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Nonconvex cost optimal control problems for semilinear second order evolution equations
2階半線形発展方程式の非凸コスト最適制御問題

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1 Introduction

In this paper we study the optimal control problem for the control system described by the semilinear evolution problem in Hilbert space of the form

$$\begin{cases} y'' + A_2(t)y' + A_1(t)y = f(t,y,y') + B_3v_3 \ \text{ in } \ (0,T) \\ y(0) = y_0 + B_1v_1, \quad y'(0) = y_1 + B_2v_2, \end{cases} \tag{1.1}$$

where $A_1(t), A_2(t)$ are time varying operators on Hilbert spaces V_1, V_2 embedded in a pivot Hilbert space H, $f(t,y,y')$ is a nonlinear function, y_0, y_1 are given initial values, v_1, v_2, v_3 are control variables, and B_1, B_2, B_3 are controllers. Under appropriate conditions on $A_1(t), A_2(t)$, y_0, y_1 and $f(t,y,y')$ in (1.1), we establish the wellposedness result and the Fréchet differentiability of solutions with respect to $v = (v_1, v_2, v_3)$ by the variational setting as in Dautray and Lions [3]. The quadratic cost optimal control theory for linear hyperbolic distributed parameter systems has been completely developed by Lions [8] and his school at the middle of 60's. After that the central theme of control theory has been moved to the nonlinear problems. Also the general nonconvex cost optimal control problems are studied extensively for nonlinear systems by many researchers (see Ahmed and Teo [1], Barbu [2], Fattorini [4], Fursikov [5], Li and Yong [10] and the references cited therein). However, in practical applications to partial differential equations, there is a few researches involving initial value controls and the attached cost functional is not necessary convex. Taking into account of this matter, we study the nonconvex cost optimal control problems for (1.1). Let $F = F(v,y)$ and $G = G(t,v,y)$ be real valued (not necessary convex in y) functions. The cost $J(v)$ attached to (1.1) is given by the following general integral cost

$$J(v) = F(v, y(v;T)) + \int_0^T G(t, v, y(v;t))dt, \tag{1.2}$$

where $y = y(v)$ is the solution of (1.1). Under the Fréchet differentiability on F, G in the argument for y and the Gâteaux differentiability on F, G in the argument for v, we establish the necessary optimality condition for optimal controls by using the Fréchet differentiability of $y(v)$ in the control variable $v = (v_1, v_2, v_3)$.

2 Semilinear second order evolution equations

Let H be a real pivot Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$. For $i=1,2$, let V_i be a real separable Hilbert space with the norm $\|\cdot\|_{V_i}$. The dual space of V_i is denoted by V_i' and the duality pairing between V_i' and V_i is denoted by $\langle\cdot,\cdot\rangle_{V_i',V_i}$. Assume that each pair (V_i,H) is a Gelfand triple space and that V_1 is continuously embedded in V_2. Let $0<T<\infty$ and let $a_i(t;\phi,\varphi), t\in[0,T]$ be a family of symmetric bilinear forms on $V_i\times V_i$, $i=1,2$. We suppose that there exist $c_{i1}>0$ such that

$$|a_i(t;\phi,\varphi)|\le c_{i1}\|\phi\|_{V_i}\|\varphi\|_{V_i} \quad \text{for all} \quad \phi,\psi\in V_i \quad \text{and} \quad t\in[0,T]; \tag{2.1}$$

and there exist $\alpha_i>0$ and $\lambda_i\in\mathbf{R}$ such that

$$a_i(t;\phi,\phi)+\lambda_i|\phi|_H^2\ge\alpha_i\|\phi\|_{V_i}^2 \quad \text{for all} \quad \phi\in V_i \quad \text{and} \quad t\in[0,T]. \tag{2.2}$$

Further, we suppose that the function $t\to a_1(t;\phi,\varphi)$ is continuously differentiable in $[0,T]$ and there exists a $c_{12}>0$ such that

$$|a_1'(t;\phi,\varphi)|\le c_{12}\|\phi\|_{V_1}\|\varphi\|_{V_1} \quad \text{for all} \quad \phi,\psi\in V_1 \quad \text{and} \quad t\in[0,T]. \tag{2.3}$$

By (2.1) we can define the operators $A_i(t)\in\mathcal{L}(V_i,V_i')$ by the relation $a_i(t;\phi,\varphi)=\langle A_i(t)\phi,\varphi\rangle_{V_i',V_i}$. In what follows, we shall write $V_1=V$ for notational simplicity.

