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Second order Nonlinear Difference Equations whose Eigenvalues are 1

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1 Introduction

At first we consider the following second order nonlinear difference equation,

$$\begin{cases} u(t+1) = U(u(t), v(t)), \\ v(t+1) = V(u(t), v(t)), \end{cases} \tag{1.1}$$

where $U(u,v)$ and $V(u,v)$ are entire functions for u and v. We suppose that the equation (1.1) admits an equilibrium point $(u^*, v^*) = (0,0)$. Furthermore we suppose that U and V are written in the following form

$$\begin{pmatrix} u(t+1) \\ v(t+1) \end{pmatrix} = M \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} U_1(u(t), v(t)) \\ V_1(u(t), v(t)) \end{pmatrix},$$

where $U_1(u,v)$ and $V_1(u,v)$ are higher order terms of u and v. Let λ_1, λ_2 be characteristic values of matrix M. For some regular matrix P which decided by M, put $\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$, then we can transform the system (1.1) into the following simultaneous system of first order difference equations (1.2):

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \tag{1.2}$$

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where $X(x,y)$ and $Y(x,y)$ are supposed to be holomorphic and expanded in a neighborhood of $(0,0)$ in the form

$$\begin{cases} X(x,y) = \lambda_1 x + \sum_{i+j \geqq 2} c_{ij} x^i y^j = \lambda_1 x + X_1(x,y), \\ Y(x,y) = \lambda_2 y + \sum_{i+j \geqq 2} d_{ij} x^i y^j = \lambda_2 y + Y_1(x,y), \end{cases} \tag{1.3}$$

or

$$\begin{cases} X(x,y) = \lambda x + y + \sum_{i+j \geqq 2} c'_{ij} x^i y^j = \lambda x + X'_1(x,y), \\ Y(x,y) = \lambda y + \sum_{i+j \geqq 2} d'_{ij} x^i y^j = \lambda y + Y'_1(x,y), \end{cases} \tag{1.4}$$

where $\lambda = \lambda_1 = \lambda_2$.

In this note we consider analytic solutions of difference system (1.2), making use of Theorems in [1] and [4]. We will seek an analytic solution of (1.2) under the conditions $\lambda_1 = \lambda_2 = 1$ and definition (1.3). Further we suppose that

$$\begin{cases} X(x,y) = x + \sum_{i+j \geqq 2,\ i \geqq 1} c_{ij} x^i y^j = x + X_1(x,y), \\ Y(x,y) = y + \sum_{i+j \geqq 2,\ j \geqq 1} d_{ij} x^i y^j = y + Y_1(x,y), \end{cases} \tag{1.5}$$

where $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$. For the case $|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$, we obtained analytic general solutions of (1.2) in [5] and [6], For a long time we could not treat the equation (1.2) under the condition $|\lambda_1| = |\lambda_2| = 1$, because it is difficult to have an analytic solution of the equation (1.2). For analytic solutions of a nonlinear first order difference equations, Kimura [1] and Yanagihara [7] studied the cases in which the absolute value of the eigenvalue equal to 1.

Next we consider a functional equation

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{1.6}$$

where $X(x,y)$ and $Y(x,y)$ are holomorphic functions in $|x| < \delta_1$, $|y| < \delta_1$. We assume that $X(x,y)$ and $Y(x,y)$ are expanded there as in (1.5).

As far as $\frac{dx}{dt} \neq 0$, an existence of solutions of (1.2) is equivalent to an existence of solution Ψ of (1.6). Furthermore we can reduce (1.2) to the following first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))), \tag{1.7}$$

Hereafter we consider t to be a complex variable, and concentrate on the difference system (1.2). Our aim in this paper is to show the following Theorem 1.

