated and \(\langle H(\omega_2), \in, NS_{\omega_1}\rangle \models \forall \phi\) holds.

In the above theorem scheme, the phrase that \((1), (\omega)\) and \((\omega_1)\) are all \(\Omega_{\text{ZFC}}\)-consistent are usually considered as the slightly stronger following statement:

\((\text{ZFC} + \Diamond)\) Both \((1), (\omega)\) and \((\omega_1)\) hold.

One of important conclusions of \(\mathbb{P}_{\text{max}}^\phi\) extensions is \(\Pi_2\)-maximality. To show this, we need a more technical lemma. For a sentence \(\Phi\) in the language of set theory, the iteration lemma for \(\phi\) from \(\Phi\) is defined as follows:

Lemma Scheme; The Iteration Lemma for \(\phi\) from \(\Phi\) \(\text{(ZFC + \Phi)}\) If

1. \((M, I)\) is an iterable pair,
2. \(a \in H(\omega_2)^M\) and \(\langle H(\omega_2), \in, I\rangle^M \models \forall b \phi_0(a, b)\)
3. \(J\) is a normal uniform ideal on \(\omega_1\), and
4. \(\langle H(\omega_2), \in, J\rangle \models \forall \phi\),

then there exists an iteration \(j : (M, I) \rightarrow (M^*, I^*)\) of length \(\omega_1\) such that

1. \(I^* = J \cap M^*,\) and
2. \(\langle H(\omega_2), \in, J\rangle \models \forall v \phi_0(j(a), v)\).
Of course, the case that $\Phi$ contradicts $\phi$ does not make sense. In [10], the simple iteration lemma for $\phi$ is the iteration lemma for $\phi$ from $\diamond$, and the optimal iteration lemma for $\phi$ is the iteration lemma for $\phi$ from any trivial statement. We note that if (under ZFC) player $I$ has a winning strategy in $G_1^\phi$, then the optimal iteration lemma for $\phi$ holds. We should notice that for some $\Sigma_2$-sentence $\phi$ in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$, the simple iteration lemma for $\phi$ fails. For example, the simple iteration lemma for CH, and for the statement that the almost disjointness number is $\aleph_1$ fail (see [10, §1.3] and [11, Lemma 5.29]).

**Theorem Scheme 2** ([11, Chapter 4], [1, §§3-5], [7, §§5-7], [10, §1]) (ZFC + LC) Let $\phi$ be a $\Sigma_2$-sentence in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$ and $\Phi$ a sentence in the language of set theory such that the iteration lemma for $\phi$ from $\Phi$ holds. Assume that both $(\omega)$ and $(\omega_1)$ are $\Omega_{\text{ZFC}}$-consistent. Let $G$ be a $(L(\mathbb{R}), \mathbb{P}_{\text{max}}^\phi)$-generic filter. Then in $L(\mathbb{R})[G]$, ZFC holds, $c = \aleph_2$, $NS_{\omega_1}$ is saturated and $\langle H(\omega_2), \in, NS_{\omega_1} \rangle \models \phi$ holds, and for any $\Pi_2$-sentence $\psi$ in the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ for some set $R$ of reals in $L(\mathbb{R})$, if the statement $\Phi + \langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \models \phi$ holds, then $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \models \psi$ holds.

Therefore under the assumption in Theorem Scheme 1, if the optimal iteration lemma for $\phi$ holds, then $\phi$ is $\Pi_2$-compact in the extension by $\mathbb{P}_{\text{max}}^\phi$ over $L(\mathbb{R})$. We have some examples of $\Sigma_2$-statements for which the optimal iteration lemma fails, e.g. for the existence of a Suslin tree. (See [10, §1.3].) However we should notice that even if the optimal iteration lemma for $\phi$ fail, we cannot conclude that $\phi$ cannot be $\Pi_2$-compact.

In this note, we prove the optimal iteration lemma for $\mathcal{G} = \aleph_1$. This proof is prototypical for any other $\mathbb{P}_{\text{max}}$ variations of $\mathcal{G} = \aleph_1$ where $\mathcal{G}$ is a cardinal invariant which is the smallest size of the cofinality of some ordered structure, or some ideal on the reals. The point whether we can adopt the proof for $\mathcal{G} = \aleph_1$ to the optimal iteration lemma for $\mathcal{G} = \aleph_1$ is whether we have a Suslin ccc Amoeba forcing for this structure and we can show the subgenericity lemma (i.e. a variation of Proposition 2.3).
2 The optimal iteration lemma for $\mathcal{D} = \aleph_1$

We don't prove the optimal iteration lemma for $\mathcal{D} = \aleph_1$ usually. We find an equivalent statement of $\mathcal{D} = \aleph_1$ and we show the optimal iteration lemma for it.