Now we consider the following semilinear damped second order evolution equation

$$\begin{cases} y''+A_2(t)y'+A_1(t)y=f(t,y,y') \quad \text{in} \quad (0,T), \\ y(0)=y_0\in V, \quad y'(0)=y_1\in H, \end{cases} \tag{2.4}$$

where $f:[0,T]\times V_2\times H\to V_2'$. The solution Hilbert space $W(0,T)$ of (2.4) is defined by

$$W(0,T)=\{w|w\in L^2(0,T;V), w'\in L^2(0,T;V_2), w''\in L^2(0,T;V')\},$$

endowed with the norm

$$\|w\|_{W(0,T)}=\left(\|w\|^2_{L^2(0,T;V)}+\|w'\|^2_{L^2(0,T;V_2)}+\|w''\|^2_{L^2(0,T;V')}\right)^{\frac{1}{2}}.$$

A function $t\to y(t)$ is said to be a weak solution of (2.4) if $y\in W(0,T)$ and y satisfies

$$\langle y''(\cdot),\phi\rangle_{V',V}+a_2(\cdot;y'(\cdot),\phi)+a_1(\cdot;y(\cdot),\phi)=\langle f(\cdot,y(\cdot),y'(\cdot)),\phi\rangle_{V_2',V_2}$$
$$\text{for all} \quad \phi\in V \quad \text{in the sense of} \quad \mathcal{D}'(0,T)$$

$$y(0)=y_0\in V, \quad \frac{dy}{dt}(0)=y_1\in H,$$

where $\mathcal{D}'(0,T)$ is the space of distributions on $(0,T)$ (cf. Dautray and Lions [3]).

We impose the following assumptions on the nonlinear term $f:[0,T]\times V_2\times H\to V_2'$ in (2.4).

(A1) The mapping $t\to f(t,y,z)$ is strongly measurable in V_2' for all $y\in V_2$ and $z\in H$.

(A2) There exists a $\beta \in L^2(0,T;\mathbf{R}^+)$ such that

$$\|f(t,y_1,z_1)-f(t,y_2,z_2)\|_{V_2'} \le \beta(t)(\|y_1-y_2\|_{V_2}+|z_1-z_2|_H) \quad \text{a.e. } t\in[0,T]$$
$$\text{for } \ y_1,y_2\in V_2 \text{ and } \ z_1,z_2\in H.$$

(A3) There exists a $\gamma \in L^2(0,T;\mathbf{R}^+)$ such that

$$\|f(t,0,0)\|_{V_2'} \le \gamma(t) \quad \text{a.e. } t\in[0,T].$$

The following theorem on existence, uniqueness, regularity and energy equality of solutions to (2.4) holds (for a proof see [7]).

Theorem 2.1 *Assume that both $a_i, i=1,2$ satisfy* (2.1)-(2.3) *and $f(t,y,z)$ satisfy* (A1)-(A3). *Then there exists a unique weak solution $y\in W(0,T)\cap C([0,T];V)\cap C^1([0,T];H)$ of* (2.4). *Moreover, for each $t\in[0,T]$, y satisfies the energy equality*

$$\begin{aligned}
& a_1(t;y(t),y(t))+|y'(t)|_H^2+2\int_0^t a_2(\sigma;y'(\sigma),y'(\sigma))d\sigma \\
= \ & a_1(0;y_0,y_0)+|y_1|_H^2+\int_0^t a_1'(\sigma;y(\sigma),y(\sigma))d\sigma \\
& +2\int_0^t \langle f(\sigma,y(\sigma),y'(\sigma)),y'(\sigma)\rangle_{V_2',V_2}d\sigma.
\end{aligned} \tag{2.5}$$

The following energy inequality follows from the assumptions (A1)-(A3) and the energy equality (2.5): For each $t\in[0,T]$

$$\|y(t)\|_V^2+|y'(t)|_H^2+\int_0^t \|y'(\sigma)\|_{V_2}^2 d\sigma \le c(\|y_0\|_V^2+|y_1|_H^2+\|\gamma\|_{L^2(0,T;\mathbf{R}^+)}^2), \tag{2.6}$$

where c is a proper constant depending only on β in $(A.2)$.