Theorem 1 *Suppose $X(x,y)$ and $Y(x,y)$ are expanded in the forms (1.5) such that $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$.*

(1) We define domains $D_1(\kappa_0, R_0)$ *by*

$$D_1(\kappa_0, R_0) = \{t\ :\ |t| > R_0, |\arg[t]| < \kappa_0\}, \tag{1.8}$$

where κ_0 *is any constant such that* $0 < \kappa_0 \leqq \frac{\pi}{4}$ *and* R_0 *is sufficiently large number which may depend on* X *and* Y*. Further define*

$$D^*(\kappa, \delta) = \{x\,;\ |\arg[x]| < \kappa,\ 0 < |x| < \delta\}, \tag{1.9}$$

where δ *is a small constant and* κ *is a constant such that* $\kappa = 2\kappa_0$*, i.e.,* $0 < \kappa \leqq \frac{\pi}{2}$*. Suppose that* $kc_{20} = d_{11} < 0$ *for some* $k \in \mathbb{N}$*,* $k \geqq 2$*, and* $A = c_{20}$*, then we have a formal solution* $x(t)$ *of (1.2) the following form*

$$\frac{1}{At}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{1.10}$$

where $\hat{q}_{jk}$ *are constants which are defined by* X *and* Y*.*
(2) Suppose $R_1 = \max(R_0, 2/(|A|\delta))$*, then there is a solution* $x(t)$ *of (1.2) such that* $x(t) \in D^*(\kappa, \delta)$ *for* $t \in D_1(\kappa_0, R_1)$*, which the solution satisfying the following conditions:*
(i) $x(t)$ *is holomorphic in* $D_1(\kappa_0, R_1)$*.*
(ii) $x(t)$ *is expressible in the form*

$$x(t) = -\frac{1}{At}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{1.11}$$

where $b(t, \eta)$ *is holomorphic for* $t \in D_1(\kappa_0, R_1)$*,* $|\eta| < r$*, and in the expansion* $b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t)\eta^k$*,* $b_k(t)$ *is asymptotically develop-able into* $b_k(t) \sim \sum_{j+k\geqq 1}^{\infty} b_{jk} t^{-j}$*, as* $t \to \infty$ *through* $D_1(\kappa_0, R_1)$*, where* b_{jk} *are constants which are defined by* X *and* Y*.*

2 Proof of Theorem 1

In [1], Kimura considered the following first order difference equation

$$w(t + \lambda) = F(w(t)), \tag{D1}$$

where F is represented in a neighborhood of ∞ by a Laurent series

$$F(z) = z\Big(1 + \sum_{j=1}^{\infty} b_j z^{-j}\Big),\ b_1 = \lambda \neq 0. \tag{2.1}$$

He defined the following domains

$$D(\epsilon, R) = \{t\ :\ |t| > R, |\arg[t] - \theta| < \frac{\pi}{2} - \epsilon, \text{ or } \mathbf{Im}(e^{i(\theta-\epsilon)}t) > R, \text{ or } \mathbf{Im}(e^{i(\theta+\epsilon)}t) < -R\}, \tag{2.2}$$

$$\hat{D}(\epsilon, R) = \{t : |t| > R, |\arg[t] - \theta - \pi| < \frac{\pi}{2} - \epsilon \text{ or } \mathbf{Im}(e^{-i(\theta+\pi-\epsilon)}t) > R \text{ or } \mathbf{Im}(e^{-i(\theta+\pi+\epsilon)}t) < -R\}, \quad (2.3)$$

where ϵ is an arbitrarily small positive number and R is a sufficiently large number which may depend on ϵ and F, $\theta = \arg\lambda$, (in this present paper, we consider the case $\lambda = 1$ in (D1)). He proved the following theorems A and B.

Theorem A. *Equation (D1) admits a formal solution of the form*

$$t\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right) \quad (2.4)$$

containing an arbitrary constant, where $\hat{q}_{jk}$ *are constants defined by* F.