**Definition 2.1** ([10, Lemma 2.6.]). Let $I$ be a normal uniform ideal on $\omega_1$. A sequence $\langle f_\xi; \xi \in \omega_1 \rangle$ of functions in $\omega^\omega$ is an $I$-good scale if

- it is a scale, i.e. a well-ordered with respect to the eventually dominance, and

- for every $f \in \omega^\omega$, the set $\{ \xi \in \omega_1; f_\xi$ dominates $f$ everywhere $(f \leq f_\xi)\}$ is $I$-positive.

**Proposition 2.2.** Assume that $I$ is a normal uniform ideal on $\omega_1$. $\mathcal{D} = \aleph_1$ holds iff there exists an $I$-good scale.

**Proof.** Suppose that $\mathcal{D} = \aleph_1$ holds, and let $\langle g_\xi; \xi \in \omega_1 \rangle$ be a scale, i.e.

- if $\xi < \eta$ in $\omega_1$, then $g_\xi \leq^* g_\eta$, and

- for any $h \in \omega^\omega$, there exists $\xi \in \omega_1$ such that $h \leq^* g_\xi$.

Let

$$\langle X_{s,\alpha}; s \in \omega^{<\omega} \& \alpha \in \omega_1 \rangle$$

be a sequence of pairwise disjoint $I$-positive subsets of $\omega_1$.

By recursion on $\xi \in \omega_1$, we construct $f_\xi \in \omega^\omega$ such that

- $f_\xi \leq^*$-dominates $g_\eta$ and $f_\eta$ for all $\eta < \xi$, and

- if $\xi$ is in some $X_{s,\alpha}$, then $f_\xi \sqsubset$-dominates the function $s^{\wedge}(g_\alpha \upharpoonright |s|, \infty))$.

Then we note that $\langle f_\xi; \xi \in \omega_1 \rangle$ is a scale. So what we need to check is $I$-goodness.

Let $f \in \omega^\omega$. Then since $\langle g_\xi; \xi \in \omega_1 \rangle$ is a scale, we can find $\alpha \in \omega_1$ so that $f$ is $\leq^*$-dominated by $g_\alpha$. Let $n \in \omega$ be such that $f(i) \leq g_\alpha(i)$ for every $i \geq n$ and let $s := f \upharpoonright n$. Then

$$\{ \xi \in \omega_1; f \leq g_\xi \} \supseteq X_{s,\alpha},$$

that is, the set $\{ \xi \in \omega_1; f \leq g_\xi \}$ is $I$-positive.

The other direction is trivial. $\square$
We have a Suslin ccc Amoeba forcing for the structure \( \langle \omega^\omega, \leq^* \rangle \), the Hechler forcing \( \mathbb{D} := \omega^{<\omega} \times \omega^\omega \). For \( p = \langle s^p, f^p \rangle \) and \( q = \langle s^q, f^q \rangle \), \( p \leq \mathbb{D} q \) if \( s^p \supseteq s^q \), \( f^q \leq f^p \) and for every \( i \in [|s^q|, |s^p|) \), \( s^p(i) \geq f^q(i) \). For a condition \( p \in \mathbb{D} \), we define

\[
\text{body}(p) := s^p \upharpoonright [|s^p|, \infty),
\]

and let \( \mathbb{D} \upharpoonright f := \{ p \in \mathbb{D}; \text{body}(p) \leq f \} \).

**Proposition 2.3.** Suppose that \( M \) is a model of a large enough fragment of ZFC. (ZFC – Powerset + \( \exists \mathcal{P}(2^\omega) \) is sufficient.) Suppose that \( f \) eventually dominates all functions in \( \omega^\omega \cap M \), and \( D \in M \) is such that \( D \) is dense in \( \mathbb{D} \) in \( M \). Then \( D \cap (\mathbb{D} \upharpoonright f) \) is dense in \( (\mathbb{D} \upharpoonright f) \cap M \).