Note here that we will omit writing the integral variables in the definite integral if there are some confusions. For example, in (2.6) we will express $\int_0^t \|y'\|_{V_2}^2 d\sigma$ instead of $\int_0^t \|y'(\sigma)\|_{V_2}^2 d\sigma$.

3 Continuity and Fréchet differentiability

Throughout this section we assume that (2.1)-(2.3) and (A1)-(A3) hold without any indication. In this section we establish the continuity and Gâteaux differentiability of the solution mapping for (2.4) on the initial values and forcing functions. Let $\mathcal{F}$ be a product space defined by

$$\mathcal{F}=V\times H\times L^2(0,T;V_2'). \tag{3.1}$$

The norm of $\mathcal{F}$ is defined by

$$\|(y_0,y_1,g)\|_{\mathcal{F}}=(\|y_0\|_V^2+|y_1|_H^2+\|g\|_{L^2(0,T;V_2')}^2)^{1/2} \quad \text{for } (y_0,y_1,g)\in\mathcal{F}.$$

For each $q = (y_0, y_1, g) \in \mathcal{F}$ we consider the following semilinear damped second order system:

$$\begin{cases} y''(q) + A_2(t)y'(q) + A_1(t)y(q) = f(t, y(q), y'(q)) + g \ \text{ in } \ (0,T), \\ y(q;0) = y_0 \in V, \ y'(q;0) = y_1 \in H, \end{cases} \tag{3.2}$$

Here in (3.2), $A_1(t), A_2(t)$ and $f(t,y,z)$ are differential operators and the nonlinear function satisfying the assumptions given in Section 2.

By virtue of Theorem 2.1, we can define uniquely the solution mapping $q = (y_0, y_1, g) \to y(q)$ of $\mathcal{F}$ into $W(0,T)$, because $f(t,y,z) + g(t)$ satisfies the assumptions (A1)-(A3).

Theorem 3.1 *The solution mapping $q = (y_0, y_1, g) \to y(q)$ of $\mathcal{F}$ into $W(0,T)$ is strongly continuous. Further, for each $q_1 = (y_0^1, y_1^1, g_1) \in \mathcal{F}$ and $q_2 = (y_0^2, y_1^2, g_2) \in \mathcal{F}$ we have the inequality*

$$\begin{aligned} &\|y(q_1;t) - y(q_2;t)\|_V^2 + |y'(q_1;t) - y'(q_2;t)|_H^2 + \int_0^t \|y'(q_1) - y'(q_2)\|_{V_2}^2 d\sigma \\ \leq\ & c(\|y_0^1 - y_0^2\|_V^2 + |y_1^1 - y_1^2|_H^2 + \|g_1 - g_2\|_{L^2(0,T;V_2)}^2), \ \textit{for all}\ t \in [0,T], \end{aligned} \tag{3.3}$$

where $c > 0$ depends only on β in (A3).

In turn, we raise the problem of differentiability of solution map $q = (y_0, y_1, g) \in \mathcal{F} \to y(q) \in W(0,T)$. The Fréchet differentiability of solution map is desirable for many applications, and then we can establish the Fréchet differentiability of solution mapping $q = (y_0, y_1, g) \in \mathcal{F} \to y(q) \in W(0,T)$ and characterize the Fréchet derivatives as the solutions of linearized second order evolution equations for (3.2).

Let X and Y be Banach spaces, and let $\mathcal{L}(X,Y)$ be a set of all bounded linear operators from X to Y. We denote the Banach space $\mathcal{L}(X,Y)$ endowed with the strong operator topology by $\mathcal{L}_s(X,Y)$, and endowed with the operator norm topology by $\mathcal{L}_u(X,Y)$.

We recall the following defintion of Fréchet differentiability of the mapping $\Phi : X \to Y$:

Definition 3.1 Let $\Phi : X \to Y$. The function Φ is said to be Fréchet differentiable at $x = x_0$, if there exists a $T \in \mathcal{L}(X,Y)$ such that

$$\frac{\|\Phi(x_0 + h) - \Phi(x_0) - Th\|_Y}{\|h\|_X} \to 0 \ \text{ as } \ \|h\|_X \to 0. \tag{3.4}$$

If Φ is Fréchet differentiable at each $x_0 \in X$, Φ is said to be Fréchet differentiable on X.

The operator T in (3.4) is called the Fréchet derivative of $\Phi(x)$ at $x = x_0$ and is denoted by $\Phi_x(x_0)$.