Theorem B. *Given a formal solution of the form (2.4) of (D1), there exists a unique solution* $w(t)$ *satisfying the following conditions:*
(i) $w(t)$ *is holomorphic in* $D(\epsilon, R)$,
(ii) $w(t)$ *is expressible in the form*

$$w(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right), \quad (2.5)$$

where the domain $D(\epsilon, R)$ *is defined by (2.2) and* $b(t, \eta)$ *is holomorphic for* $t \in D(\epsilon, R)$, $|\eta| < 1/R$, *and in the expansion* $b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t)\eta^k$, $b_k(t)$ *is asymptotically developable into* $b_k(t) \sim \sum_{j+k\geqq 1}^{\infty} \hat{q}_{jk} t^{-j}$, *as* $t \to \infty$ *through* $D(\epsilon, R)$, *where* $\hat{q}_{jk}$ *are constants which are defined by* X *and* Y.

Also there exists a unique solution $\hat{w}$ *which is holomorphic in* $\hat{D}(\epsilon, R)$ *and satisfies a condition analogous to (ii), where the domain* $\hat{D}(\epsilon, R)$ *is defined by (2.3).*

In Theorem A and B, he defined the function F as in (2.1). But in our method, we can not have a Laurent series of the function F. Hence we derive following Propositions.

In the following, A denotes the constant $A = c_{20}$ in Theorem 1, where c_{20} is the coefficient in (1.5) .

Proposition 2. *Suppose* $\tilde{F}(t)$ *is holomorphic and expanded asymptotically in* $\{t;\ -1/(At) \in D^*(\kappa, \delta),\ A < 0\}$ *as*

$$\tilde{F}(t) \sim t\left(1 + \sum_{j=1}^{\infty} b_j t^{-j}\right), \qquad b_1 = \lambda \neq 0,$$

where $D^*(\kappa, \delta)$ *is defined in (1.9). Then the equation*

$$\psi(\tilde{F}(t)) = \psi(t) + \lambda \quad (2.6)$$

has a formal solution

$$\psi(t) = t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right), \tag{2.7}$$

where q_1 can be arbitrarily prescribed while other coefficients are uniquely determined by b_j, $(j = 1, 2, \cdots)$, independently of q_1.

Proposition 3. *The equation (2.6) has a solution $w = \psi(t)$, which is holomorphic in $\{t;\ -1/(At) \in D^*(\kappa/2, \delta/2),\ A < 0\}$ and has asymptotic expansion* (2.7) *there.*

These Propositions are proved as in [1] pp. 212–222. Since $A = c_{20} < 0$ and $\kappa_0 = \kappa/2$, we see that $x = -1/(At) \in D^*(\kappa/2, \delta/2)$ equivalent to $t \in D_1(\kappa/2, 2/(|A|\delta)) = D_1(\kappa_0, 2/(|A|\delta))$, where $D_1(\kappa_0, R_0)$ is defined in (1.8). Further, as in [1] pp.206 and pp.228–232, we have following Proposition 4.

Proposition 4. *Suppose a function ϕ is the inverse of ψ such that $w = \psi^{-1}(t) = \phi(t)$. Then we have $\phi \circ \psi(w) = w, \psi \circ \phi(t) = t$, furthermore ϕ is holomorphic and asymptotically expanded in $\{t;\ t \in D_1(\kappa_0, 2/(|A|\delta))\}$ as*

$$\phi(t) \sim t\left\{1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right\}. \tag{2.8}$$

This function $\phi(t)$ is a solution of difference equation of (D1).

In [4], we proved the following theorem C.

Theorem C. *Suppose $X(x, y)$ and $Y(x, y)$ are defined in (1.5). Then*

(1) if $kc_{20} \neq d_{11}$ for any $k \in \mathbb{N}$, $k \geqq 2$, then the formal solution $\Psi(x)$ of (1.6) of the following form

$$\Psi(x) = \sum_{m=1}^{\infty} a_m x^m, \tag{2.9}$$

is identical to 0, i.e., $a_1 = a_2 = \cdots = 0$.
(2) if $kc_{20} = d_{11}$ for some $k \in \mathbb{N}$, $k \geqq 2$, then we have a formal solution $\Psi(x)$ of (1.6) such the following form

$$\Psi(x) = \sum_{m=k}^{\infty} a_m x^m, \tag{2.10}$$

i.e., $a_1 = a_2 = \cdots = a_{k-1} = 0$.
(3) suppose

$$kc_{20} = d_{11} < 0 \quad \textit{for some} \quad k \in \mathbb{N},\ k \geqq 2. \tag{2.11}$$