**Proof.** Let \( p_0 = \langle s_0, f_0 \rangle \in (\mathbb{D} \upharpoonright f) \cap M \). Working in \( M \), we choose \( p_i = \langle s_i, f_i \rangle \in \mathbb{D} \cap M \) by induction on \( i \in \omega \) such that

- \( p_{i+1} \in D \), and
- \( p_{i+1} \leq \mathbb{D} \langle \text{body}(p_0) \upharpoonright |s_i|, f_i \rangle \). (We must note that \( \langle \text{body}(p_0) \upharpoonright |s_i|, f_i \rangle \) is a condition in \( \mathbb{D} \cap M \) which extends \( p_0 \).)

Then we define \( g \in \omega^\omega \) such that

\[
g(i) := \begin{cases} s_0(i) & \text{if } i < |s_0| \\ s_{k+1}(i) & \text{if } |s_k| \leq i < |s_{k+1}| \text{ for some } k \in \omega \end{cases}
\]

Since \( \langle p_i; i \in \omega \rangle \) is in \( M \), \( g \) is also in \( M \). Thus \( g \leq^* f \) holds, hence for large enough \( k \in \omega \), \( g \upharpoonright [|s_k|, \infty) \leq f \upharpoonright [|s_k|, \infty) \). Then for a fixed such a \( k \), \( p_{k+1} \) is in \( D \cap (\mathbb{D} \upharpoonright f) \). \( \square \)

**Corollary 2.4.** Suppose that \( \mathbb{P} \) is a forcing notion and \( \dot{g} \) is a \( \mathbb{P} \)-name such that

1. \( \models_{\mathbb{P}} \" \dot{g} \in \omega^\omega \& \dot{g} \) eventually dominates all functions in \( \omega^\omega \cap \mathbb{V} \" \), (where \( \mathbb{V} \) is the ground model) and

2. for every condition \( r \in \text{SLOC} \), \( \| \text{body}(\check{r}) \leq \dot{g} \|_{\text{ro}(\mathbb{P})} \) is non-zero.

Then \( \mathbb{D} \) is completely embeddable into \( \mathbb{Q} := \text{ro}(\mathbb{P}) \ast ((\mathbb{D} \upharpoonright \dot{g}) \cap \mathbb{V}) \) such that

\[
\models_{\mathbb{Q}} \" \dot{f}_{\mathbb{G}} := \bigcup_{p \in \mathbb{G}} s^p \leq \dot{g} \".
\]
Proof. We show that the embedding $i$ from $\mathbb{D}$ into $\mathbb{Q}$, defined by

$$i(r) := \langle \|\text{body}(\check{r}) \leq \check{g}\|_{\text{ro}(P)}, \check{r} \rangle$$

for each $r \in \mathbb{D}$, is a complete embedding.

To prove this, we show that for any dense subset $D$ in $\mathbb{D}$ (in the ground model), the set $\{i(r); r \in D\}$ is predense in $\mathbb{Q}$. Let $\langle p, \check{q} \rangle \in \mathbb{Q}$, i.e., $p \leq_P \|\text{body}(\check{q}) \leq \check{g}\|_{\text{ro}(P)}$.

Since, by the previous proposition,

$$\models_P " \tilde{D} \cap (\mathbb{D} \upharrow \tilde{g}) \text{ is dense in } (\mathbb{D} \upharrow \tilde{g}) \cap V " ,$$

we can find $p' \leq_P p$ and $q' \leq_D q$ such that $q' \in D$ and

$$p' \models_P " \tilde{q}' \in \tilde{D} \cap (\mathbb{D} \upharrow \tilde{g}) " .$$

Then

$$\langle p', \check{q}' \rangle \leq_Q \langle \|\text{body}(\check{q}') \leq \check{g}\|_{\text{ro}(P)}, \check{q}' \rangle = i(q').$$

Assume that $i$ is not a complete embedding, i.e. there exists $\langle p, \check{q} \rangle$ in $\mathbb{Q}$ such that the set

$$D := \{ r \in \mathbb{D}; i(r) \text{ and } \langle p, \check{q} \rangle \text{ are incompatible in } \mathbb{Q} \}$$

is dense in $\mathbb{D}$. Then the set $\{i(r); r \in D\}$ is predense in $\mathbb{Q}$. However then, there exists $r \in D$ so that $i(r)$ and $\langle p, \check{q} \rangle$ are incompatible in $\mathbb{Q}$, which is a contradiction. \qed

Lemma 2.5. Suppose that $M$ is a countable model of (a large enough fragment of) ZFC, $P$ and $\check{g}$ satisfy the hypothesis of Corollary 2.4 in $M$, $p \in P \cap M$ and $f \in \omega^\omega$. (We may not assume $f \in M$.) Then there exists a $(M, P)$-generic filter $G$ containing $p$ such that $f$ is $\leq^*$-dominated by $\check{g}[G]$.