Assume that $\Phi : X \to Y$ is Fréchet differentiable on X. If the Fréchet derivative $\Phi_x(\xi)$ is continuous in $\xi \in X$ with respect to the norm topology of $\mathcal{L}_u(X,Y)$, Φ is said to be continuously Fréchet differentiable, or of C^1-class. The space of all continuously Fréchet differentiable functions $\Phi : X \to Y$ is denoted by $C^1(X,Y)$.

By Definition 3.4, the solution mapping $q \to y(q)$ of $\mathcal{F}$ into $W(0,T)$ is Fréchet differentiable if for any $q = (y_0, y_1, g) \in \mathcal{F}$ and any $w = (y_0^*, y_1^*, g^*) \in \mathcal{F}$ there exists a $dy(q) \in \mathcal{L}(\mathcal{F}, W(0,T))$ such that

$$\frac{\|y(q + w) - y(q) - T(q)w\|_{W(0,T)}}{\|w\|_{\mathcal{F}}} \to 0 \ \text{ as } \ \|w\|_{\mathcal{F}} \to 0. \tag{3.5}$$

The operator $dy(q)$ is called the Fréchet derivative of $y(q)$ and the function $dy(q)w \in W(0,T)$ is called the Fréchet derivative of $y(q)$ in the direction $w \in \mathcal{F}$.

Now, in order to obtain the Fréchet differentiability of the solution mapping, we impose the following assumptions on the nonlinear term $f(t,y,z)$.

(A4) For each $t \in [0,T]$ and $z \in H$, $f(t,y,z) \in C^1(V_2, V_2')$, and for each $t \in [0,T]$, $f_y(t,y,z) \in C(V_2 \times H, \mathcal{L}(V_2, V_2'))$ and there is $\beta_1 \in L^2(0,T;\mathbf{R}^+)$ such that

$$\|f_y(t,y,z)\|_{\mathcal{L}(V_2,V_2')} \le \beta_1(t)(\|y\|_{V_2} + |z|_H + 1) \quad a.e.\ t \in [0,T].$$

(A5) For each $t \in [0,T]$ and $y \in V_2$, $f(t,y,z) \in C^1(H, V_2')$ and $f_z(t,y,z) \in C(H \times V_2, \mathcal{L}(H, V_2'))$, and there is $\beta_2 \in L^2(0,T;\mathbf{R}^+)$ such that

$$\|f_z(t,y,z)\|_{\mathcal{L}(H,V_2')} \le \beta_2(t)(\|y\|_{V_2} + |z|_H + 1) \quad a.e.\ t \in [0,T].$$

Theorem 3.2 *Assume that (A4) and (A5) hold. Then the mapping $q = (y_0, y_1, g) \to y(q)$ of $\mathcal{F}$ into $W(0,T)$ is Fréchet differentiable and such the Fréchet derivative of $y(q)$ at $q = \overline{q}$ in the direction $w = (y_0^*, y_1^*, g^*) \in \mathcal{F}$, say $z = dy(\overline{q})w$, is a unique weak solution satisfying the following equation*

$$\begin{cases} z'' + A_2(t)z' + A_1(t)z = f_y(t, y(\overline{q}), y'(\overline{q}))z + f_z(t, y(\overline{q}), y'(\overline{q}))z' + g^* \quad in \quad (0,T), \\ z(0) = y_0^*, \quad z'(0) = y_1^*. \end{cases} \tag{3.6}$$

The Fréchet derivative $dy(q)$ is norm continuous in q.

Theorem 3.3 *Assume that (A4) and (A5) hold true. Then the Fréchet derivative $dy(q)$ is continuous on $\mathcal{F}$ with respect to the norm topology of $\mathcal{L}_u(\mathcal{F}, W(0,T))$.*

Remark 3.1 The Gâteaux differentiability of the mapping $q \to y(q)$ of $\mathcal{F}$ into $W(0,T)$ is proved in [7] under the same assumptions (A4) and (A5).