For any κ, $0 < \kappa \leqq \frac{\pi}{2}$ and small δ, we define the following domain $D^*(\kappa, \delta)$,

$$D^*(\kappa, \delta) = \{x\,;\, |\arg[x]| < \kappa,\ 0 < |x| < \delta\}. \tag{1.10}$$

there is a constant $\delta > 0$ and a solution $\Psi(x)$ of (1.6), which is holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j. \tag{2.12}$$

Proof of Theorem 1. We prove (1) of Theorem 1. We assume that $kc_{20} = q_{11} < 0$ for some $k \in \mathbb{N}$, we suppose that $R_0 > R$ and $\kappa_0 < \frac{\pi}{4} - \epsilon$. Since $\theta = \arg[\lambda] = \arg[1] = 0$, we have

$$D_1(\kappa_0, R_0) \subset D(\epsilon, R). \tag{2.13}$$

From Theorem C, for a $x \in D^*(\kappa, \delta)$ we have a solution $\Psi(x)$ of (1.6) which is holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j. \tag{2,12}$$

On the other hand putting $A = c_{20}$ and $w(t) = -\frac{1}{Ax(t)}$ in (1.7), then we have

$$w(t+1) = -\frac{1}{AX\left(-\frac{1}{Aw(t)}, \Psi\left(-\frac{1}{Aw(t)}\right)\right)}. \tag{2.14}$$

If we can have $-\frac{1}{Aw} = x \in D^*(\kappa, \delta)$, then making use of Theorem C, we have a solution $\Psi(x)$ of (1.6) such that $\Psi(x) = \Psi\left(-\frac{1}{Aw}\right) \sim \sum_{m=k}^{\infty} a_j \left(-\frac{1}{Aw}\right)^m$, $(k \geqq 2)$. Further from (1.5), we have

$$-\frac{1}{AX\Big(x, \Psi(x)\Big)} \sim w\Big[1 + c_{20}\frac{1}{A}w^{-1} + \sum_{k \geqq 2} \tilde{c}_k(w)^{-k}\Big], \tag{2.15}$$

where $\tilde{c}_k$ are defined by c_{ij} and a_k $(i + j \geqq 2,\ i \geqq 1,\ k \geqq 2)$. From (2.15) and definition of A, we can write (2.15) into the following form (2.16),

$$w(t+1) = \tilde{F}(w(t)) \sim w(t)\Big\{1 + w(t)^{-1} + \sum_{k \geqq 2} \tilde{c}_k(w(t))^{-k}\Big\}. \tag{2.16}$$

On the other hand, putting $\lambda = 1$ and $m = 1$ in (2.1), i.e. $\theta = 0$, then making use of the Theorem A, we have the following first order difference equation (D1, $\lambda = 1$)

$$w(t+1) = F(w(t)) = w(t)\Big(1 + w(t)^{-1} + \sum_{j=2}^{\infty} b_j w(t)^{-j}\Big), \tag{D1, $\lambda = 1$}$$

admits a formal solution of the form $t\left(1+\sum_{j+k\geqq 1}\hat{q}_{jk}t^{-j}\left(\frac{\log t}{t}\right)^k\right)$.

Similarly for the first order difference equation (2.16), making use of Proposition 2, we have a formal solution (2.17) of it such that,

$$w(t)=t\left(1+\sum_{j+k\geqq 1}b_{jk}t^{-j}\left(\frac{\log t}{t}\right)^k\right), \tag{2.17}$$

where b_{jk} are defined by $\tilde{F}$ in (2.16).