Proof. We fix a complete embedding from $\text{ro}(\mathbb{D})$ into $\mathbb{Q} := \text{ro}(P) \ast ((\mathbb{D} \upharrow \tilde{g}) \cap V)$ as in the previous corollary, and let $p'$ be a projection of $p$ via this embedding.

Let $N$ be a countable model of a large enough fragment of ZFC containing $M \cup \{S\}$. Since $N$ is a countable model, there exists a $(N, \mathbb{D})$-generic filter $F'$ containing $p'$. We let $F := F' \cap M$. Since $\mathbb{D}$ is a Suslin ccc forcing notion, all maximal antichains on $\mathbb{D}$ belonging to $M$ are still maximal in $N$. Thus $F$
is \((M, \mathbb{D})\)-generic and \(f \leq^* f_F\). We take a \((M, \mathbb{Q})\)-generic filter \(H\) extending \(F\) (via the fixed embedding) with \(p \in H\) and let \(G := \text{ro}(\mathbb{P}) \cap H\). We note that \(G\) is \((M, \text{ro}(\mathbb{P}))\)-generic. Then

\[
f \leq^* f_F \subseteq \dot{g}[H] = \dot{g}[G].
\]

\(\square\)

**Theorem 2.6** (The optimal iteration lemma for the existence of a good scale). (ZFC) If

- \((M, I)\) is an iterable pair,
- \(a \in H(\omega_2)^M\) and \(H(\omega_2)^M \models \text{"}a\ \text{is\ an\ } I\text{-good\ scale\ "}\),
- \(J\) is a normal uniform ideal on \(\omega_1\), and
- \(\text{cof}(\mathbb{N}) = \aleph_1\),

then there exists an iteration \(j : (M, I) \rightarrow (M^*, I^*)\) of length \(\omega_1\) such that

- \(I^* = J \cap M^*\), and
- \(j(a)\) is a \(J\)-good scale.

**Proof.** Suppose that \((M, I)\) is an iterable pair, i.e.

- \(M\) is a countable transitive model of ZFC, and
- \(I \in M\) and \(M \models \text{"}I\ \text{is\ a\ normal\ uniform\ ideal\ on\ } \omega_1^M\ ",

Let \(\langle f_\xi; \xi \in \omega_1^M \rangle\) be in \(M\) such that

\[
M \models \text{"} \langle f_\xi; \xi \in \omega_1^M \rangle\ \text{is\ an\ } I\text{-good\ scale\ "}\,
\]

and \(\langle g_\xi; \xi \in \omega_1 \rangle\) be a (\(J\)-good) scale. (We don't need \(J\)-goodness of the sequence \(\langle g_\xi; \xi \in \omega_1 \rangle\).) Let \(\langle X_{n,\alpha}; n \in \omega \& \alpha \in \omega_1 \rangle\) be a sequence of \(J\)-positive subsets of \(\omega_1\) which are pairwise disjoint.

We build an iteration \(\langle M_\gamma, G_\beta, j_{\gamma, \delta}; \beta \leq \gamma \leq \delta \leq \omega_1 \rangle\) of \((M, I)\) of length \(\omega_1\) such that

- for each \(\alpha \in \omega_1\), we fix a sequence \(\langle Y_{n,\alpha}; n \in \omega \rangle\) of all \(j_{0,\alpha}(I)\)-positive subsets of \(\omega_1^{M_\alpha}\),
• if $\alpha \leq \gamma$ in $\omega_1$, $n \in \omega$ and $\omega_1^{M_\alpha} \in X_{n,\alpha}$, then $j_{\alpha,\gamma}(Y_{n,\alpha}) \in G_\gamma$, and

• for every $\alpha \in \omega_1$, $g_\alpha \leq^* f_{\omega_1^{M_\alpha}}^{\alpha+1}$ ($= f_{\omega_1^{M_\alpha}}^{\omega_1}$), where for each $\alpha \leq \omega_1$, we write

$$j_{0,\alpha} \left( \langle f_\xi; \xi \in \omega_1^{M} \rangle \right) = \langle f_\xi^{\alpha}; \xi \in \omega_1^{M_\alpha} \rangle.$$

(We note that if $\alpha \leq \beta$ in $\omega_1 + 1$ and $\xi \in \omega_1^{M_\alpha}$, then $f_\xi^\alpha = f_\xi^\beta$.)