4 Nonconvex cost optimal control problems

Let $\mathcal{U}_i$, $i = 1,2,3$ be the Hilbert spaces of control variables v_i, $i = 1,2,3$, respectively. We define the product space

$$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \tag{4.1}$$

as the Hilbert space of control variables $v = (v_1, v_2, v_3)$. We consider the following control system

$$\begin{cases} y'' + A_2(t)y' + A_1(t)y = f(t,y,y') + B_3 v_3 \quad \text{in} \quad (0,T) \\ y(0) = y_0 + B_1 v_1 \in V, \quad y'(0) = y_1 + B_2 v_2 \in H, \end{cases} \tag{4.2}$$

in which three control variables are involved in forcing terms and initial conditions (cf. Lions and Magenes II [9; Chapter 6]). Here in (4.2), $B_1 \in \mathcal{L}(\mathcal{U}_1, V)$, $B_2 \in \mathcal{L}(\mathcal{U}_2, H)$ and $B_3 \in \mathcal{L}(\mathcal{U}_3, L^2(0,T;V_2'))$ and are controllers, $y_0 \in V$, $y_1 \in H$, $f(t,y,y')$ is a nonlinear forcing function satisfying the conditions (A1)-(A5), $v = (v_1, v_2, v_3)$ is a control variable and $y(v)$ denotes the solution state for $v = (v_1, v_2, v_3) \in \mathcal{U}$. We put $B = (B_1, B_2, B_3) \in$

$\mathcal{L}(\mathcal{U}_1, V) \times \mathcal{L}(\mathcal{U}_2, H) \times \mathcal{L}(\mathcal{U}_2, L^2(0,T;V_2'))$. By Theorem 2.1, for any $v \in \mathcal{U}$ there is a unique weak solution $y = y(v) \in W(0,T) \cap C([0,T];V)$. Hence we have the solution mapping $v \to y(v) : \mathcal{U} \to W(0,T)$. Since the mapping $\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \to \mathcal{F}$ defined by

$$(v_1, v_2, v_3) \to (y_0 + B_1 v_1, y_1 + B_2 v_2, B_3 v_3) \in \mathcal{F}$$

is affine and continuous, the following theorem follows from Theorem 3.2 and Theorem 3.3 (cf. Ha and Nakagiri [6]).

Theorem 4.1 *Assume that (A4) and (A5) hold true. Then the mapping $v \to y(v)$ of $\mathcal{U}$ into $W(0,T)$ is Fréchet differentiable on $\mathcal{U}$ and the Fréchet derivative of $y(v)$ at $v = u$ in the direction $w = (w_1, w_2, w_3) \in \mathcal{U}$, say $\xi = dy(u)w$, is a unique weak solution satisfying the following equation*

$$\begin{cases} \xi'' + A_2(t)\xi' + A_1(t)\xi = f_y(t, y(u), y'(u))\xi + f_z(t, y(u), y'(u))\xi' + B_3 w_3 \quad in \quad (0,T), \\ z(0) = B_1 w_1, \quad z'(0) = B_2 w_2. \end{cases} \tag{4.3}$$

Further the Fréchet derivative $dy(v)$ is continuous on $\mathcal{U}$ with respect to the norm topology of $\mathcal{L}_u(\mathcal{U}, W(0,T))$.

The nonconvex cost function associated with the control system (4.2) is given by

$$J(v) = F(v, y(v;T)) + \int_0^T G(t, v, y(v;t))dt, \quad \forall v \in \mathcal{U}, \tag{4.4}$$

where $F : \mathcal{U} \times V \to \mathbf{R}$, $G : [0,T] \times \mathcal{U} \times V \to \mathbf{R}$. We assume the following conditions on F and G in (4.4).

(B1) The mapping $(v, y) \to F(v, y)$ is continuous on $\mathcal{U} \times V$.

(B2) The mapping $t \to G(t, v, y)$ is measurable for all $(v, y) \in \mathcal{U} \times V$.

(B3) The mapping $y \to G(t, v, y)$ is measurable for all $(t, y) \in [0,T] \times \mathcal{U}$.

(B4) For any $v \in \mathcal{U}$ and arbitrary bounded set $K \subset V$, there exists an $m = m_{v,K} \in L^1(0,T)$ such that

$$\sup_{y \in K} |G(t, v, y)| \leq m_{v,K}(t), \quad a.e.\ t \in [0,T].$$

Let $\mathcal{U}_{ad} = \mathcal{U}_{ad}^1 \times \mathcal{U}_{ad}^2 \times \mathcal{U}_{ad}^3$ be a closed convex subset of $\mathcal{U}$, which is called the admissible set. An element $u = (u_1, u_2, u_3) \in \mathcal{U}$ is said to be the optimal control of $J(v)$ over $\mathcal{U}_{ad}$ if $u \in \mathcal{U}_{ad}$ and u satisfies $J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v)$.

On the existence of an optimal control for the cost J, we have to suppose some compactness conditions to obtain the existence of an optimal control.