From $x(t)=-\frac{1}{Aw(t)}$, we have a formal solution of (1.2) such that

$$x(t)=-\frac{1}{At}\left(1+\sum_{j+k\geqq 1}b_{jk}t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}. \tag{2.18}$$

Conversely, if we have a formal function $x(t)$ such that in (2.18) exist in the domain $D^*(\kappa,\delta)$, then we can prove that the formal function (2.18) is a formal solution of (1.2), as $t\to\infty$ through $D_1(\kappa_0,R_0)$. At first we take a small $\delta>0$. For sufficiently large R, since $R_0>R$, we can have

$$|x(t)|=\left|\frac{1}{At}\right|\left|1+\sum_{j+k\geqq 1}b_{jk}t^{-j}\left(\frac{\log t}{t}\right)^k\right|^{-1}<\frac{1}{|A|R}(1+1)<\delta. \tag{2.19}$$

for $t\in D_1(\kappa_0,R_0)$. Since $A=c_{20}<0$, if we take sufficiently large R_0, then we have

$$\left|\arg\left[1+b\left(t,\frac{\log t}{t}\right)\right]\right|<\kappa_0,\quad \text{for } t\in D_i(\kappa_0,R_0).$$

Hence we have $-\kappa_0-\kappa_0\leqq\arg[x(t)]\leqq\kappa_0+\kappa_0$. From the assumption of $\kappa=2\kappa_0$, we have

$$|\arg[x(t)]|<\kappa\leqq\frac{\pi}{2}\ \text{ for } t\in D_1(\kappa_0,R_0). \tag{2.20}$$

From (2.19) and (2.20), we have that $x(t)\in D^*(\kappa,\delta)$ for a some κ, $(0<\kappa\leqq\frac{\pi}{2})$. Hence we have a $\Psi(x(t))$ which satisfies the equation (1.6) and we prove that the function $x(t)$ is a formal solution of (1.2) and holomorphic in $D_1(\kappa_0,R_0)$. Therefore we see that the function $x(t)$ in the (2.18) is a formal solution of (1.2).

Next we prove (2). Suppose that $R_1=\max(R_0,2/(|A|\delta))$, making use of Proposition 4, then we have a holomorphic solution $w(t)$ of (2.16) for $t\in D_1(\kappa_0,R_1)$, i.e., we have a solution $x(t)$ of (1.2) for t at there, in which satisfying following conditions:
(i) $x(t)$ is holomorphic in $D_1(\kappa_0,R_1)$,
(ii) $w(t)$ is expressible in the form

$$x(t)=-\frac{1}{At}\left(1+b\left(t,\frac{\log t}{t}\right)\right)^{-1}, \tag{2.21}$$

where $b(t,\eta)$ is holomorphic for $t \in D_1(\kappa_0, R_1)$, $|\eta| < r$, and in the expansion $b(t,\eta) \sim \sum_{k=1}^{\infty} b_k(t)\eta^k$, $b_k(t)$ is asymptotically develop-able into $b_k(t) \sim \sum_{j+k\geq 1}^{\infty} b_{jk}t^{-j}$, as $t \to \infty$ though $D_1(\kappa_0, R_1)$. □

Finally, we have a solution $u(t)$, $v(t)$ of (1.1) by the transformation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = P \begin{pmatrix} x(t) \\ \Psi(x(t)) \end{pmatrix}.$$

References

[1] T. Kimura, *On the Iteration of Analytic Functions*, Funkcialaj Ekvacioj, **14**, (1971), 197-238.

[2] M. Suzuki, *Holomorphic solutions of some functional equations*, Nihonkai Math. J., **5** , (1994), 109-114.

[3] M. Suzuki, *Holomorphic solutions of some system of n functional equations with n variables related to difference systems*, Aequationes Mathematicae, **57**, (1999), 21-36.

[4] M. Suzuki, *Holomorphic solutions of some functional equations II*, Southeast Asian Bulletin of Mathematics, **24** , (2000), 85-94.

[5] M. Suzuki, *Analytic General Solutions of Nonlinear Difference Equations* , preprint.

[6] M. Suzuki, *Holomorphic solutions of a Functional Equation*, preprint.

[7] N. Yanagihara, *Meromorphic solutions of some difference equations*, Funkcialaj Ekvacioj, **23**, (1980), 309-326.