This can be done by the following claim:

Claim Assume that we have constructed $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \alpha \rangle$ and $Z \in (P (\omega_1^{M_\alpha}) \setminus j_{0,\alpha}(I))^{M_\alpha}$. Then there is a $(P (\omega_1^{M_\alpha}) \setminus j_{0,\alpha}(I))^{M_\alpha}$-generic filter $G_\alpha$ with $Z \in G_\alpha$ such that $g_\alpha \leq^* f_{\omega_1^{M_\alpha}}^{\alpha+1}$.

Proof of Claim. We have to notice that

• in a generic extension of $M_\alpha$ with $(P (\omega_1^{M_\alpha}) \setminus j_{0,\alpha}(I))^{M_\alpha}$, $f_\xi^{\alpha} \leq^* f_{\omega_1^{M_\alpha}}^{\alpha+1}$ holds, hence $f_{\omega_1^{M_\alpha}}^{\alpha+1}$ $\leq^*$-dominates all slaloms in $S \cap M_\alpha$, and

• for each $p \in \mathbb{D} \cap M_\alpha$, the set

$$\{ \xi \in \omega_1^{M_\alpha}; \text{body}(p) \leq f_\xi^{\alpha} \}$$

is $j_{0,\alpha}(I)$-positive.

(We note that $f_{\omega_1^{M_\alpha}}^{\alpha+1}$ is in $M_{\alpha+1}$ which is a subuniverse of $M_\alpha[G]$ and it is not changed by the transitive collapse and the relation $\leq^*$ is absolute.) So by Lemma 2.5, we can find a desired $G_\alpha$.

By the construction (and the standard argument, e.g. [11, Lemma 4.36.] or [7, 2.8 Lemma]), $j_{0,\omega_1}(I) = J \cap M_\omega$ and $j_{0,\omega_1} \left( \langle f_\xi; \xi \in \omega_1^{M} \rangle \right)$ is a scale. What we need to check is $J$-goodness of the scale.

To see $J$-goodness, take any $p \in \mathbb{D}$. Then there is $\alpha \in \omega_1$ such that $\text{body}(p) \leq^* g_\alpha$, so we can find $n \in \omega$ such that $\text{body}(p) \leq^n f_{\omega_1^{M_\alpha}}^{\alpha+1}$. Let $g \in S$ be such that $g := (\text{body}(p) \restriction n) \cup f_{\omega_1^{M_\alpha}}^{\alpha+1} \downarrow [n, \infty)$. We note that $g$ is in $M_{\alpha+1}$. Since

$$M \models " \langle f_\xi; \xi \in \omega_1^{M} \rangle \text{ is an } I\text{-good scale } " ,$$

by elementarity of $j_{0,\alpha+1}$

$$M_{\alpha+1} \models " j_{0,\alpha+1} \left( \langle f_\xi; \xi \in \omega_1^{M} \rangle \right) \text{ is an } j_{0,\alpha+1}(I)\text{-good scale } " ,$$
Therefore the set
\[ \{ \xi \in \omega_1^{M_{\alpha+1}} ; g \leq f^\alpha_\xi \} \]
belongs to $M_{\alpha+1}$ and is $j_{0,\omega_1}(I)$-positive. Since $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ and
\[
\begin{align*}
j_{\alpha+1,\omega_1}(\{ \xi \in \omega_1^{M_{\alpha+1}} ; g \leq f^\alpha_\xi \}) &= \{ \xi \in \omega_1 ; g \leq f^{\omega_1}_\xi \} \\
&\subseteq \{ \xi \in \omega_1 ; \text{body}(p) \leq f^{\omega_1}_\xi \},
\end{align*}
\]
the set $\{ \xi \in \omega_1; \text{body}(p) \leq f^{\omega_1}_\xi \}$ is $J$-positive.

We can show the strategic iteration lemma for the existence of a good scale using arguments of the previous proof and [10, Lemma 2.8]. So we can conclude Shelah–Zapletal’s theorem that $\varnothing = \aleph_1$ is $\Pi_2$-compact.

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**References**


[12] T. Yorioka. $\mathbb{P}_{\max}$ variations related to slaloms, to appear in MLQ.