(C1) The admissible set $\mathcal{U}_{ad}$ is compact in $\mathcal{U}$.

(C2) The controller $B = (B_1, B_2, B_3)$ is a compact operator.

Theorem 4.2 *Assume that (B1)-(B4) hold true. If (C1) or (C2) is satisfied, then there exists at least one optimal control u for the cost $J(v)$ in (4.4) subject to the control system (4.2).*

This existence theorem follows from the strong continuity of $y(v)$ in v in the space $W(0,T)$.

In order to give the necessary conditions for the optimal control u, we require the following additional conditions on F and G:

(D1) for fixed $v \in \mathcal{U}$ the Fréchet derivative $F_y(v,y) \in \mathcal{L}(V,\mathbf{R})$ exists and $F_y(v,y)$ is strong continuous in $(v,y) \in \mathcal{U} \times V$;

(D2) for fixed $y \in V$ the Gâteaux derivative $F_v(v,y) \in \mathcal{L}(\mathcal{U},\mathbf{R})$ exists and $F_v(v,y)$ is strong continuous in $v \in \mathcal{U}$;

(D3) for fixed $(t,v) \in [0,T] \times \mathcal{U}$ the Fréchet derivative $G_y(t,v,y) \in \mathcal{L}(V,\mathbf{R})$ exists and $G_y(t,v,y)$ is strong continuous in $(v,y) \in \mathcal{U} \times V$;

(D4) for any bounded set $K \subset \mathcal{U} \times V$, there exists an $m_K^1(t) \in L^1(0,T)$ such that

$$\sup_{(v,y)\in K} \|G_y(t,v,y)\|_{\mathcal{L}(V,\mathbf{R})} \le m_K^1(t) \quad a.e.\ t \in [0,T];$$

(D5) for fixed $(t,v) \in [0,T] \times V$ the Gâteaux derivative $G_v(t,v,y) \in \mathcal{L}(\mathcal{U},\mathbf{R})$ exists and $G_v(t,v,y)$ is strong continuous in $v \in \mathcal{U}$;

(D6) for any bounded set $K \subset \mathcal{U} \times V$, there exists an $m_K^2(t) \in L^1(0,T)$ such that

$$\sup_{(v,y)\in K} \|G_v(t,v,y)\|_{\mathcal{L}(\mathcal{U},\mathbf{R})} \le m_K^2(t) \quad a.e.\ t \in [0,T].$$

In what follows we suppose the existence of an optimal control $u = (u_1,u_2,u_3)$ of the cost (4.4). It is well known (cf. Lions [8]) that the optimality condition for u is given by the variational inequality

$$J'(u)(v-u) \ge 0 \quad \text{for all} \quad v \in \mathcal{U}_{ad}, \tag{4.5}$$

where $J'(u)$ denotes the Gâteaux derivative of $J(v)$ in (4.4) at $v = u$. In order to give the exact form of $J'(u)(v-u)$, we give the following proposition.

Proposition 4.1 *Assume that (A1)-(A5) hold and that F and G satisfy (B1)-(B4) and (C1)-(C4). Then $J(v)$ is Gâteaux differnetiable and the derivative $J'(u)(v-u)$ at the direction $v-u$ is given by*

$$\begin{aligned} J'(u)(v-u) &= F_y(u,y(u;T))\xi(T) + \int_0^T G_y(t,u,y(u;t))\xi(t)dt \\ &\quad + F_v(u,y(u;T))(v-u) + \int_0^T G_v(t,u,y(u;t))(u-v)dt, \end{aligned} \tag{4.6}$$

where ξ is the Fréchet derivative $dy(u)(v-u)$ in Theorem 4.1.

It is desirable to write down the necessary condition in terms of adjoint state equations. However, the well-posedness of adjoint system can not be verified under the conditions $(D1)$ on F_y and $(D3)$ on G_y. Hence, as in Ha and Nakagiri [6] we employ the transposition method develped by Lions [8] and Lions and Magenes [9] to define the *transposed* adjoint system.

Let $\Lambda_{\mathcal{U}_i}$ be the cannonical isomorphism of $\mathcal{U}_i$ onto $\mathcal{U}_i'$, $i = 1,2,3$. The following main theorem follows from Proposition 4.1 via the transposition method.

Theorem 4.3 *Assume all conditions in Proposition 4.1 hold. Then the optimal control* $u \in \mathcal{U}_{ad}$ *for* (4.4) *is characterized by the following system of equations and inequality:*

$$\begin{cases} y''(u) + A_2(t)y'(u) + A_q(t)y(u) = f(t, y(u), y'(u)) + B_3u_3 \text{ in } (0,T) \\ y(u;0) = y_0 + B_1u_1 \in V, \quad y'(u;0) = y_1 + B_2u_2 \in H. \end{cases}$$

$$\begin{cases} \langle p_T(u), \psi(0)\rangle + (p_T'(u), \psi'(0))_V \\ + \int_0^T \langle p(u;t), \psi'' + A_2(t)\psi' + A_1(t)\psi - f_y(t, y(u), y'(u))\psi - f_z(t, y(u), y'(u))\psi'\rangle_{V_2, V_2'} dt \\ = \langle F_y(u, y(u;T)), \psi(T)\rangle + \int_0^T \langle G_y(t, u, y(u;t)), \psi(t)\rangle dt \\ \forall \psi \in W(0,T) \text{ such that} \\ \quad \psi'' + A_2(t)\psi' + A_1(t)\psi - f_y(t, y(u), y'(u))\psi - f_z(t, y(u), y'(u))\psi' \in L^2(0,T;V_2'), \\ \quad \psi(0) \in V, \quad \psi'(0) \in H. \end{cases}$$

$$\begin{aligned} &(\Lambda_1^{-1}B_1^*p_T(u), v_1 - u_1)_{\mathcal{U}_1} + (\Lambda_2^{-1}B_2^*p_T'(u), v_2 - u_2)_{\mathcal{U}_2} + (\Lambda_3^{-1}B_3^*p(u), v_3 - u_3)_{\mathcal{U}_3} \\ &+ F_v(u, y(u;T))(v-u) + \int_0^T G_v(t, u, y(u))(v-u)dt \geq 0, \\ &\qquad\qquad \forall v = (v_1, v_2, v_3) \in \mathcal{U}_{ad} = \mathcal{U}_{ad}^1 \times \mathcal{U}_{ad}^2 \times \mathcal{U}_{ad}^3. \end{aligned}$$

Remark 4.1 The *transposed* solution $p_u = (p_T(u), p_T'(u), p(u;\cdot))$ of the adjoint system in Theorem 4.3 is verified to satisfy formally the equation

$$\begin{cases} p'' - A_2(t)p + (A_1(t) - A_2'(t))p \\ \qquad = f_y(t, y(u), y'(u))^*p + (f_z(t, y(u;t), y'(u;t))^*p)' + G_y(t, u, y(u;t)) \quad \text{in } (0,T) \\ p(u;T) = F_y(u, y(u;T)), \\ p'(u;T) = 0, \end{cases}$$

and $p_T(u) = p(u;0)$, $p_T'(u) = p'(u;0)$.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and let $Q = [0,T] \times \Omega$ and $\Sigma = [0,T] \times \partial\Omega$. We can give an application of the above Theorem 4.3 to the nonconvex cost optimal control problems for the coupled sine-Gordon equations studied in Nakagiri and Ha [11].

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2} + \alpha_{11}\dfrac{\partial y_1}{\partial t} + \alpha_{12}\dfrac{\partial y_1}{\partial t} - \beta_1 \Delta y_1 + \gamma_1 \sin y_1 + k_{11}y_1 + k_{12}y_2 = B_1 v_1(t,x) \text{ in } Q, \\ \dfrac{\partial^2 y_2}{\partial t^2} + \alpha_{21}\dfrac{\partial y_1}{\partial t} + \alpha_{22}\dfrac{\partial y_2}{\partial t} - \beta_2 \Delta y_2 + \gamma_2 \sin y_2 + k_{21}y_1 + k_{22}y_2 = B_2 v_2(t,x) \text{ in } Q, \\ y_i = 0 \text{ on } \Sigma, \\ y_i(0,x) = E_0^i w_0^i(x), \quad \dfrac{\partial y_i}{\partial t}(0,x) = E_1^i w_1^i(x) \text{ in } \Omega, \quad i = 1,2. \end{cases} \tag{4.7}$$

Here in (4.7) α_{ij}, $\beta_i > 0$, γ_i and k_{ij} are constants, v_i and w_0^i, w_1^i are control variables, and B_i and E_0^i, E_1^i are controllers defined on appropriate Hilbert spaces of control variables.

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